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TOPOLOGICAL APPROACH TO CHAIN RECURRENCE IN CONTINUOUS DYNAMICAL SYSTEMS

Abstract. In this paper we present equivalent definitions of chain recurrent set for continuous dynamical systems. This definitions allow us to define chain recurrent set in topological spaces.

1. INTRODUCTION

In his paper [4] Conley defined weak form of recurrence called chain recurrence. Theory of chain recurrence is much developed, especially in the theory of discrete dynamical systems [1, 2, 3] and it is essential part of stability theory. When we make computer simulation, calculating orbit of a point in each step we have rounding errors, so each time we obtain pseudo-orbit (ε-chain) instead of true orbit of observed point. So we will rather detect chain recurrent points than periodic orbits in computer experiments.

Chain recurrent set of continuous dynamical system on compact metric space has many interesting properties: it is closed, flow invariant and restriction of the flow to chain recurrent set does not change chain recurrent points. It is also well known that flow restricted to limit set of any point in X is chain recurrent.

To define chain recurrent point in a way proposed by Conley we need to have flow on metric space. However authors of [3] proved that in a case of discrete dynamical systems given by homeomorphisms f chain recurrent points and chain recurrent set may be defined in a equivalent way, using asymptotically stable sets or topological properties of f. We will show that similar results may be obtained in a case of

continuous dynamical systems. This result show that chain recurrent set depends only on the topology of X.

In this paper we will give two equivalent definitions of chain recurrent set. We will show that point x is chain recurrent iff it belongs to the set given by intersection of all asymptotically stable sets containing positive limit set $L^+(x)$ of x. We will also show that point is not chain recurrent if and only if it has open neighborhood of a special kind (see Definition 16).

2. CHAIN RECURRENCE

Let X be a compact metric space with metric d. We say that a continuous function $\varphi \colon \mathbb{R} \times X \to X$ is a *flow* (*dynamical system*) if $\varphi(0,x) = x$ and $\varphi(s,\varphi(t,x)) = \varphi(s+t,x)$ for any s,t,x. We will usually write $\varphi(I \times A) = \varphi(I,A)$ and $\varphi_t = \varphi(t,\cdot)$. If $I = \{t\}$ we will write $\varphi(t,A)$.

By *an orbit* of point x we mean the set $o(x) = \{\varphi(t,x) \colon t \in \mathbb{R}\}$ and by *positive (negative) semiorbit* we mean the set $o^+(x) = \{\varphi(t,x) \colon t \geq 0\}$ ($o^-(x) = \{\varphi(t,x) \colon t \leq 0\}$).

For a given point x we define *positive limit set* (or ω-*limit set*) of x as $L^+(x) = \{y \in X \colon \varphi(t_n,x) \to y \text{ for some } t_n \to \infty\}$.

A set A is said to be *positively (resp. negatively) invariant* if $o^+(x) \subset A$ (resp. $o^-(x) \subset A$) for every $x \in A$. If A is both positively and negatively invariant then we say that it is *invariant*.

Let (X,φ) be a flow and let $x,y \in X$. Given $\varepsilon > 0$ and $T > 0$ an (ε,T)-*chain* from x to y is a pair of finite sets of points $\{x_0,\ldots,x_{p+1}\}$ and $\{t_0,\ldots,t_p\}$ such that $x = x_0$, $y = x_{p+1}$, $t_j > T$ and $d(\varphi(t_j,x_j),x_{j+1}) < \varepsilon$ for $j = 0,\ldots,p$.

If for any $\varepsilon > 0$ and $T > 0$ there exists (ε,T)-chain from x to y then we write xPy. Point x is chain-recurrent if xPx and the set $CR(\varphi) = \{x \mid xPx\}$ is said to be *chain recurrent set* of φ.

We recall that chain recurrent set is closed and invariant. It is also known that that connected components of $CR(\varphi)$ are equivalence classes of the relation $\sim$ where $x \sim y$ iff xPy and yPx (see [4, Thm. 3.1]). Moreover set of chain recurrent points does not change if we restrict flow to the set $CR(\varphi)$ [6, Prop. 2.1]. Similar theory is developed for discrete dynamical systems (see [2, 1]).

