Opuscula Mathematica • Vol. 25 • No. 2 • 2005

M. O. Otieno

ON INTERTWINING AND *w*-HYPONORMAL OPERATORS

Abstract. Given $A, B \in B(H)$, the algebra of operators on a Hilbert Space H, define $\delta_{A,B}\colon B(H) \to B(H)$ and $\Delta_{A,B}\colon B(H) \to B(H)$ by $\delta_{A,B}(X) = AX - XB$ and $\Delta_{A,B}(X) = AXB - X$. In this note, our task is a twofold one. We show firstly that if A and B^* are contractions with $C.o$ completely non unitary parts such that $X \in \ker \triangle_{A,B}$, then $X \in \ker \triangle_{A^*,B^*}$. Secondly, it is shown that if A and B^* are w-hyponormal operators such that $X \in \ker \delta_{A,B}$ and $Y \in \ker \delta_{B,A}$, where X and Y are quasi-affinities, then A and B are unitarily equivalent normal operators. A w-hyponormal operator compactly quasi-similar to an isometry is unitary is also proved.

Keywords: w-hyponormal operators, contraction operators and quasi-similarity.

Mathematics Subject Classification: 47B20.

1. INTRODUCTION

Let H be an infinite dimensional Complex Hilbert space and let $B(H)$ denote the algebra of operators from H to itself (= bounded linear transformations).

Given $A, B \in B(H)$, define $\delta_{A,B}\colon B(H) \to B(H)$ and $\Delta_{A,B}\colon B(H) \to B(H)$ by

$$\delta_{A,B}(X) = AX - XB \quad \text{and} \quad \Delta_{A,B}(X) = AXB - X.$$

The classical Putnam–Fuglede Theorem [21, p. 104] says that if A and B^* are normal operators, then $\ker \delta_{A,B} = \ker \delta_{A^*,B^*}$.

Analoguesly, if A and B^* are normal operators, then $\ker \Delta_{A,B} = \ker \Delta_{A^*,B^*}$.

A number of generalisations of the Putnam–Fuglede Theorem, and its $\Delta_{A,B}$ analogue, are to be found in the extant literature, amongst them generalisations where the normal operators A and B^* are replaced by larger classes than the normal operators. The particular classes which have drawn alot of attention are those consisting of either subnormal or hyponormal or M-hyponormal or dominant or k-quasi-hyponormal operators as well as p-hyponormal operators.

It is well known that $\ker \delta_{A,B} \subset \ker \delta_{A^*,B^*}$ ($\ker \Delta_{A,B} \subset \ker \Delta_{A^*,B^*}$) for A and B^* belonging to many a pair of these classes ([8, 12, 13, 14, 19, 23, 27, 28, 29] and some of the references there) except for when both A and B^* are dominant (see [12, 14, 15]).

In the first part of this note, using the operator equation $\Delta_{A,B}(X)$, we show among other results that Putnam–Fuglede Theorem holds true for contractions A and B^* with $C_{\cdot}o$ completely non unitary and one can easily deduce that a w-hyponormal contraction operator is unitary.

For $p > 0$, recall that ([1, 2, 12, 17]) an operator A is said to be p-hyponormal if $(A^*A)^p \geq (AA^*)^p$, where A^* denotes the adjoint of A. A p-hyponormal is called hyponormal if $p = 1$, semi-hyponormal if $p = \frac{1}{2}$. An invertible operator A is called log-hyponormal if $\log(A^*A) \geq \log(AA^*)$.

An operator A is said to be Paranormal if $\|Ax\|^2 \leq \left\|A^2x\right\| \|x\|$, for all $x \in H$, k-paranormal if $\|Ax\|^k \leq \left\|A^kx\right\| \|x\|^{k-1}$ for all $x \in H$ and k ≥ 2 is some integer and is said to be k-quasi-hyponormal if $A^{*k}(A^*A - AA^*)A^k \geq 0$ for all $x \in H$ and $k \geq 1$. Of cousre it is well known that neither the class of k-quasi-hyponormal operators nor the class of k-paranormal operators contain each other and are therefore independent.

Let $A = U|A|$ be the polar decomposition of A, then following ([1, 2]), we define the first **Aluthge transform** of A by $\widetilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$ and define the second **Aluthge transform** of A by $\widetilde{\widetilde{A}} = |\widetilde{A}|^{\frac{1}{2}}\widetilde{U}|\widetilde{A}|^{\frac{1}{2}}$, where $\widetilde{A} = \widetilde{U}|\widetilde{A}|$ is the polar decomposition of $\widetilde{A}$.

