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EXISTENCE OF SOLUTIONS OF THE DIRICHLET PROBLEM FOR AN INFINITE SYSTEM OF NONLINEAR DIFFERENTIAL-FUNCTIONAL EQUATIONS OF ELLIPTIC TYPE

Abstract. The Dirichlet problem for an infinite weakly coupled system of semilinear differential-functional equations of elliptic type is considered. It is shown the existence of solutions to this problem. The result is based on Chaplygin's method of lower and upper functions.

Keywords: infinite systems, elliptic differential-functional equations, monotone iterative technique, Chaplygin's method, Dirichlet problem.

Mathematics Subject Classification: Primary 35J45, Secondary 35J65, 35R10, 47H07.

1. INTRODUCTION

Let S be an infinite set. Let $G \subset \mathbb{R}^m$ be an open bounded domain with $C^{2+\alpha}$ boundary ($\alpha \in (0,1)$). Let $\mathcal{B}(S)$ be the Banach space of all bounded functions $w\colon S \to \mathbb{R}$, $w(i) = w^i$ ($i \in S$) with the norm

$$\|w\|_{\mathcal{B}(S)} := \sup_{i\in S} \left|w^i\right|.$$

In $\mathcal{B}(S)$ there is a partly order $w \le \tilde{w}$ defined as $w^i \le \tilde{w}^i$ for every $i \in S$. Elements of $\mathcal{B}(S)$ will be denoted by $(w^i)_{i\in S}$, too.

Let $C(\bar{G})$ be the space of all continuous functions $v\colon \bar{G} \to \mathbb{R}$ with the norm

$$|v|_{C(\bar{G})} := \max_{x\in\bar{G}} |v(x)|.$$

In this space $v \le \tilde{v}$ means that $v(x) \le \tilde{v}(x)$ for every $x \in \bar{G}$. By $C^{l+\alpha}(\bar{G})$, where $l = 0, 1, 2, ..$ and $\alpha \in (0,1)$, we denote the space of all continuous functions in $\bar{G}$ whose derivatives of order less or equal l exist and are Hölder continuous with exponent α in G (see [5], pp. 52–53). By $H^{l,p}(G)$ we denote the Sobolev space of all functions whose weak derivatives of order l are included in $L^p(G)$ (see [1], pp. 44–46). A notation $g \in C^{l+\alpha}(\partial G)$ (resp. $g \in H^{l,p}(\partial G)$) means that there exists a function $\mathbf{g} \in C^{l+\alpha}(\bar{G})$ (resp. $\mathbf{g} \in H^{l,p}(G) \cap C(\bar{G})$) such that $\mathbf{g}(x) = g(x)$ for every $x \in \partial G$. In these spaces norms are defined as

$$|g|_{C^{l+\alpha}(\partial G)} := \inf\{|\mathbf{g}|_{C^{l+\alpha}(\bar{G})} : \mathbf{g} \in C^{l+\alpha}(\bar{G}) : \forall x \in \partial G : \mathbf{g}(x) = g(x)\}$$

and

$$|g|_{H^{2,p}(\partial G)} := \inf\{|\mathbf{g}|_{H^{2,p}(G)} : \mathbf{g} \in H^{2,p}(G) \cap C(\bar{G}) : \forall x \in \partial G : \mathbf{g}(x) = g(x)\}.$$

We denote $z = (z^i)_{i \in S} \in C_S(\bar{G})$ if $z \colon \bar{G} \to \mathcal{B}(S)$ and $z^i \colon \bar{G} \to \mathbb{R}$ $(i \in S)$ is a continuous function and $\sup_{i \in S} \left|z^i\right|_{C(\bar{G})} < \infty$. The space $C_S(\bar{G})$ is a Banach space with the norm

$$\|z\|_{C_S(\bar{G})} := \sup_{i \in S} \left|z^i\right|_{C(\bar{G})}$$

and the partly order $z \leq \tilde{z}$ defined as $z^i(x) \leq \tilde{z}^i(x)$ for every $x \in \bar{G}$, $i \in S$. The space $C_S^{l+\alpha}(\bar{G})$ is a space of all functions $(z^i)_{i \in S}$ such that $z^i \in C^{l+\alpha}(\bar{G})$ for every $i \in S$ and $\sup_{i \in S} \left|z^i\right|_{C^{l+\alpha}(\bar{G})} < \infty$. In this space the norm is defined as

$$\|z\|_{C_S^{l+\alpha}(\bar{G})} = \sup_{i \in S} \left|z^i\right|_{C^{l+\alpha}(\bar{G})}.$$