3. ASYMPTOTICAL STABILITY

In this section we will prove some properties of asymptotically stable sets. This properties will be useful in the next section.

Let X be a compact metric space with metric d and let $A \subset X$ be closed and positively invariant. Set A is said to be *stable* if for every open neighborhood U of A exists open neighborhood V of A such that for every $x \in V$ positive semiorbit $o^+(x) \subset A$. If A is stable and there exists its open neighborhood U_0 such that $L^+(x) \subset A$ for every $x \in U_0$ then A is called *asymptotically stable*.

Theorem 1. *Let A be asymptotically stable. Then there exists open neighborhood U_0 of A such that if U is any open neighborhood of A then exists $T_U > 0$ with property that*

$$\forall \, x \in U_0 \quad o^+(\varphi(T_U, x)) \subset U.$$

Proof. Set A is asymptotically stable so there exists open neighborhood W of A such that $L^+(x) \subset A$ for any $x \in W$. Sets A and $X \backslash W$ are compact, so there exists open set U_0 with property that $A \subset U_0 \subset \overline{U_0} \subset W$. Observe that $L^+(x) \subset A$ for any point $x \in \overline{U_0}$.

Let us take any open neighborhood U of A. We may also assume that $U \subset U_0$. There exists open neighborhood V of A such that $o^+(x) \subset U$ for any $x \in V$. Furthermore

$$\forall \, x \in \overline{U_0} \quad \exists s_x > 0 \quad \varphi(s_x, x) \in V.$$

Map φ_{s_x} is continuous so there exists open neighborhood W_x of x such that $\varphi_{s_x}(W_x) \subset V$. Set $\overline{U_0}$ is compact and family $\{W_x\}$ is its open cover, so we may choose finite cover $W_{x_1}, \ldots, W_{x_p}$ of $\overline{U_0}$. As for any $y \in W_{x_i}$ and $t > s_{x_i}$ we have $\varphi(t, y) \in U$ it is enough to take $T_U = \max_{i=1,\ldots,p} \{s_{x_i}\}$. $\quad\square$

Theorem 2. *Let A be closed and positively invariant set. If there exists open neighborhood U_0 of A such that for any open neighborhood U of A exists time $T_U > 0$ with property*

$$\forall \, x \in U_0 \quad o^+(\varphi(T_U, x)) \subset U$$

then A is asymptotically stable.

Proof. We must only show that A is stable. Let us assume that it is false. There exists open neighborhood U of A and sequences $\{t_k\}, \{x_k\}$ such that

$$x_k \to x_0 \in A, \ t_k > 0, \ \varphi(t_k, x_k) \notin U.$$

However $t_k < T_U$ and we may assume that $t_k \to t_0 \in [0, T_u]$. Set A is positively invariant and φ is continuous, so $\varphi(x_k, t_k) \to \varphi(x_0, t_0) \in A \cap (X \backslash U) = \emptyset$. $\quad\square$

Theorem 3. *Let A be closed and positively invariant set. If there exists open neighborhood V of A fulfilling following conditions:*

$$\exists \, T_V > 0 \quad \forall \, x \in \overline{V} \quad o^+(\varphi(T_V, x)) \subset V \qquad (3.1)$$

$$\bigcap_{n \in \mathbb{N}} \varphi(nT_V, \overline{V}) \subset A \qquad (3.2)$$

then A is asymptotically stable.

Proof. Let $U_0 = V$. We will show that U_0 fulfills assumptions of Theorem 2.

Let us take any open neighborhood U of A and observe that $\varphi(T_V, \overline{V}) \subset V \subset \overline{V}$. This implies that $\varphi(nT_V, \overline{V}) \subset \varphi((n-1)T_V, \overline{V})$ and so sets $\varphi(nT_V, \overline{V})$ forms

decreasing family. Then by condition (3.2) exists positive integer n such that $\varphi(nT_V, \overline{V}) \subset U$ and so for any $x \in U_0$

$$o^+\big(\varphi((n+1)T_V, x)\big) = \varphi\big(nT_V, o^+(\varphi(T_V, x))\big) \subset \varphi\big(nT_V, \overline{V}\big) \subset U.$$

It is enough to take $T_U = nT_V$ to finish the proof. $\square$

Corollary 4. *Let V be open set which fulfills condition* (3.1). *In this case set*

$$A = \bigcap_{n \in \mathbb{N}} \varphi\big(nT_V, \overline{V}\big)$$

is asymptotically stable.