An operator A is said to be w-hyponormal if

$$|\widetilde{A}| \geq |A| \geq |\widetilde{A}^*|.$$

The classes of log-and w-hyponormal were introduced and their properties studied in [3, 4, 5, 25, 31, 32] and other references there. In particular, it was shown in [3] and [5] that the class of w-hyponormal contains both the log- and p-hyponormal operators.

The class of log-hyponormal operators were independently introduced by Tanahashi in his paper [31]. There, he gave an interesting example ([31, Example 12]) of a log-hyponormal operator which is not p-hyponormal for $p > 0$. Thus the class of p-hyponormal operators are totally independent of the class of log- hyponormal operators.

Since the class of w-hyponormal operators contains both log- and p-hyponormal operators, it therefore provides a unified approach in studying the latter classes. Indeed, Tanahashi's example can be used to show that the class of w-hyponormal operators properly contains the classes of log- and p-hyponormal operators. For if $A \in B(H)$ is the Tanahashi operator ([31, Example 12]), then $A \oplus 0$ defined on $H \oplus H$ is w-hyponormal operator but is neither log- nor p- hyponormal operator. Thus in general, if B is a non invertible p-hyponormal operator, then $A \oplus B$ is w-hyponormal but is neither log-nor p-hyponormal operator.

It is well known that if an operator A is w-hyponormal, then $\widetilde{A}$ is semi-hyponormal and $\widetilde{\widetilde{A}}$ is hyponormal.

Also if an operator A is p-hyponormal, then $\ker A \subset \ker A^*$ and if A is log-hyponormal, then $\ker A = \ker A^*$. However, if A is w-hyponormal, then it is not known whether the kernel condition $\ker A \subset \ker A^*$ holds. Nevertheless, there are several properties that p-hyponormal operators share with w-hyponormal operators A or w-hyponormal operators A with $\ker A \subset \ker A^*$ ([3] and [5]).

Recall that an operator $A \in B(H)$ is said to be dominant if for each $\lambda \in \mathbf{C}$, there exists a positive number M_λ such that

$$(A-\lambda)(A-\lambda)^* \leq M_\lambda (A-\lambda)^*(A-\lambda).$$

If the constants M_λ are bounded by a positive operator M, then A is said to be M-hyponormal.

Clearly the following inclusions hold.

$$\textit{Hyponormal} \subset p\textit{-Hyponormal}(0<p<1) \subset w\textit{-Hyponormal} \subset \textit{Paranormal} \subset K\textit{-paranormal},$$

$$\textit{Hyponormal} \subset \textit{Log-hyponormal} \subset w\textit{-Hyponormal} \subset \textit{Paranormal} \subset K\textit{-paranormal},$$

$$\textit{Hyponormal} \subset k\textit{-quasi} - \textit{hyponormal},$$

and

$$\textit{Hyponormal} \subset M\textit{-hyponormal} \subset \textit{Dominant}.$$

An operator $X \in B(H)$ is called a quasi-affinity if X is both injective and has a dense range. Two operators A and B are said to quasi-similar if $\exists$ quasiaffinities X and Y such that $X \in \ker \delta_{A,B}$ and $Y \in \ker \delta_{B,A}$.

The operator A is said to be pure if there exists no non-trivial reducing subspace N of H such that the restriction of A to N ($A\mid_N$) is normal and is completely hyponormal if it is pure.

Recall that every $A \in B(H)$ has a direct sum decomposition $A = A_1 \oplus A_2$, where A_1 and A_2 are normal and pure parts respectively. Of course in the sum decomposition, either A_1 or A_2 may be absent.

We say that the contraction $A \in$ to class $C_{.0}$ of contractions (A$\in C_{.0}$) if $A^{*n} \to 0$ strongly as n $\to \infty$. The contraction A is said to be completely non unitary (*c.n.u.*) if there exists no non-trivial reducing subspace U of H such that A restricted to U is unitary. Every contraction A has a direct sum decomposition $A = A_1 \oplus A_2$, where A_1 is unitary and A_2 is c.n.u. and of course either A_1 or A_2 may be absent. Clearly a pure contraction is completely non unitary.

Jeon and Duggal [17] have shown among other results that the normal parts of quasi-similar p-hyponormal operators are unitarily equivalent and that a p-hyponormal operator compactly quasi-similar to an isometry is unitary.

