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THE EXACT VALUES OF NONSQUARE CONSTANTS FOR A CLASS OF ORLICZ SPACES

Abstract. We extend the $M_\triangle$-condition from [10] and introduce the $\Phi_\triangle$-condition at zero. Next we discuss nonsquare constant in Orlicz spaces generated by an N-function $\Phi(u)$ which satisfy $\Phi_\triangle$-condition. We obtain exact value of nonsquare constant in this class of Orlicz spaces equipped with the Luxemburg norm.

Keywords: nonsquare constant Orlicz space, $\Phi_\triangle$-condition.

Mathematics Subject Classification: 46B20, 46E30.

1. INTRODUCTION

Let $(X, \|\cdot\|)$ be a Banach space, $S(X) = \{x\colon \|x\| = 1, x \in X\}$ denote the unit sphere of X. Following James [1], X is called *uniformly nonsquare* if there is a constant $0 < c < 1$ such that for $x, y \in S(X)$, we have

$$\|x + y\| \leq 2 - 2c, \quad \text{or} \quad \|x - y\| \leq 2 - 2c.$$

To discuss the property of uniform nonsquareness, Gao and Lau [2] introduce the following concept.

Definition 1. *For a Banach space X, the parameter $J(X)$ is termed a nonsquare constant (in the sense of James) where*

$$J(X) = \sup\{\min(\|x - y\|, \|x + y\|)\colon x, y \in S(X)\} \tag{1}$$

It is easy to deduce that (cf. Gao and Lau [2]) X is uniformly nonsquare iff $J(X) < 2$.

Let:

$$\Phi(u) = \int_0^{|u|} \phi(t)dt, \qquad \Psi(v) = \int_0^{|v|} \psi(s)ds$$

be a pair of complementary N-functions, i.e., $\phi(t)$ is right continuous, $\phi(0) = 0$, and $\phi(t) \nearrow \infty$ as $t \nearrow \infty$ and the same properties has ψ . Let $(\Omega, \sum, \mu)$ be a measure space. The Orlicz space is defined by

$$L^{\Phi}(\Omega) = \{x \colon \Omega \to \mathbf{R}, \ \text{ measurable}, \ \rho_{\Phi}(\lambda x) d\mu < \infty \text{ for some } \lambda > 0\}.$$

Luxemburg norm (*gauge norm*) and *Orlicz norm* in $L^{\Phi}(G)$ are defined, respectively, by

$$\|x\|_{(\Phi)} = \inf \left\{ c > 0 \colon \rho_{\Phi}\left(\frac{x}{c}\right) \leq 1 \right\}$$

and

$$\|x\|_{\Phi} = \inf_{k>0} \frac{1}{k}[1 + \rho_{\Phi}(kx)].$$

For simplicity, we use the notations L^{Φ} and $L^{(\Phi)}$ for the Orlicz spaces equipped with the Orlicz norm $(L^{\Phi}(\Omega), \|\cdot\|_{\Phi})$ and the Orlicz spaces equipped with the Luxemburg norm $(L^{\Phi}(\Omega), \|\cdot\|_{(\Phi)})$,respectively. i.e. we denote $L^{(\Phi)} = (L^{\Phi}, \|\cdot\|_{(\Phi)})$ and $L^{\Phi} = (L^{\Phi}, \|\cdot\|_{\Phi})$.

An N-function $\Phi(u)$ is said to satisfy the $\triangle_2$-*condition* for small u (for all $u \geq 0$ or for large u), in symbol $\Phi \in \triangle_2(0)$ ($\Phi \in \triangle_2$ or $\Phi \in \triangle_2(\infty)$), if there exists $u_0 > 0$ and $c > 0$ such that $\Phi(2u) \leq c\Phi(u)$ for $0 \leq u \leq u_0$ (for all $u \geq 0$ or for $u \geq u_0$). An $\mathcal{N}$-function $\Phi(u)$ is said to satisfy the ∇_2-*condition* for small u (for all $u \geq 0$ or for large u), in symbol $\Phi \in \nabla_2(0)$ ($\Phi \in \nabla$ or $\Phi \in \nabla_2(\infty)$), if its complementary $\mathcal{N}$-function $\Psi \in \triangle_2(0)$ ($\Psi \in \triangle_2$ or $\Psi \in \triangle_2(\infty)$). The basic facts on Orlicz spaces can be found in Krasnoselskii and Rutickii [11], Lindenstrauss and Tzafriri [12], Rao and Ren [4] and Chen [3].