Theorem 5. *Let A be asymptotically stable. There exists open neighborhood W of A fulfilling conditions:*

$$\exists\, T_W > 0 \quad \forall\, x \in \overline{W} \quad o^+\big(\varphi(T_W, x)\big) \subset W \tag{3.3}$$

$$\bigcap_{n \in \mathbb{N}} \varphi\big(nT_W, \overline{W}\big) \subset A \tag{3.4}$$

Furthermore, if U is open neighborhood of A, then W may be chosen in a way that $A \subset W \subset \overline{W} \subset U$.

Proof. Theorem 1 implies that there exists open neighborhood U_0 of A such that for any open neighborhood U of A exists $T_U > 0$ with property

$$\forall\, x \in U_0 \quad o^+\big(\varphi(T_U, x)\big) \subset U.$$

Sets A and $X \backslash U_0$ are compact, so there exists open neighborhood W of A such that $A \subset W \subset \overline{W} \subset U_0$. If U is fixed we may choose W in a way that $\overline{W} \subset U$.

Observe that there exists $T_W > 0$ such that

$$\forall\, x \in U_0 \quad o^+\big(\varphi(T_W, x)\big) \subset W.$$

This implies that W fulfills condition (3.3) and family $\big\{\varphi(nT_W, \overline{W})\big\}$ is decreasing. Then to prove (3.4) it is enough to find N large enough to have $\varphi(NT_W, \overline{W}) \subset U$.

By the definition of U_0 there exists time $T_U > 0$ such that

$$\forall\, x \in U_0 \quad o^+\big(\varphi(T_U, x)\big) \subset U \tag{3.5}$$

Let us take positive integer N_U such that $N_U T_W > T_U$. Condition (3.5) implies that $\varphi(N_U T_W, \overline{W}) \subset U$. $\square$

Corollary 6. *Let sets A_1 be A_2 asymptotically stable. Then set $A = A_1 \cap A_2$ is also asymptotically stable.*

Remark 7. *Let A be closed and positively invariant set. Theorems 1 and 2 give equivalent definition of asymptotical stability of A. Theorems 3 and 5 give another one.*

4. EQUIVALENT DEFINITIONS OF CHAIN RECURRENT SET

In this section we will assume that (X, d) is a compact metric space and φ is a flow. However most of techniques used in this and previous section will work in the case that X is T_2 topological space and φ is semi-dynamical system.

Definition 8. *Let $\mathcal{A}(x)$ denote the family of the sets*

$$\mathcal{A}(x) = \left\{ A \mid L^+(x) \subset A \text{ and } A \text{ is asymptotically stable} \right\}.$$

By $Q(x)$ we will denote the set

$$Q(x) = \bigcap_{A \in \mathcal{A}(x)} A.$$

Remark 9. *Set $Q(x)$ is closed and positively invariant.*

Definition 10. *The set of all points $y \in X$ such that there exists $\left(\frac{1}{n}, n\right)$-chain from x to y will be denoted by $R_n(x)$.*

Lemma 11. *Set $R_n(x)$ has following properties:*

$$R_n(x) \text{ is open} \tag{4.6}$$

$$y \in \overline{R_n(x)} \implies o^+\big(\varphi(4n, y)\big) \subset R_n(x) \tag{4.7}$$

$$o^+\big(\varphi(4n, x)\big) \subset R_n(x) \text{ and } L^+(x) \subset R_n(x) \tag{4.8}$$

Proof. Let sequences

$$\{x, x_1, \ldots, x_p, y\}, \ \{t_1, \ldots, t_p\}$$

form $\left(\frac{1}{n}, n\right)$-chain from x to y. By the definition $d(\varphi(t_p, x_p), y) < \frac{1}{n}$, so there exists $\delta > 0$ such that $d(\varphi(t_p, x_p), y) + \delta < \frac{1}{n}$. Then for any point $z \in K(y, \delta)$ sequences

$$\{x, x_1, \ldots, x_p, z\}, \ \{t_1, \ldots, t_p\}$$

form $\left(\frac{1}{n}, n\right)$-chain from x to y. This implies that ball $B(y, \delta) \subset R_n(x)$ and $R_n(x)$ is open. This proves condition (4.6).