Jeon,Tanahashi and Uchiyama [25] proved that similar results of ([17]) hold true for the class of log-hyponormal operators.

In the second part of this paper, we use the second Aluthge transform operator $\widetilde{\widetilde{A}}$ and the kernel condition $\ker A \subset \ker A^*$ as major tools to show that these results ([17] and [25]) still hold true to the more general case of w-hyponormal operators.

2. INTERTWINING OF w-HYPONORMAL OPERATORS

We begin by proving results on contraction operators with $C_{.}o$ completely non unitary parts.

The following result shows that contraction operators A and B^* with $C_{.}o$ completely non unitary parts such that $X \in \ker \Delta_{A,B}$ are unitarily equivalent unitary operators.

Theorem 1. *If the contractions A and $B^* \in B(H)$ have $C_{.}o$ completely non unitary parts and $X \in \ker \Delta_{A,B}$ for some $X \in B(H)$, then $X \in \ker \Delta_{A^*,B^*}$, $\overline{\operatorname{ran} X}$ reduces A, $\ker^{\perp} X$ reduces B and $A\mid_{\overline{\operatorname{ran} X}}$ and $B\mid_{\ker^{\perp} X}$ are unitarily equivalent unitary operators.*

Proof. Decompose A and B^* into their unitary and $C_{.}o$ completely non unitary parts, $A = A_1 \oplus A_2$ and $B^* = B_1^* \oplus B_2^*$. Let $X = [X_{ij}]_{i,j=1}^2$.

Since A_2 and B_2^* both belong to $C_{.}o$ completely non unitary parts,

$$\|X_{12}x\| = \|A_1^n X_{12} B_2^n x\| \le \|X_{12}\| \, \|B_2^n x\| \to 0 \text{ as } n \to \infty$$

for all $x \in H$. Using a similar arguements to the equations $X_{21}^* \in \ker \Delta_{B_1^*,A_2^*}$ and $X_{22} \in \ker \Delta_{A_2,B_2}$, $X_{22} = X_{21} = 0$.

Consequently applying Putnam–Fuglede Theorem to $X_{11} \in \ker \Delta_{A_1,B_1}$ where A_1 and B_1 are unitary operators, $X_{11} \in \ker \Delta_{A_1^*,B_1^*}$ and the result follows. □

Corollary 2 ([13]). *If A and B^* are contractions with $C_{.}o$ completely non unitary parts such that $X \in \ker \Delta_{A,B}^n$ for some $X \in B(H)$, then the conclusions in Theorem 1 above hold.*

Proof. Let $X \in \ker \Delta_{A,B}^{n-1} = Y$, then clearly $Y \in \ker \Delta_{A,B}$ and by the Theorem, $Y \in \ker \Delta_{A^*,B^*}$. Thus $\overline{\operatorname{ran} Y}$ reduces A, $\ker^{\perp} Y$ reduces B and $A\mid_{\overline{\operatorname{ran} Y}}$ and $B\mid_{\ker^{\perp} Y}$ are unitarily equivalent unitary operators.

Let X has a matrix representation as in the proof of the Theorem. Now if $A = C_1 \oplus C_2$ and $B = D_1 \oplus D_2$ with $H = \overline{\operatorname{ran} Y} \oplus (\overline{\operatorname{ran} Y})^{\perp}$ and $H = \ker^{\perp} Y \oplus (\ker^{\perp} Y)^{\perp}$ respectively, then C_1 and D_1 are unitarily equivalent unitary operators and

$$\mathbf{Y} = \mathbf{X} \in \ker \boldsymbol{\Delta}_{\mathbf{A},\mathbf{B}}^{\mathbf{n-1}} = \begin{bmatrix} X_{11} \in \ker \Delta_{C_1,D_1}^{n-1} & X_{12} \in \ker \Delta_{C_1,D_2}^{n-1} \\ X_{21} \in \ker \Delta_{C_2,D_1}^{n-1} & X_{22} \in \ker \Delta_{C_2,D_2}^{n-1} \end{bmatrix}.$$

Clearly,

$$X_{12} \in \ker \Delta^{n-1}_{C_1,D_2} = X_{21} \in \ker \Delta^{n-1}_{C_2,D_1} = X_{22} \in \ker \Delta^{n-1}_{C_2,D_2} = 0.$$

Now $X_{11} \in \ker \Delta^{n-1}_{C_1,D_1}$ and so is $X_{11} \in \ker \Delta_{C_1,D_1}$. $X_{11} \in \ker \Delta_{C_1,D_1}$ means

$$C_1X_{11}D_1 = X_{11} = C_1X_{11}D_1 - X_{11} = C_1X_{11}D_1 - C_1C_1^*X_{11} = 0.$$

Consequently $(-1)C_1[C_1^*X_{11} - X_{11}D_1] = 0$ and $(-1)C_1(X_{11} \in \ker \delta_{C_1^*,D_1})$.