We will write that $z = (z^i)_{i \in S} \in L_S^p(G)$ if $z^i \in L^p(G)$ for every $i \in S$ and $\sup_{i \in S} \left|z^i\right|_{L^p(G)} < \infty$. A notation $z = (z^i)_{i \in S} \in H_S^{l,p}(G)$ means that $z^i \in H^{l,p}(G)$ for every $i \in S$ and $\sup_{i \in S} \left|z^i\right|_{H^{l,p}(G)} < \infty$. In these spaces the norms are defined as

$$\|z\|_{L_S^p(G)} = \sup_{i \in S} \left|z^i\right|_{L^p(G)}$$

and

$$\|z\|_{H_S^{l,p}(G)} = \sup_{i \in S} \left|z^i\right|_{H^{l,p}(G)}.$$

We consider the Dirichlet problem for an infinite weakly coupled system of semilinear differential-functional equations of the following form

$$-\mathcal{L}^i[u^i](x) = f^i(x, u(x), u), \qquad \text{for } x \in G,\ i \in S \tag{1}$$

and

$$u^i(x) = h^i(x), \qquad \text{for } x \in \partial G,\ i \in S, \tag{2}$$

where

$$\mathcal{L}^i[u^i](x) := \sum_{j,k=1}^{m} a_{jk}^i(x) u_{x_j x_k}^i(x) + \sum_{j=1}^{m} b_j^i(x) u_{x_j}^i(x),$$

which are strongly uniformly elliptic in $\bar{G}$,

$$f^i \colon \bar{G} \times \mathcal{B}(S) \times C_S(\bar{G}) \ni (x, y, z) \mapsto f^i(x, y, z) \in \mathbb{R}$$

for every $i \in S$. The notation $f(x, u(x), u)$ means that the dependence of f on the second variable is a function-type dependence and $f(x, u(x), \cdot)$ is a functional-type dependence.

A function u is said to be *regular in* $\bar{G}$ if $u \in C_S(\bar{G}) \cap C^2_S(G)$. A function u is said to be a *classical* (*regular*) *solution* of the problem (1), (2) in $\bar{G}$ if u is regular in $\bar{G}$ and fulfills the system of equations (1) in G with the condition (2). A function u is said to be a *weak solution* of the problem (1), (2) in $\bar{G}$ if $u \in L^2_S(G)$ such that $\mathcal{L}^i[u^i] \in L^2(G)$ and

$$-\int_G \mathcal{L}^i[u^i](x)\xi(x)\,dx = \int_G f^i(x, u(x), u)\xi(x)\,dx$$

for every $i \in S$ and for any test function $\xi \in C^\infty_0(\bar{G})$.

We would like to find assumptions which guarantee existence of the classical solutions of the problem (1), (2)

$$u \colon \bar{G} \to \mathcal{B}(S).$$

Regular functions $u_0 = u_0(x)$ and $v_0 = v_0(x)$ in $\bar{G}$ satisfying the infinite systems of inequalities:

$$\begin{cases} -\mathcal{L}^i[u^i_0](x) \le f^i(x, u_0(x), u_0) & \text{for } x \in G, i \in S, \\ u^i_0(x) \le h^i(x) & \text{for } x \in \partial G, i \in S, \end{cases} \tag{3}$$

$$\begin{cases} -\mathcal{L}^i[v^i_0](x) \ge f^i(x, v_0(x), v_0) & \text{for } x \in G, i \in S, \\ v^i_0(x) \ge h^i(x) & \text{for } x \in \partial G, i \in S \end{cases} \tag{4}$$

are called a *lower* and an *upper function* for the problem (1), (2), respectively.