Let $y \in \overline{R_n(x)}$. Then there exists sequence $\{x_k\} \subset R_n(x)$ such that $x_k \to y \in R_n(x)$. Map φ_{2n} is continuous so there exists $\delta > 0$ such that

$$\forall x \in X \qquad d(x, y) < \delta \implies d(\varphi_{2n}(x), \varphi_{2n}(y)) < \frac{1}{n}.$$

Let us take $K \in \mathbb{N}$ large enough to have $d(x_K, y) < \delta$. Point $x_K \in R_n(x)$ so there exists sequences

$$\{x, x_1, \ldots, x_p, x_K\}, \ \{t_0, \ldots, t_p\}$$

forming $\left(\frac{1}{n}, n\right)$-chain from x to x_K.

Observe that sequences

$$\{x, x_1, \ldots, x_p, x_K, \varphi(2n, y), \varphi(2n + t, y)\}, \ \{t_0, \ldots, t_p, 2n, t\}$$

form $\left(\frac{1}{n}, n\right)$-chain from x to $\varphi(t, y)$ for any $t > n$. This proves condition (4.7).

Sequences $\{x, \varphi(t, x)\}$, $\{t\}$ form $\left(\frac{1}{n}, n\right)$-chain iff $t > n$. This implies that for any fixed $T > 0$ we have inclusion $o^+(\varphi_T(x)) \subset R_n(x)$. If $y \in L^+(x)$ then there exists $T > 0$ such that $d(\varphi(T, x), y) < \frac{1}{n}$. Then sequences $\{x, y\}$, $\{T\}$ form $\left(\frac{1}{n}, n\right)$-chain from x to y and we obtain condition (4.8). $\qquad \square$

Theorem 12. *If $y \in Q(x)$ then xPy.*

Proof. It is enough to show that for any $n \in \mathbb{N}$ point $y \in R_n(x)$. Let us take any $n \in \mathbb{N}$ and let $V = R_n(x)$. Set V is open and for $T_V = 4n$ we have

$$\forall \, x \in \overline{V} \quad o^+\big(\varphi(T_V, x)\big) \subset V.$$

The set $A = \bigcap_{n \in \mathbb{N}} \varphi(nT_V, \overline{V})$ is, by Corollary 4, asymptotically stable. By (4.8) we have inclusion $L^+(x) \subset A$. Observe that $y \in Q(x) \subset A \subset R_n(x)$. $\qquad \square$

Lemma 13. *If U is open neighborhood of $Q(x)$ then there exists asymptotically stable set A such that*

$$Q(x) \subset A \subset U.$$

Proof. Let $\mathcal{A}$ be the family of asymptotically stable sets containing $L^+(x)$. Observe that $\mathcal{A} \neq \emptyset$ as it contains X.

Suppose that there exists open neighborhood U of $Q(x)$ such that $A \cap (X \backslash U) \neq \emptyset$ for any $A \in \mathcal{A}$. Let us denote by $\mathbf{S}$ the family

$$\mathbf{S} = \{S_A = A \cap (X \backslash U) \colon A \in \mathbf{A}\}.$$

By Corollary 6 family $\mathbf{S}$ is centered, so there exists $y \in X$ such that

$$\forall \, A \in \mathbf{A} \qquad y \in S_A.$$

It is a contradiction, as otherwise $y \in Q(x) \cap (X \backslash U) = \emptyset$. $\qquad \square$