Similarly

$$X_{11} \in \ker \Delta^2_{C_1,D_1} = (-1)^2C_1^2(X_{11} \in \ker \delta^2_{C_1^*,D_1})$$

and in general

$$X_{11} \in \ker \Delta^n_{C_1,D_1} = (-1)^nC_1^n(X_{11} \in \ker \delta^n_{C_1^*,D_1}).$$

Hence by Lemma 2 of [28],

$$\lim_{n\to\infty} \left\|X_{11} \in \ker \Delta^n_{C_1,D_1}\right\|^{\frac{1}{n}} = \lim_{n\to\infty} \left\|X_{11} \in \ker \delta^n_{C_1^*,D_1}\right\|^{\frac{1}{n}} = 0.$$

Thus $X_{11} \in \ker \Delta^n_{C_1,D_1}$ is a zero operator and so $X_{11} \in \ker \Delta^{n-1}_{C_1,D_1}$.

Consequently $X \in \ker \Delta^{n-1}_{A,B}$ and $X \in \ker \Delta_{A,B}$ is a zero operator and again by the Theorem, $X \in \ker \Delta_{A^*,B^*}$ and the result follows.

Corollary 3. *If A is a k-paranormal or dominant or k-quasihyponormal contractions operator and B^* a contraction operator with $C_{.}o$ c.n.u. parts, such that $X \in \ker \Delta_{A,B}$, then $X \in \ker \Delta_{A^*,B^*}$, $\overline{\operatorname{ran} X}$ reduces A, $\ker^\perp X$ reduces B and $A\,|_{\overline{\operatorname{ran} X}}$ and $B\,|_{\ker^\perp X}$ are unitarily equivalent unitary operators.*

Clearly if in Corollary 3, X is quasiaffinity, then A and B are unitarily equivalent unitary operators.

Similarly if in Theorem 1, the same is true, then we have the following Corollary.

Corollary 4. *If the contractions A and $B^* \in B(H)$ have $C_{.}o$ completely non unitary parts such that $X \in \ker \Delta_{A,B}$ where X is quasiaffinity, then A and B are unitarily equivalent unitary operators.*

We now prove a Putnam–Fuglede Theorem $\Delta_{A,B}(X)$ analogue for w-hyponormal operators.

Theorem 5. *Let A, $B^* \in B(H)$ be w-hyponormal operators with $\ker A(B^*) \subset \ker A^*(B)$. If $X \in \ker \Delta_{A,B}$ for some $X \in B(H)$, then $X \in \ker \Delta_{A^*,B^*}$, $\overline{\operatorname{ran} X}$ reduces A, $\ker^\perp X$ reduces B and $A\,|_{\overline{\operatorname{ran} X}}$ and $B\,|_{\ker^\perp X}$ are normal operators.*

To prove the theorem, we need auxiliary lemmas.

The following lemma is well known.

Lemma 6. *If* $\ker \Delta_{A,B} \subset \ker \Delta_{A^*,B^*}$*, then, for all* $X \in \ker \Delta_{A,B}$, $\overline{\operatorname{ran} X}$ *reduces* A , $\ker^{\perp} X$ *reduces* B *and* $A\,|_{\overline{\operatorname{ran} X}}$ *and* $B\,|_{\ker^{\perp} X}$ *are normal operators.*

The next result was proved in [3, Theorem 2.4].

Lemma 7. *If* A *is* w*-hyponormal, then* $\widetilde{A}$ *is semi-hyponormal and* $\widetilde{\widetilde{A}}$ *is hyponormal.*

The following result is Theorem 2.6 of [3].

Lemma 8. *Let* A *be* w*-hyponormal with* $\ker A \subset \ker A^*$*. If* $\widetilde{A}$ *is normal, then* $A = \widetilde{A}$*.*

Proof of Theorem 5. Let $\widetilde{\widetilde{X}} = \left|\widetilde{A}\right|^{\frac{1}{2}} |A|^{\frac{1}{2}}\, X \left|\widetilde{B^*}\right|^{\frac{1}{2}} |B^*|^{\frac{1}{2}}$. Since $X \in \ker \Delta_{A,B}$, $\widetilde{\widetilde{X}} \in \ker \Delta_{\widetilde{\widetilde{A}},\widetilde{\widetilde{B}}}$, where $\widetilde{\widetilde{A}}$ and $\widetilde{\widetilde{B^*}}$ are hyponormal operators by Lemma 7.