If $u_0 \le v_0$, we define

$$\mathcal{K} := \{(x, y, z) \colon x \in \bar{G}, y \in [m_0, M_0], z \in \langle u_0, v_0 \rangle\},$$

where $m_0 := (m^i_0)_{i \in S}$, $m^i_0 := \min_{x \in \bar{G}} u^i_0(x)$, $M_0 := (M^i_0)_{i \in S}$, $M^i_0 := \max_{x \in \bar{G}} v^i_0(x)$ and $\langle u_0, v_0 \rangle := \{\zeta \in C_S(\bar{G}) \colon u_0(x) \le \zeta(x) \le v_0(x) \text{ for } x \in \bar{G}\}$.

Assumptions. We make the following assumptions.

(a) $\mathcal{L}$ is a strongly uniformly elliptic operator in $\bar{G}$, i.e., there exists a constant $\mu > 0$ such that

$$\sum_{j,k=1}^{m} a^i_{jk}(x)\xi_j\xi_k \ge \mu \sum_{j=1}^{m} \xi^2_j, \quad i \in S,$$

for all $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m$, $x \in G$.

(b) The functions a^i_{jk}, b^i_j for $i \in S$, $j, k = 1, \dots, m$ are functions of class $C^{0+\alpha}(\bar{G})$ and $a^i_{jk}(x) = a^i_{kj}(x)$ for every $x \in \bar{G}$.

(c) $h^i \in C^{2+\alpha}(\partial G)$ for every $i \in S$ and $\sup_{i\in S} \left|h^i\right|_{C^{2+\alpha}(\partial G)} < \infty$.

(d) There exists at least one ordered pair u_0, v_0 of a lower and an upper function for the problem (1), (2) in $\bar{G}$ such that

$$u_0(x) \le v_0(x) \qquad \text{for } x \in \bar{G}.$$

(e) $f(\cdot, y, z) \in C^{0+\alpha}_S(\bar{G})$ for $y \in \mathcal{B}(S)$, $z \in C_S(\bar{G})$.

(f) For every $i \in S$, $x \in \bar{G}$, $y, \tilde{y} \in \mathcal{B}(S)$, $z \in C_S(\bar{G})$

$$\left|f^i(x, y, z) - f^i(x, \tilde{y}, z)\right| \le L_f \left\|y - \tilde{y}\right\|_{\mathcal{B}(S)},$$

where $L_f > 0$ is a constant independent of i and

$$\left|f^i(x, y^1, \dots, y^{i-1}, y^i, y^{i+1}, \dots, z) - f^i(x, y^1, \dots, y^{i-1}, \tilde{y}^i, y^{i+1}, \dots, z)\right| \le \\ \le k^i \left|y^i - \tilde{y}^i\right|,$$

where $k^i > 0$ is a constant and there exists $\mathbf{k} < \infty$ such that $k^i \le \mathbf{k}$ for every $i \in S$.

(g) $f(x, \cdot, \cdot)$ is a continuous function for every $x \in \bar{G}$.

(h) $f^i(x, \cdot, z)$ is a quasi-increasing function for every $i \in S$, $x \in \bar{G}$, $z \in C_S(\bar{G})$ i.e., for every $i \in S$ for arbitrary $y, \tilde{y} \in \mathcal{B}(S)$ if $y^j \le \tilde{y}^j$ for all $j \in S$ such that $j \ne i$ and $y^i = \tilde{y}^i$, then $f^i(x, y, z) \le f^i(x, \tilde{y}, z)$ for $x \in \bar{G}$, $z \in C_S(\bar{G})$.

(i) $f^i(x, y, \cdot)$ is an increasing function for every $i \in S$, $x \in \bar{G}$, $y \in \mathcal{B}(S)$.