For nonsquare constant for the Orlicz function and sequence spaces equipped with Luxemburg norm with Φ satisfying the $\triangle_2$-condition, Ji and Wang [5] and Ji and Zhan [6] gave some expressions. Letter on, Y. Q. Yan [7] gave the corresponding results for the Orlicz spaces equipped with Orlicz norm with Φ satisfying the $\triangle_2$-condition. Using this expression, it is not easy to compute nonsquare constant for specific Orlicz spaces. For computation, Rao and Ren [9] and Y. Q. Yan [7, 8] gave estimates of nonsquare constants by Semenov and Simonenko indices of Φ, and obtain its exact value in some class of Orlicz spaces.

In view of some their results for latter use, we review the Semenov indices of Φ here:

$$\alpha_{\Phi} = \liminf_{u\to\infty} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \qquad \beta_{\Phi} = \limsup_{u\to\infty} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)};$$

$$\alpha^0_{\Phi} = \liminf_{u\to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \qquad \beta^0_{\Phi} = \limsup_{u\to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)};$$

$$\overline{\alpha}_\Phi = \inf_{u>0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \qquad \overline{\beta}_\Phi = \sup_{u>0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}.$$

Using the Semenov indices, Ren gave the following estimate of nonsquare constant.

Lemma 1 (Rao and Ren [9, p. 54]). *Let Φ and Ψ be a pair of complementary N-function. Then:*

$$\max\left(\frac{1}{\alpha_\Phi}, 2\beta_\Phi\right) \le J\left(L^{(\Phi)}[0,1]\right),$$

$$\max\left(\frac{1}{\overline{\alpha}_\Phi}, 2\overline{\beta}_\Phi\right) \le J(L^{(\Phi)}[0,\infty)),$$

$$\max\left(\frac{1}{\alpha^0_\Phi}, 2\beta^0_\Phi\right) \le J(l^{(\Phi)}).$$

Lemma 2 (Rao and Ren [9, p. 66]). *Let Φ be an N-function, and Φ_s be the inverse of:*

$$\Phi_s^{-1}(u) = \left[\Phi^{-1}(u)\right]^{1-s}\left[\Phi_0^{-1}(u)\right]^s, \quad u \ge 0,\ 0 < s \le 1$$

with $\Phi_0(u) = u^2$. If (Ω, Σ, μ) is a σ-finite space, then:

(i) *for $L^{(\Phi_s)}(\Omega)$ on (Ω, Σ, μ) with Luxemburg norm,*

$$J(L^{(\Phi_s)}(\Omega)) \le 2^{1-\frac{s}{2}},$$

(ii) *the same result holds also for $L^{\Phi_s}(\Omega)$ with Orlicz norm.*

Definition 2. *An N-function Φ is said to satisfy $\Phi_\triangle(0)$, written as $\Phi \in \Phi_\triangle(0)$, if $p = \lim\limits_{u\to 0} \frac{\ln \Phi(u)}{\ln u} < \infty$. An N-function Φ is said to satisfy $\Phi_\triangle(\infty)$, written as $\Phi \in \Phi_\triangle(\infty)$, if $p = \lim\limits_{u\to\infty} \frac{\ln \Phi(u)}{\ln u} < \infty$.*

By the definition of N-function, we easily see that $p \ge 1$. Using a similar method as [10], we have

Lemma 3.

(i) *If $\Phi \in \Phi_\triangle(0)$ and $\lim\limits_{u\to 0} \frac{\ln \Phi(u)}{\ln u} = p > 1$, then*

$$\lim_{u\to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} = 2^{-\frac{1}{p}},$$

where $\Phi^{-1}(u)$ is an inverse function of Φ.

(ii) *If $\Phi \in \Phi_\triangle(\infty)$ and $\lim\limits_{u\to\infty} \frac{\ln \Phi(u)}{\ln u} = p > 1$, then*

$$\lim_{u\to\infty} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} = 2^{-\frac{1}{p}},$$

Proof. (ii) was proved in [10]. The authors use the definition of limit on $\lim_{u\to 0}\frac{\ln\Phi(u)}{\ln u}=p$, and then give the estimate of $\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}$ to get the result. (i) is similar to (ii). But we adjust the proof in [10] and prove (ii) here.