Theorem 14. *If xPy, then $y \in Q(x)$.*

Proof. Suppose that $y \notin Q(x)$ and observe that then there exists open neighborhood U of $Q(x)$ such that $y \notin U$. Lemma 13 guaranties that there exists asymptotically stable set A such that $Q(x) \subset A \subset U$. By Theorem 5 there exists open set W and time T_W such that $A \subset W \subset \overline{W} \subset U$ and conditions (3.3) and (3.4) are fulfilled. For any $z \in \overline{W}$ we have inclusion

$$o^+(\varphi(2T_W, z)) = \varphi\big(T_W, o^+(\varphi(T_W, z))\big) \subset \varphi(T_W, W) \subset \varphi(T_W, \overline{W}) \subset W.$$

If we take $\varepsilon > 0$ small enough to have $d(\varphi(T_W, \overline{W}), X \backslash W) > \varepsilon$ then for any $T > 2T_W$ every (ε, T)-chain starting in x must end in W and it is in contradiction with xPy. $\qquad\square$

Corollary 15.
$$x \in CR(\varphi) \iff x \in Q(x)$$

Proof. Consequence of Theorem 12 and 14. $\qquad\square$

Definition 16. *We define set $CR_T(\varphi)$ in a following way. Point $x \in CR_T(\varphi)$ iff there do not exist open set U and time $T_U > 0$ such that*

$$\forall \, y \in \overline{U} \qquad o^+\big(\varphi(T_U, y)\big) \subset U \tag{4.9}$$

$$x \notin \overline{U} \ and \ \varphi(T_U, x) \in U \tag{4.10}$$

Remark 17. *It is enough to assume that $x \notin U$ in Definition 16.*

Proof. Suppose that $x \in \partial U$ and U fulfills conditions (4.9) and (4.10).
Then

$$\forall \, y \in \overline{U} \quad o^+\big(\varphi(2T_U, y)\big) = \varphi\big(T_U, o^+(\varphi(T_U, y))\big) \subset \varphi(T_U, U) \subset \varphi(T_U, \overline{U}) \subset U$$

and there exists open set W such that

$$\varphi(T_U, \overline{U}) \subset W \subset \overline{W} \subset U.$$

Let us take $T_W = 2T_U$ and observe that

$$\varphi(T_W, x) = \varphi\big(T_U, \varphi(T_U, x)\big) \in \varphi(T_U, U) \subset W.$$

We may replace U by W to keep both conditions (4.9) and (4.10) fulfilled. $\qquad\square$

Theorem 18.
$$x \in CR_T(\varphi) \iff x \in Q(x)$$

Proof. Suppose that $x \notin CR_T(\varphi)$. There exists open set V and time $T_V > 0$ such that
$$\forall \, y \in \overline{V} \qquad o^+\big(\varphi(T_U, y)\big) \subset V.$$

By Corollary 4 we obtain that the set

$$A = \bigcap_{n \in \mathbb{N}} \varphi(nT_V, \overline{V})$$

is asymptotically stable.

We have inclusion $L^+(x) \subset A$ and then $Q(x) \subset A \subset \overline{V}$. But $x \notin V$ and so $x \notin Q(x)$.

To prove second implication let us assume that $x \notin Q(x)$. Then there exists asymptotically stable set A such that $x \notin A$ and $L^+(x) \subset A$. By Theorem 5 there exists open neighborhood U of A such that conditions (3.3) and (3.4) are fulfilled and $x \notin U$. Inclusion $L^+(x) \subset A$ implies that there exists $T > 0$ such that $\varphi(x, T) \subset U$. Takeing this U to Definition 16 we obtain that $x \notin CR_T(\varphi)$. The proof is ended. $\square$

Corollary 19. *Sets $CR(\varphi)$ and $CR_T(\varphi)$ are equal.*

Remark 20. *Proof of Theorem 18 and definitions of sets $Q(x)$ and $CR_T(\varphi)$ need only that X is T_2 (Hausdorf) topological space. This allows us to define chain recurrent sets in topological spaces, and in metric spaces this definitions are equivalent to classical Conley definition.*

Remark 21. *All presented results stay true when φ is a semi-dynamical system.*

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