2. AUXILIARY RESULTS

From the assumption (f) we have $k := (k^i)_{i\in S} \in \mathcal{B}(S)$. Let $\beta = (\beta^i)_{i\in S} \in C^{0+\alpha}_S(\bar{G})$. We define the operator

$$\mathcal{P} \colon C^{0+\alpha}_S(\bar{G}) \ni \beta \mapsto \gamma \in C^{2+\alpha}_S(\bar{G}),$$

where $\gamma = (\gamma^i)_{i\in S}$ is the solution (supposedly unique) of the following problem

$$\begin{cases} -(\mathcal{L}^i - k^i\mathcal{I})[\gamma^i](x) = f^i(x, \beta(x), \beta) + k^i\beta^i(x) & \text{for } x \in G,\ i \in S, \\ \gamma^i(x) = h^i(x) & \text{for } x \in \partial G,\ i \in . \end{cases} \tag{5}$$

We remark that the problem (5) is a system of separate problems with only one equation, so $\mathcal{P}[\beta]$ is a collection of solutions of these problems.

Lemma 1. *The operator* $\mathcal{P}\colon C_S^{0+\alpha}(\bar{G}) \to C_S^{2+\alpha}(\bar{G})$ *is a continuous and bounded operator. If the operator* $\mathcal{P}$ *maps* $C_S^{0+\alpha}(\bar{G})$ *into* $C_S^{0+\alpha}(\bar{G})$*, then it is a compact operator.*

Proof. Let $\beta \in C_S^{0+\alpha}(\bar{G})$, so

$$\left|\beta^i(x) - \beta^i(\tilde{x})\right| \le H_\beta \left\|x^j - \tilde{x}^j\right\|_{\mathbb{R}^m}^\alpha ,$$

where $H_\beta > 0$ is some constant independent of i and $\|x\|_{\mathbb{R}^m} = (\sum_{j=1}^m x_j^2)^{1/2}$.

We define the operator

$$\mathbf{F} = (\mathbf{F}^i)_{i\in S}\colon C_S^{0+\alpha}(\bar{G}) \ni \beta \mapsto \delta \in C_S^{0+\alpha}(\bar{G})$$

such that for every $i \in S$

$$\mathbf{F}^i[\beta](x) = \delta^i(x) := f^i(x, \beta(x), \beta) + k^i\beta^i(x).$$

For arbitrary $i \in S$ we have:

$$\begin{aligned} \left|\delta^i(x) - \delta^i(\tilde{x})\right| &= \left|f^i(x,\beta(x),\beta) + k^i\beta^i(x) - f^i(\tilde{x},\beta(\tilde{x}),\beta) - k^i\beta^i(\tilde{x})\right| \le \\ &\le \left|f^i(x,\beta(x),\beta) - f^i(\tilde{x},\beta(x),\beta)\right| + \left|f^i(\tilde{x},\beta(x),\beta) - f^i(\tilde{x},\beta(\tilde{x}),\beta)\right| + \\ &\quad + k^i\left|\beta^i(x) - \beta^i(\tilde{x})\right| \le (H_f + L_f H_\beta + \mathbf{k}H_\beta)\left\|x - \tilde{x}\right\|_{\mathbb{R}^m}^\alpha , \end{aligned}$$

where $H_f + L_f H_\beta + \mathbf{k}H_\beta$ is some constant independent of i.

By the properties of f, we see that the operator $\mathbf{F}$ is a continuous and bounded operator.

Now, we have our problem for arbitrary $i \in S$ in the following form

$$\begin{cases} -(\mathcal{L}^i - k^i\mathcal{I})[\gamma^i](x) = \delta^i(x) & \text{for } x \in G, \\ \gamma^i(x) = h^i(x) & \text{for } x \in \partial G, \end{cases} \tag{6}$$

which satisfies the assumptions of the Schauder theorem ([7], p. 115), so the problem (6) for every $i \in S$ has a unique solution $\gamma^i \in C^{2+\alpha}(\bar{G})$ and the following estimate

$$\left|\gamma^i\right|_{C^{2+\alpha}(\bar{G})} \le C\left(\left|\delta^i\right|_{C^{0+\alpha}(\bar{G})} + \left|h^i\right|_{C^{2+\alpha}(\partial G)}\right) \tag{7}$$

holds, where $C > 0$ is independent of δ, h and i.