Applying Putnam–Fuglede Theorem for hyponormal operators analogue to $\Delta_{A,B}(X)$ [15, Theorem 2], it follows that $\widetilde{\widetilde{X}} \in \ker \Delta_{\widetilde{\widetilde{A^*}},\widetilde{\widetilde{B^*}}}$. Hence by Lemma 6,

$$\overline{\operatorname{ran} \widetilde{\widetilde{X}}} \text{ reduces } \widetilde{\widetilde{A}} \text{ and } \ker^{\perp} \widetilde{\widetilde{X}} \text{ reduces } \widetilde{\widetilde{B}} \text{ and } \widetilde{\widetilde{A}}\,|_{\overline{\operatorname{ran} \widetilde{\widetilde{X}}}} \text{ and } \widetilde{\widetilde{B}}\,|_{\ker^{\perp} \widetilde{\widetilde{X}}}$$

are normal operators.

Consequently, $\widetilde{\widetilde{A}}$ and $\widetilde{\widetilde{B}}$ must be normal operators [9] and by Lemma 8, A and B are normal operators. Thus $\ker \Delta_{A,B} \subset \ker \Delta_{A^*,B^*}$, and the result follows. □

3. w-HYPONORMAL OPERATORS AND QUASI-SIMILARITY

Douglas ([11]) proved that quasi-similar normal operators are unitarily equivalent.This result was extended by Clary ([10]) who proved that quasi-similar hyponormal operators are unitarily equivalent.

In this section, we extend the result of Clary ([10]) to the class of w-hyponormal operators.

The following lemma is due to Williams [34, Lemma 1.1].

Lemma 9. *Let* A *and* B *be normal operators. If there exist injective operators* X *and* Y *such that* $X \in \ker \delta_{A,B}$ *and* $Y \in \ker \delta_{B,A}$*, then* A *and* B *are unitarily equivalent.*

Theorem 10. *Let* A *and* B^* *be* w*-hyponormal operators with* $\ker A \subset \ker A^*$ *and* $\ker B \subset \ker B^*$ *respectively. If there* $\exists$ *quasiaffinities* X *and* Y *such that* $X \in \ker \delta_{A,B}$ *and* $Y \in \ker \delta_{B,A}$*, then* A *and* B *are unitarily equivalent normal operators.*

Proof. First decompose A and B^* into their normal and pure parts by $A = A_1 \oplus A_2$ and $B^* = B_1^* \oplus B_2^*$. Let $\widetilde{\widetilde{X}} = \left|\widetilde{A}_2\right|^{\frac{1}{2}} |A_2|^{\frac{1}{2}}\, X \left|\widetilde{B_2^*}\right|^{\frac{1}{2}} |B_2^*|^{\frac{1}{2}}$. Since $X \in \ker \delta_{A_2,B_2}$, $\widetilde{\widetilde{X}} \in \ker \delta_{\widetilde{\widetilde{A}}_2,\widetilde{\widetilde{B_2}}}$, where $\widetilde{\widetilde{A}}_2$ and $\widetilde{\widetilde{B_2^*}}$ are hyponormal operators by Lemma 7 and $\widetilde{\widetilde{X}}$ is quasi-affinity.

Now by Putnam–Fuglede Theorem for hyponormal operators,

$$\widetilde{\widetilde{X}} \in \ker \delta_{\widetilde{\widetilde{A_2^*}}, \widetilde{\widetilde{B_2^*}}}$$

and

$$\overline{\operatorname{ran} \widetilde{\widetilde{X}}} \text{ reduces } \widetilde{\widetilde{A}}_2 \text{ and } \ker^{\perp} \widetilde{\widetilde{X}} \text{ reduces } \widetilde{\widetilde{B_2}} \text{ and } \widetilde{\widetilde{A}}_2 \mid_{\overline{\operatorname{ran} \widetilde{\widetilde{X}}}} \text{ and } \widetilde{\widetilde{B_2}} \mid_{\ker^{\perp} \widetilde{\widetilde{X}}}$$

are unitarily equivalent normal operators. Since $\widetilde{\widetilde{X}}$ is quasiaffinity,

$$\overline{\operatorname{ran} \widetilde{\widetilde{X}}} = H \quad \text{and} \quad \ker^{\perp} \widetilde{\widetilde{X}} = H$$

and $\widetilde{\widetilde{A}}_2$ and $\widetilde{\widetilde{B_2}}$ are unitarily equivalent normal operators. In particular $\widetilde{\widetilde{A}}_2$ and $\widetilde{\widetilde{B_2}}$ are normal operators and by Lemmas 8 and 9, the result follows. □

From the Theorem, the following corollaries are immediate.