Noting $\lim_{u\to 0}\frac{\ln\Phi(u)}{\ln u}=p$ iff $\lim_{u\to 0}\frac{\ln\Phi^{-1}(u)}{\ln u}=\frac{1}{p}$. Let $\beta(u)=\ln\Phi^{-1}(u)-\frac{1}{p}\ln u$. Then $\lim_{u\to 0}\frac{\beta(u)}{\ln u}=0$ and

$$\Phi^{-1}(u)=u^{\frac{1}{p}}e^{\beta(u)}.$$

So

$$\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}=\frac{u^{\frac{1}{p}}e^{\beta(u)}}{(2u)^{\frac{1}{p}}e^{\beta(2u)}}=2^{-\frac{1}{p}}\frac{\left(e^{\frac{\beta(u)}{\ln u}}\right)^{\ln u}}{\left(e^{\frac{\beta(2u)}{\ln 2u}}\right)^{\ln 2u}}.$$

Noting that $\ln u-\ln 2u=\ln\frac{1}{2}$ and $\lim_{u\to 0}e^{\frac{\beta(u)}{\ln u}}=\lim_{u\to 0}e^{\frac{\beta(2u)}{\ln 2u}}=1$, we have

$$\lim_{u\to 0}\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}=2^{-\frac{1}{p}}.$$ □

The next lemma, which has been proved in [10], is useful for our goal.

Lemma 4. *Let $\Phi\in\Phi_{\triangle}(\infty)$ and Ψ be its complementary N-function, $\lim_{u\to\infty}\frac{\ln\Phi(u)}{\ln u}=p>1$ and $\Phi_0(u)=u^{p_0}$ where $p_0>1$. Then:*

(i) $\lim_{v\to\infty}\frac{\ln\Psi(v)}{\ln v}=q>1$, *where* $\frac{1}{p}+\frac{1}{q}=1$,

(ii) $\lim_{u\to\infty}\frac{\ln\Phi^{-1}(u)}{\ln u}=\frac{1}{p}$,

(iii) $\lim_{u\to\infty}\frac{\ln\Phi(u)\Phi_0(u)}{\ln u}=p+p_0$,

(iv) $\lim_{u\to\infty}\frac{\Phi_0(\Phi(u))}{\ln u}=pp_0$.

Lemma 4 is also true for $\Phi\in\Phi_{\triangle}(0)$.

2. NONSQUARE CONSTANTS FOR ORLICZ SPACES WITH LUXEMBURG NORM

Now we give our main results.

Theorem 1. *Let $\Phi\in\Phi_{\triangle}(\infty)$ and $\lim_{u\to\infty}\frac{\ln\Phi(u)}{\ln u}=p>1$. Then*

$$J\left(L^{(\Phi)}[0,1]\right)=\max\left(2^{\frac{1}{p}},2^{1-\frac{1}{p}}\right). \tag{2}$$

Proof. First, by Lemma 1, we have

$$J\left(L^{(\Phi)}[0,1]\right) \geq \max\left(\frac{1}{\alpha_\Phi}, 2\beta_\Phi\right).$$

By Lemma 3, we have $\alpha_\Phi = \beta_\Phi = 2^{-\frac{1}{p}}$. Hence

$$J\left(L^{(\Phi)}[0,1]\right) \geq \max\left(2^{\frac{1}{p}}, 2^{1-\frac{1}{p}}\right).$$

Next, we will show the inequality $\leq$ in (2). Now let $1 < p \leq 2$. We choose l such that $1 < l < p \leq 2$ and take $s = \frac{2(p-l)}{p(2-l)}$. Obviously, $0 < s < 1$. Let M be the inverse function of $M^{-1}(u) = u^{-\frac{s}{2(1-s)}}[\Phi^{-1}(u)]^{\frac{1}{1-s}}$ and $\Phi_0(u) = u^2$. Then $\Phi^{-1}(u) = [M^{-1}(u)]^{1-s}[\Phi_0^{-1}(u)]^s$, i.e. $\Phi_s^{-1} = [M^{-1}]^{1-s}[\Phi_0^{-1}]^s = \Phi^{-1}$. Therefore, by Lemma 2, we have

$$J\left(L^{(\Phi)}[0,1]\right) = J(L^{(\Phi_s)}[0,1]) < 2^{1-\frac{s}{2}} = 2^{1-\frac{p-l}{p(2-l)}}.$$

Since $\lim\limits_{l\to 1} \frac{p-l}{p(2-l)} = \frac{p-1}{p}$, we get

$$J\left(L^{(\Phi)}[0,1]\right) \leq 2^{\frac{1}{p}} = \max\left(2^{\frac{1}{p}}, 2^{1-\frac{1}{p}}\right).$$