Let us introduce the operator

$$\mathbf{G} = (\mathbf{G}^i)_{i\in S}\colon C_S^{0+\alpha}(\bar{G}) \ni \delta \mapsto \gamma \in C_S^{2+\alpha}(\bar{G}).$$

The function

$$\gamma^i = \mathbf{G}^i[\delta^i] = \mathbf{G}_1^i[h^i] + \mathbf{G}_2^i[\delta^i],$$

where

$$\mathbf{G}_1^i\colon C^{2+\alpha}(\bar{G}) \ni h^i \mapsto \mathbf{G}_1^i[h^i] \in C^{2+\alpha}(\bar{G})$$

and $\mathbf{G}_1^i[h^i]$ is the unique solution of the problem (6) with $\delta^i(x) = 0$ in $\bar{G}$, and

$$\mathbf{G}_2^i\colon C^{0+\alpha}(\bar{G}) \ni \delta^i \mapsto \mathbf{G}_2^i[\delta^i] \in C^{2+\alpha}(\bar{G})$$

and $\mathbf{G}_2^i[\delta^i]$ is the unique solution of the problem (6) with $h^i(x) = 0$ on ∂G. The operator $\mathbf{G}^i$ is a continuous operator because the operator $\mathbf{G}_1^i$ is independent of δ^i, and $\mathbf{G}_2^i$ is a continuous operator (from (7)) with respect to δ^i. By (7), we have

$$\left|\gamma^i\right|_{C^{2+\alpha}(\bar{G})} = \left|\mathbf{G}^i \circ \mathbf{F}^i[\beta^i]\right|_{C^{2+\alpha}(\bar{G})} \leq C\left(\left|\delta^i\right|_{C^{2+\alpha}(\bar{G})} + \left|h^i\right|_{C^{2+\alpha}(\partial G)}\right),$$

where $C > 0$ is independent of δ, h and i. Thus the operator $\mathbf{G} \circ \mathbf{F}$ is a continuous and bounded operator.

Since $\partial G \in C^{2+\alpha}$, the imbedding operator

$$\mathbf{I}\colon C_S^{2+\alpha}(\bar{G}) \to C_S^{0+\alpha}(\bar{G})$$

is a compact operator ([1], p. 11). So $\mathcal{P} = \mathbf{I} \circ \mathbf{G} \circ \mathbf{F}$ is a compact operator. □

Next, let us consider the operator $\mathcal{P}$ as a operator mapping $L_S^p(G)$.

Lemma 2. *The operator $\mathcal{P}$ is a compact operator mapping $L_S^p(G)$ into $L_S^p(G)$.*

Proof. We define δ and the operator $\mathbf{F}$ such in the proof of Lemma 1 but on an element of $L_S^p(G)$.

The operator $\mathbf{F}\colon L_S^p(G) \to L_S^p(G)$ and $\mathbf{F}$ is a continuous and bounded operator by arguing as [6] (Th. 2.1, Th. 2.2 and Th. 2.3, pp. 31–37) and [9] (Th. 19.1, p. 204).

Now, we have our problem for arbitrary $i \in S$ in the following form

$$\begin{cases} -(\mathcal{L}^i - k^i\mathcal{I})[\gamma^i](x) = \delta^i(x) & \text{for } x \in G, \\ \gamma^i(x) = h^i(x) & \text{for } x \in \partial G, \end{cases} \tag{8}$$

which satisfies the assumptions of Agmon–Douglis–Nirenberg theorem for arbitrary $i \in S$, so the problem (8) has a unique weak solution $\gamma^i \in H^{2,p}(G)$ and the following estimate

$$\left|\gamma^i\right|_{H^{2,p}(G)} \leq C\left(\left|\delta^i\right|_{L^p(G)} + \left|h^i\right|_{H^{2,p}(\partial G)}\right) \tag{9}$$

holds, where $C > 0$ is independent of δ, h and i.