Corollary 11. *If a w-hyponormal operator A with* $\ker A \subset \ker A^*$ *is quasi-similar to a normal operator B, then A and B are unitarily equivalent normal operators.*

Corollary 12 ([17, Corollary 6] and [25, Corollary 7]). *If a p-hyponormal or log-hyponormal A is quasi-similar to a normal operator B, then A and B are unitarily equivalent normal operators.*

During the early days of operator theory, Berberian S. K. [9] posed a very interesting question on the class of hyponormal operators: "Does there exist a completely hyponormal operator which is not normal?". While studying the concept of hyponormal operators, Ando [7] gave a negative answer to this question. That is to say, that every completely hyponormal operator is normal.

From Theorem 10, it is easy to deduce that a pure w-hyponormal operator is normal, which therefore generalises Ando's result [7].

However in the sequel, we wish to give an alternative proof of this result.

Theorem 13. *If A is w-hyponormal operator, then* $\|A^n\| = \|A\|^n$ *for all n.*

Proof. A is w-hyponormal implies

$$\|\widetilde{A}\| = \left\|\widetilde{|A|}\right\| \geq \||A|\| = \|A\|.$$

But

$$\|A\| \geq \|\widetilde{A}\| \geq \|\widetilde{\widetilde{A}}\|$$

is always true. Hence $\|A\| = \|\widetilde{A}\|$. Similarly, $\|\widetilde{A}\| = \|\widetilde{\widetilde{A}}\|$. Now since $\widetilde{\widetilde{A}}$ is hyponormal, by [7]

$$\|A\|^n = \|\widetilde{\widetilde{A}}\|^n = \|\widetilde{\widetilde{A^n}}\| = \|A^n\|$$

for all n. □

Corollary 14. *Every non-zero w-hyponormal operator has a non-zero element in its spectrum.*

Corollary 15. *A pure w-hyponormal operator is normal.*

Stampfli and Wadhwa ([30]) proved that if A is dominant and B is a normal operator such that $X \in \ker \delta_{A,B}$ where X has a dense range, then A is normal.

Recently, Duggal and Jeon ([17]) and Jeon,Tanahashi and Uchiyama ([25]) extended this result to a more general case of p-hyponormal and log-hyponormal respectively.

In the sequel, we try to extend the results of ([17]) and ([25]) to the class of w-hyponormal operators.

Theorem 16 (Generalised Putnam–Fuglede). *Let A be w-hyponormal with $\ker A \subset \ker A^*$ and B be a normal operator. If there exists an operator $X \in B(H)$ with a dense range such that $X \in \ker \delta_{A,B}$, then A is normal.*

Proof. Decompose $A = A_1 \oplus A_2$ into normal and pure parts respectively. Let $A_2 = U_2 |A_2|$, $\widetilde{A}_2 = |A_2|^{\frac{1}{2}} U |A_2|^{\frac{1}{2}}$ and $\widetilde{\widetilde{A}}_2 = \left|\widetilde{A}_2\right|^{\frac{1}{2}} \widetilde{U} \left|\widetilde{A}_2\right|^{\frac{1}{2}}$.

A_2 being pure, it is injective and $|A_2|^{\frac{1}{2}}$ is quasiaffinity. Also since A_1 is normal, $\widetilde{\widetilde{A}} = \widetilde{\widetilde{A}}_1 \oplus \widetilde{\widetilde{A}}_2 = A_1 \oplus \widetilde{\widetilde{A}}_2$.

Now if we let T $= \left|\widetilde{A}_2\right|^{\frac{1}{2}} |A_2|^{\frac{1}{2}}$, then by a simple computation, $\widetilde{\widetilde{A}}_2 T = T A_2$ and T is quasiaffinity.

Also if we let $Z = I_H \oplus T$, then clearly Z is also quasiaffinity such that $\widetilde{\widetilde{A}} Z = ZA$, where $\widetilde{\widetilde{A}}$ is a hyponormal operator.