Let $2 < p < \infty$, we choose l such that $2 < p < l < \infty$ and take $s = \frac{2(l-p)}{p(l-2)}$. Obviously, $0 < s < 1$. Let M be the inverse function of $M^{-1}(u) = u^{-\frac{s}{2(1-s)}}[\Phi^{-1}(u)]^{\frac{1}{1-s}}$ and $\Phi_0(u) = u^2$. Then $\Phi^{-1}(u) = [M^{-1}(u)]^{1-s}[\Phi_0^{-1}(u)]^s$, i.e. $\Phi_s^{-1} = [M^{-1}]^{1-s}[\Phi_0^{-1}]^s = \Phi^{-1}$. Therefore,

$$J\left(L^{(\Phi)}[0,1]\right) = J(L^{(\Phi_s)}[0,1]) < 2^{1-\frac{s}{2}} = 2^{1-\frac{l-p}{p(l-2)}}.$$

Since $\lim\limits_{l\to\infty} \frac{l-p}{p(l-2)} = \frac{1}{p}$, we get

$$J\left(L^{(\Phi)}[0,1]\right) \leq 2^{1-\frac{1}{p}} = \max\left(2^{\frac{1}{p}}, 2^{1-\frac{1}{p}}\right).$$ □

For Orlicz function spaces $L^{(\Phi)}[0,\infty)$ and Orlicz sequence spaces $l^{(\Phi)}$, we have similar results.

Theorem 2. *Let* $\Phi \in \Phi_\triangle(\infty)$ *and* $\lim\limits_{u\to\infty} \frac{\ln\Phi(u)}{\ln u} = p > 1$. *Then*

$$J\left(L^{(\Phi)}[0,\infty)\right) = \max\left(2^{\frac{1}{p}}, 2^{1-\frac{1}{p}}\right). \tag{3}$$

Proof. By Lemma 1, we get

$$J\left(L^{(\Phi)}[0,\infty)\right) \geq \max\left(\frac{1}{\overline{\alpha}_\Phi}, 2\overline{\beta}_\Phi\right).$$

By Lemma 3, we have $\overline{\alpha}_\Phi \le \alpha_\Phi = 2^{-\frac{1}{p}}$, $\overline{\beta}_\Phi \ge \beta_\Phi = 2^{-\frac{1}{p}}$.

So

$$J\left(L^{(\Phi)}[0,\infty)\right) \ge \max\left(2^{\frac{1}{p}}, 2^{1-\frac{1}{p}}\right).$$

The rest of the proof is similar to the proof of Theorem 1. □

Theorem 3. *Let* $\Phi \in \Phi_\triangle(0)$ *and* $\lim\limits_{u\to 0} \frac{\ln \Phi(u)}{\ln u} = p > 1$.

Then

$$J\left(l^{(\Phi)}\right) = \max\left(2^{\frac{1}{p}}, 2^{1-\frac{1}{p}}\right). \tag{4}$$

Proof. The proof is similar to the proof of Theorem 1. □

Example 1. *Let* $L^p \in \{L^p[0,1], L^p[0,\infty), l^p\}$, $1 < p < \infty$.

Then

$$J(L^p) = \max\left(2^{\frac{1}{p}}, 2^{1-\frac{1}{p}}\right).$$

In fact, if we take $\Phi(u) = |u|^p$, *then the results is easy to be deduce from Theorems 1, 2 and 3.*

Example 2. *Let* $\Phi(u) = |u|^{2p} + 2|u|^p$, $1 < p < \infty$. *Then* $\lim\limits_{u\to\infty} \frac{\ln \Phi(u)}{\ln u} = 2p > 1$ *and* $\lim\limits_{u\to 0} \frac{\ln \Phi(u)}{\ln u} = p > 1$. *By Theorems 1, 2 and 3, we have* $J\big(l^{(\Phi)}\big) = \max\left(2^{\frac{1}{p}}, 2^{1-\frac{1}{p}}\right)$, $J\left(L^{(\Phi)}[0,1]\right) = J\big(L^{(\Phi)}[0,\infty)\big) = \max\left(2^{\frac{1}{2p}}, 2^{1-\frac{1}{2p}}\right)$.