Let us introduce the operator

$$\mathbf{G} = (\mathbf{G}^i)_{i\in S}\colon L_S^p(G) \ni \delta \mapsto \gamma \in H_S^{2,p}(G).$$

The function

$$\gamma^i = \mathbf{G}^i[\delta^i] = \mathbf{G}_1^i[h^i] + \mathbf{G}_2^i[\delta^i],$$

where

$$\mathbf{G}_1^i\colon H^{2,p}(G) \ni h^i \mapsto \mathbf{G}_1^i[h^i] \in H^{2,p}(G)$$

and $\mathbf{G}_1^i[h^i]$ is the unique weak solution of the problem (8) with $\delta^i(x) = 0$ in $\bar{G}$, and

$$\mathbf{G}_2^i \colon L^p(G) \ni \delta^i \mapsto \mathbf{G}_2^i[\delta^i] \in H^{2,p}(G)$$

and $\mathbf{G}_2^i[\delta^i]$ is the unique solution of the problem (8) with $h^i(x) = 0$ on ∂G. The operator $\mathbf{G}^i$ is a continuous operator because the operator $\mathbf{G}_1^i$ is independent of δ^i, and $\mathbf{G}_2^i$ is a continuous operator (from (9)) with respect to δ^i. Also we know that

$$\left|\gamma^i\right|_{H^{2,p}(G)} = \left|\mathbf{G}^i \circ \mathbf{F}^i[\beta^i]\right|_{H^{2,p}(G)} \le C\left(\left|\delta^i\right|_{L^p(G)} + \left|h^i\right|_{H^{2,p}(\partial G)}\right),$$

where $C > 0$ is independent of δ, h and i. Thus the operator $\mathbf{G} \circ \mathbf{F}$ is a continuous and bounded operator. Since $\partial G \in C^{2+\alpha}$, the imbedding operator

$$\mathbf{I} \colon H_S^{2,p}(G) \to L_S^p(G)$$

is a compact operator ([1], p. 97), and $\mathcal{P} = \mathbf{I} \circ \mathbf{G} \circ \mathbf{F}$ is also a compact operator. □

Now, we prove next some properties of the operator $\mathcal{P}$.

Lemma 3. *The operator $\mathcal{P}$ is an increasing operator.*

Proof. Let $\beta(x) \le \tilde{\beta}(x)$ in $\bar{G}$, so for all $i \in S$, $\beta^i(x) \le \tilde{\beta}^i(x)$ in $\bar{G}$. Let $\gamma := \mathcal{P}[\beta]$ and $\tilde{\gamma} := \mathcal{P}[\tilde{\beta}]$. For arbitrary $i \in S$

$$\begin{cases} -(\mathcal{L}^i - k^i\mathcal{I})[\tilde{\gamma}^i - \gamma^i](x) = & \\ \quad = f^i\left(x, \tilde{\beta}(x), \tilde{\beta}\right) - f^i(x, \beta(x), \beta) + k^i\left(\tilde{\beta}^i(x) - \beta^i(x)\right) & \text{for } x \in G, \\ (\tilde{\gamma}^i - \gamma^i)(x) = 0, & \text{for } x \in \partial G. \end{cases}$$

By assumption (h), (i) and (f),

$$\begin{aligned} &-(\mathcal{L}^i - k^i\mathcal{I})[\tilde{\gamma}^i - \gamma^i](x) \ge \\ &\quad \ge \left[f^i\left(x, \beta^1(x), \dots, \beta^{i-1}(x), \tilde{\beta}^i(x), \beta^{i+1}(x), \dots, \beta\right) - f^i(x, \beta(x), \beta)\right] + \\ &\qquad + k^i\left(\tilde{\beta}^i(x) - \beta^i(x)\right) \ge 0. \end{aligned}$$

So for every $i \in S$

$$\begin{cases} -(\mathcal{L}^i - k^i\mathcal{I})[\tilde{\gamma}^i - \gamma^i](x) \ge 0 & \text{for } x \in G, \\ \tilde{\gamma}^i(x) - \gamma^i(x) = 0 & \text{for } x \in \partial G. \end{cases}$$

By the maximum principle ([8], p. 64)

$$\tilde{\gamma}^i(x) - \gamma^i(x) \ge 0 \text{ in } \bar{G}.$$

We have for all $i \in S$ $\gamma^i(x) \le \tilde{\gamma}^i(x)$, so

$$\gamma(x) \le \tilde{\gamma}(x) \text{ in } \bar{G}.$$

□

Lemma 4. *If β is an upper (resp. a lower) function for the problem* (1), (2) *in $\bar{G}$, then $\mathcal{P}[\beta] \le \beta$ (resp. $\mathcal{P}[\beta] \ge \beta$) in $\bar{G}$ and $\mathcal{P}[\beta]$ is an upper (resp. a lower) function for problem* (1), (2) *in $\bar{G}$.*