Thus $\widetilde{\widetilde{A}} ZX = ZAX = ZXB$ and by ([30]), $\widetilde{\widetilde{A}}$ is normal. Hence by Lemma 8, we get the result .

Thus from Theorem 16, we immediately recapture Corollary 11 again. However, the following Corollary says more than this.

Corollary 17. *Let A be w-hyponormal with $\ker A \subset \ker A^*$ and B be a normal operator. If there exists a quasiaffinity $X \in B(H)$ such that $X \in \ker \delta_{A,B}$, then A and B are unitarily equivalent normal operators.*

The following example due to Clary ([10]) shows that it is not possible to replace a normal operator in Corollary 11 with an isometry.

Example 18. *Let U denote the unweighted unilateral shift with multiplicity 1, and let S_n be the unilateral weighted shift with weights $\frac{1}{n}, 1, 1, 1, \ldots$. Let $U_\infty := \sum^{\infty} \oplus U$ and $S := \sum_{i=1}^{\infty} \oplus S_i$. Then each S_n is similar to U and so S and U_∞ are quasi-similar by [22, Theorem 2.5]. Clearly U_∞ is an isometry and S is hyponormal. But since S is not bounded from below, U_∞ and S are not similar.*

This therefore gave rise to the following question.

Problem. Is it possible to replace the normality of B in Corollary 11 with an isometry?

However, in affirmative answer to this question, Duggal and Jeon ([17]) recently proved the result for the case of p-hyponormal operators under the condition that either X or Y is compact.

In the sequel, we try to extend this result to a more general case of w-hyponormal operators as follows.

Theorem 19. *Let A be w-hyponormal with $\ker A \subset \ker A^*$ and B be an isometry. If there exist quasiaffinities $X, Y \in B(H)$ such that $X \in \ker \delta_{A,B}$ and $Y \in \ker \delta_{B,A}$ where X or Y is compact, then A and B are unitarily equivalent unitary operators.*

Proof. Since $\tilde{\tilde{A}}$ is hyponormal, by [36, Theorem 2.1], $\tilde{\tilde{A}}$ is subdecomposable and so $\sigma(\tilde{\tilde{A}}) \subseteq \sigma(B) \subseteq \overline{\mathbf{D}}$, where $\overline{\mathbf{D}}$ denotes a closed unit disc.
Now since $\sigma(\tilde{\tilde{A}}) = \sigma(A)$ by [3, Corollary 2.3],

$$\|A\| = \|\tilde{\tilde{A}}\| = r(\tilde{\tilde{A}}) = r(A) \subseteq \sigma(A) \subseteq \overline{\mathbf{D}}$$

and A is a contraction. Applying theorem 2 of [6] to $Y \in \ker \delta_{B,A}$ if Y is compact, B is unitary. Similarly, by the same theorem if X is compact, then $B(YX) = YAX = = (YX)B$ and B is unitary. □

Now applying theorem 1 of [6] to the operator equation $Y \in \ker \delta_{B,A}$, A is unitary and the result follows.

Corollary 20. *If a log or p-hyponormal operator A is quasi-similar to an isometry B, then A and B are unitarily equivalent unitary operators.*

Acknowledgements

I am very gratefull to the International Science programme (ISP) of Uppsala University, Sweden for funding my visit to Lund University. Many thanks are also due to the Department of Mathematics, LTH, Lund University, Lund for the use of their facilities during the preparation of this article.

REFERENCES

[1] Aluthge A.: *On p-hyponormal operators for* $0 < p < 1$. Integral Equations and Operator Theory 13 (1990), 307–315.

[2] Aluthge A.: *Some generalized theorems on p-hyponormal operators.* Integral Equations and Operator Theory 24 (1996), 497–501.

[3] Aluthge A., Wang D.: *On w-hyponormal operators.* Integral Equations and Operator Theory 36 (2000), 1–10.

[4] Aluthge A., Wang D.: *On w-hyponormal operators II.* Integral Equations and Operator Theory 37 (3) (2000), 324–331.

[5] Aluthge A., Wang D.: *An operator inequality which implies paranormality.* Math. Inequality Appl. 2(1999), 113–119.

[6] Ando T., Takahashi K.: *On operators with unitary ρ-dilations* Ann. Polon. Math. 66 (1997), 324–331.

[7] Ando T.: *On Hyponormal Operators.* Proc. Amer. Math. Soc. 14 (1963), 290–291.