Remark 1. *Since* $\phi(u) = \Phi'(u) = 2pu^{2p-1} + 2pu^{p-1}$ *is not convex or concave on* $[0,\infty)$, *so computation method of Y. Q. Yan in* [6] *and* [8] *is not suitable for Example 2.*

Example 3. *Let* $\Phi_{p,r} = |u|^p \ln^r(1+|u|)$. *Then* $\lim\limits_{u\to\infty} \frac{\ln \Phi(u)}{\ln u} = p > 1$ *and* $\lim\limits_{u\to 0} \frac{\ln \Phi(u)}{\ln u} =$ $= p + r > 1$. *By Theorems 1, 2 and 3, we have* $J(l^{(\Phi)}) = \max(2^{\frac{1}{p}}, 2^{1-\frac{1}{p}})$, $J\left(L^{(\Phi)}[0,1]\right) = J\big(L^{(\Phi)}[0,\infty)\big) = \max(2^{\frac{1}{p+r}}, 2^{1-\frac{1}{p+r}})$.

Example 4. *Let* Φ *be a function defined as the inverse of*

$$\Phi^{-1}(u) = [\ln(1+u)]^{\frac{1}{2p}} u^{\frac{1}{4}}, \quad u \ge 0,\ 1 < p < \infty.$$

Then:

$$\lim_{u\to\infty} \frac{\ln \Phi^{-1}(u)}{\ln u} = \lim_{u\to\infty} \frac{\frac{1}{2p}\ln\ln(1+u) + \frac{1}{4}\ln u}{\ln u} = \frac{1}{4},$$

$$\lim_{u\to 0} \frac{\ln \Phi^{-1}(u)}{\ln u} = \frac{1}{4} + \frac{1}{2p} \lim_{u\to 0} \frac{\frac{1}{\ln(1+u)} \cdot \frac{1}{1+u}}{\frac{1}{u}} = \frac{1}{4} + \frac{1}{2p}.$$

By Lemma 4, we have $\lim\limits_{u\to\infty} \frac{\ln \Phi(u)}{\ln u} = 4$, $\lim\limits_{u\to 0} \frac{\ln \Phi(u)}{\ln u} = \frac{1}{\frac{1}{4} + \frac{1}{2p}}$.

So

$$J\left(L^{(\Phi)}[0,1]\right) = J(L^{(\Phi)}[0,\infty)) = \max\left\{2^{\frac{1}{4}}, 2^{1-\frac{1}{4}}\right\} = 2^{\frac{3}{4}},$$
$$J(l^{(\Phi)}) = \max\left\{2^{\frac{1}{4}+\frac{1}{2p}}, 2^{\frac{3}{4}-\frac{1}{2p}}\right\}.$$

Remark 2. *Example 4 improve the Example 8 in Chapter II of* [9, p. 71]*, because in* [9]*, the author didn't give the exact value of* $J\big(l^{(\Phi)}\big)$.

REFERENCES

[1] James R. C.: *Uniformly non-square Banach spaces.* Ann. of Math. (1964) 80, 542–550.

[2] Gao J., Lau K. S.: *On the geometry of spheres in normed linear spaces.* J. Austral. Math. Soc. Ser. A 48(1990), 101–112.

[3] Chen S. T.: *Geometry of Orlicz Spaces.* Dissertationes Math. Warszawa, 1996, 356: 1–204.

[4] Rao M. M., Ren Z. D.: *Theory of Orlicz spaces.* New York, Marcel Dekker 1991.

[5] Ji D. H., Wang T. F.: *Nonsquare constants of normed spaces.* Acta. Sci. Math. (Szeged) 59 (1994), 719–723.

[6] Ji D. H., Zhan D. P.: *Some equivalent representations of nonsquare constants and its applications.* Northeast. Math. J. (4) 15 (1999), 439–444.

[7] Yan Y. Q.: *On the Nonsquare Constants of Orlicz Spaces with Orlicz Norm.* Canad. J. Math. Vol. 55 (1) (2003), 204–224.

[8] Yan Y. Q.: *Computation of Nonsquare Constants of Orlicz Spaces.* J. Austral. Math. Soc. (to appear)

[9] Rao M. M., Ren Z. D.: *Applications of Orlicz Spaces.* Marcel Dekker, New York, 2002.

[10] Han J., Li X.: *On Exact Value of Packing for a Class of Orlicz Spaces.* (Chinese), Journal of Tongji Univ. 30 (2002) 7, 895–899.

[11] Krasnosel'skii M. A., Rutickii Ya. B.: *Convex Functions and Orlicz Space.* Groningen, P. Noordhoff Ltd. 1961.

[12] Lindenstrauss J., Tzafriri L.: *Classical Banach Spaces, I and II.* Berlin, Springer 1977 and 1979.

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