Proof. Let $\gamma = \mathcal{P}[\beta]$. By (5) we have for every $i \in S$

$$\begin{aligned} &-(\mathcal{L}^i - k^i\mathcal{I})[\gamma^i - \beta^i](x) = -(\mathcal{L}^i - k^i\mathcal{I})[\gamma^i](x) + (\mathcal{L}^i - k^i\mathcal{I})[\beta^i](x) = \\ &\quad = f^i(x,\beta(x),\beta) + k^i\beta^i(x) + \mathcal{L}^i[\beta^i](x) - k^i\beta^i(x) = f^i(x,\beta(x),\beta) + \mathcal{L}^i[\beta^i](x) \end{aligned}$$

and from (4)

$$f^i(x,\beta(x),\beta) + \mathcal{L}^i[\beta^i](x) \le 0$$

and

$$(\gamma^i - \beta^i)(x) = h^i(x) - \beta^i(x) \le 0.$$

So for every $i \in S$

$$\begin{cases} -(\mathcal{L}^i - k^i\mathcal{I})[\gamma^i - \beta^i](x) \le 0 & \text{for } x \in G, \\ (\gamma^i - \beta^i)(x) \le 0 & \text{for } x \in \partial G. \end{cases}$$

Now, by using the maximum principle ([8], th. 6, p. 64) separately for every $i \in S$

$$\gamma^i(x) - \beta^i(x) \le 0 \text{ in } \bar{G}.$$

So

$$\gamma(x) \le \beta(x) \text{ in } \bar{G}.$$

From Lemma 1 it follows that $\gamma \in C_S^{2+\alpha}(\overline{G})$ and from (5) and the assumption (f), we get for every $i \in S$

$$\begin{aligned} &-\mathcal{L}^i[\gamma^i](x) - f^i(x,\gamma(x),\gamma) = -(\mathcal{L}^i - k^i\mathcal{I})[\gamma^i](x) - f^i(x,\gamma(x),\gamma) - k^i\gamma^i(x) = \\ &\quad = f^i(x,\beta(x),\beta) + k^i\beta^i(x) - f^i(x,\gamma(x),\gamma) - k^i\gamma^i(x) \ge \\ &\quad \ge \big(f^i(x,\gamma^1(x),\ldots,\gamma^{i-1}(x),\beta^i(x),\gamma^{i+1}(x),\ldots,\gamma) - f^i(x,\gamma(x),\gamma)\big) + \\ &\qquad + k^i(\beta^i(x) - \gamma^i(x)) \ge 0 \text{ in } \bar{G}, \end{aligned}$$

so it is a upper the function for problem (1), (2) in $\bar{G}$. □

3. MAIN RESULT

Theorem. *If the assumptions (a)–(i) hold, then the problem* (1), (2) *has at least one classical solution u such that $u \in \langle u_0, v_0 \rangle$.*

Proof. By induction, we define two sequences of functions $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ by setting:

$$\begin{aligned} u_1 &= \mathcal{P}[u_0], \qquad u_n = \mathcal{P}[u_{n-1}], \\ v_1 &= \mathcal{P}[v_0], \qquad v_n = \mathcal{P}[v_{n-1}]. \end{aligned}$$

Because u_0 and v_0 are regular functions , these sequences are well defined by Lemma 1. The sequence $\{u_n\}_{n=0}^{\infty}$ is increasing and $\{v_n\}_{n=0}^{\infty}$ is decreasing by Lemma 4:

$$u_1(x) = \mathcal{P}[u_0](x) \geq u_0(x) \qquad \text{in } \bar{G},$$
$$v_1(x) = \mathcal{P}[v_0](x) \leq v_0(x) \qquad \text{in } \bar{G},$$

and by induction:

$$u_n(x) = \mathcal{P}[u_{n-1}](x) \geq u_{n-1}(x) \qquad \text{in } \bar{G},\ n = 1, 2, \dots$$
$$v_n(x) = \mathcal{P}[v_{n-1}](x) \leq v_{n-1}(x) \qquad \text{in } \bar{G},\ n = 1, 2, \dots$$