[8] Berberian S. K.: *Extensions of a theorem of Fuglede and Putnam.* Proc. Amer. Math. Soc. 71 (1978), 113–114.

[9] Berberian S. K.: *Introduction to Hilbert Spaces.* New York, Oxford Univ. Press 1961.

[10] Clary S.: *Equality of spectra of quasi-similar hyponormal operators.* Proc. Amer. Math. Soc. 53 (1975), 88–90.

[11] Douglas R. G.: *On the operator $S^*XT = X$ and related topics.* Acta Sci. Math. 30 (1969), 19–32.

[12] Duggal B. P.: *Quasi-similar p-hyponormal operators.* Integral Equations and Operator Theory 26 (1996), 338–345.

[13] Duggal B. P.: *On Intertwining operators.* Mh. Math. 106 (1988), 139–148.

[14] Duggal B. P.: *On generalised Putnam–Fuglede theorems.* Mh. Math. 107 (1989), 309–332.

[15] Duggal B. P.: *A remark on generalised Putnam–Fuglede theorems.* Proc. Amer. Math. Soc. 129 (2001), 83–87.

[16] Duggal B. P.: *On characterising contractions with C_{10} pure part.* Integral Equations and Operator Theory 27 (1997), 314–323.

[17] Duggal B. P., Jeon I. H.: *p-Hyponormal operators and quasi-similarity.* Integral Equations and Operator Theory 49 (2004), 397–403.

[18] Dunford D., Schwartz J.: *Linear operators*, part III: *Spectral operators.* New York, Interscience (1971).

[19] Fan M.: *An asymmetric Putnam–Fuglede theorem for θ-class operators and some related topics.* J. Fudan Univ. Natur. Science 26 (1987), 347–350 (Chinese).

[20] Fong C. K.: *On M-hyponormal operator*, Studia Math., 65 (1979), 1–5.

[21] Halmos P. R.: *A Hilbert Space Problem Book.* Springer-Verlag, 1982.

[22] Hoover T. B.: *Quasi-similarity of operators.* Illinois Jour. Math. 16 (1972), 678–686.

[23] Chuan Hou J., *On Putnam–Fuglede theorems for non-normal operators.* Acta Math. Sinica 28 (1985), 333–340 (Chinese).

[24] Jeon I. H.: *Weyl's theorem and quasi-similarity*. Integral Equations and Operator Theory 39 (2001), 214–221.

[25] Jeon I. H., Tanahashi K., Uchiyama A.: *On quasi-similarity for Log-hyponormal Operators*. Glassgow Math. Jour. 46 (2004), 169–176.

[26] Radjabalipour M.: *Ranges of hyponormal operators*. Illinois Jour. Math. 21 (1977), 70–75.

[27] Radjabalipour M.: *On majorization and normality of operators*. Proc. Amer. Math. Soc. 62 (1977), 105–110.

[28] Radjabalipour M.: *An extension of Putnam–Fuglede theorem for hyponormal operators*. Math. Z. 194 (1987), 117–120.

[29] Radjavi H., Rosenthal P.: *On roots of normal operators*. J. Math. Anal. Appl. 34 (1971), 653–665.

[30] Stampfli J. G., Wadhwa B. L.: *An asymmetric Putnam–Fuglede theorem for dominant operators*. Indiana Univ. Math. Jour. 25 (1976), 359–365.

[31] Tanahashi K.: *On log-hyponormal operators*. Integral Equations and Operator Theory 34 (1999), 364–372.

[32] Uchiyama A., Tanahashi K.: *Fuglede-Putnam theorem for p-hyponormal or log-hyponormal operators*. Glassgow Math. Jour. 44 (2002), 397–410.

[33] Wadhwa B. L.: *M-hyponormal operators*. Duke Math. Journal 41 (1974), 655–660.

[34] Williams L. R.: *Quasi-similarity and hyponormal operators*. Jour. Oper. Theory 5 (1981), 678–686.

[35] Xia D.: *Spectral theory for hyponormal operators*. Basel, Birkhäuser (1981).

[36] Yang L.: *Quasi-similarity of hyponormal and subdecomposable operators* Jour. Funct. Anal. 112 (1993), 204–217.

M.O. Otieno
oouma@hotmail.com, (currently) maurice@maths.lth.se

University of Nairobi
Department of Mathematics
P.O. Box 30197, Nairobi, Kenya.

Received: January 20, 2005.