Since the operator $\mathcal{P}$ is increasing and by the assumption (d) we have

$$u_1(x) = \mathcal{P}[u_0](x) \leq \mathcal{P}[v_0](x) = v_1(x) \text{ in } \bar{G}$$

and consequently by induction

$$u_n(x) \leq v_n(x) \text{ in } \bar{G}.$$

Therefore

$$u_0(x) \leq u_1(x) \leq \cdots \leq u_n(x) \leq \cdots \leq v_n(x) \leq \cdots \leq v_1(x) \leq v_0(x).$$

The sequences $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are monotone and bounded, so they have pointwise limits and we can define:

$$\underline{u}(x) := \lim_{n\to\infty} u_n(x), \qquad \bar{v}(x) := \lim_{n\to\infty} v_n(x)$$

for every $x \in \bar{G}$.

The functions $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are functions of class $L^p_S(G)$. Let be $p \in (m, \infty)$ (we need this assumption to can use a imbedding theorem). Because $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are bounded functions in $L^p_S(G)$ and $\mathcal{P}$ is an increasing compact operator in $L^p_S(G)$ (from Lemma 2), $\{\mathcal{P}u_n\}$ and $\{\mathcal{P}v_n\}$ are converging sequences in $L^p_S(G)$ and

$$\underline{u}(x) = \lim_{n\to\infty} \mathcal{P}[u_n](x) = \lim_{n\to\infty} \mathcal{P}\big[\mathcal{P}[u_{n-1}]\big](x) = \mathcal{P}[\underline{u}](x),$$
$$\bar{v}(x) = \lim_{n\to\infty} \mathcal{P}[v_n](x) = \lim_{n\to\infty} \mathcal{P}\big[\mathcal{P}[v_{n-1}]\big](x) = \mathcal{P}[\bar{v}](x).$$

Since $\underline{u}, \bar{v} \in L^p_S(G)$ and:

$$\underline{u} = \mathcal{P}[\underline{u}], \qquad \bar{v} = \mathcal{P}[\bar{v}] \tag{10}$$

by the Agmon–Douglis–Nirenberg theorem we have

$$\underline{u}, \bar{v} \in H^{2,p}_S(G).$$

Because the Sobolev space $H^{2,p}(\bar{G})$ for $p > m$ is continuously imbedding in $C^{0+\alpha}(\bar{G})$ and

$$\left|u^i\right|_{C^{0+\alpha}(\bar{G})} \leq C \left|u^i\right|_{H^{2,p}(G)},$$

where C is independed of i ([1], p. 97–98), we get

$$\underline{u}, \bar{v} \in C_S^{0+\alpha}(\bar{G}). \tag{11}$$

Applaying the Schauder theorem to (10) separately for every $s \in S$ for (11) we get

$$\underline{u}, \bar{v} \in C_S^{2+\alpha}(\bar{G}).$$

From the proof we know that

$$u_0(x) \leq \underline{u}(x) \leq \bar{v}(x) \leq v_0(x). \qquad \square$$

Corollary. *The solutions* $\underline{u}$, $\bar{v}$ *are minimal and maximal solution of the problem* (1), (2) *in* $\langle u_0, v_0 \rangle$.

Proof. If w is a solution of the problem (1), (2) then $w(x) = \mathcal{P}[w](x)$ and $u_0(x) \leq w(x) \leq v_0(x)$. Because $\mathcal{P}$ is an increasing operator, we have

$$u_1(x) = \mathcal{P}[u_0](x) \leq \mathcal{P}[w](x) = w(x) = \mathcal{P}[w](x) \leq \mathcal{P}[v_0](x) = v_1(x)$$

and by induction we get

$$u_n(x) \leq w(x) \leq v_n(x).$$

Thus

$$\underline{u}(x) = \lim_{n\to\infty} u_n(x) \leq w(x) \leq \lim_{n\to\infty} v_n(x) = \bar{v}(x). \qquad \square$$

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