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# THE STURM–LIOUVILLE INVERSE SPECTRAL PROBLEM WITH BOUNDARY CONDITIONS DEPENDING ON THE SPECTRAL PARAMETER

**Abstract.** We present the complete version including proofs of the results announced in [1]. Namely, for the problem of small transversal vibrations of a damped string of nonuniform stiffness with one end fixed we give the description of the spectrum and solve the inverse problem: find the conditions which should be satisfied by a sequence of complex numbers to be the spectrum of a damped string.

**Keywords:** damped vibrations, inhomogeneous strings, quadratic operator pencil, Hermite–Biehler functions.

**Mathematics Subject Classification:** 34A55.

## 1. INTRODUCTION

The boundary-value problem

$$\frac{\partial}{\partial s}\left(T(s)\frac{\partial u}{\partial s}\right) - \frac{\partial^2 u}{\partial t^2} - p\frac{\partial u}{\partial t} = 0, \tag{1.1}$$

$$u(0,t) = 0, \tag{1.2}$$

$$\left.\frac{\partial u}{\partial s}\right|_{s=l} + \nu \left.\frac{\partial u}{\partial t}\right|_{s=l} + \mu \left.\frac{\partial^2 u}{\partial t^2}\right|_{s=l} = 0 \tag{1.3}$$

was considered in [2]. It describes small transversal vibrations of a string of nonuniform stiffness $T(s)$ subject to constant damping proportional to $p > 0$. Here $u(s,t)$ is the transversal displacement and $l > 0$ is the length of the string. The left end of the string is fixed and the right end is equipped with a ring of mass $\mu > 0$ moving in the direction orthogonal to the equilibrium position of the string. The damping coefficient of the ring is $\nu > 0$. Substituting $u(s,t) = v(\lambda, s)\, e^{i\lambda t}$ into (1.1)–(1.3) we obtain the system for the amplitude function $v(\lambda, s)$:

$$(T(s)v'(\lambda, s))' + \lambda^2 v(\lambda, s) - ip\lambda v(\lambda, s) = 0, \tag{1.4}$$

$$v(\lambda, 0) = 0, \tag{1.5}$$

$$v'(\lambda, l) + i\nu\lambda v(\lambda, l) - \mu\lambda^2 v(\lambda, l) = 0. \tag{1.6}$$

The boundary problem (1.4)–(1.6) is invariant under the transformation $s' = rs$, $l' = rl$, $T'(s') = r^2 T(s)$, $\nu' = r^{-1}\nu$, $\mu' = r^{-1}\mu$, $p' = p$, where $r$ is an arbitrary positive number, and therefore the spectrum which is a set of normal eigenvalues (see below) does not determine the set of parameters $\{l, T(s), p, \nu, \mu\}$ uniquely. It was shown in [2] that in the case of a so-called weakly damped string, i. e. one having no purely imaginary eigenvalues, the spectrum and the length $l$ of the string uniquely determine the set $\{T(s), p, \nu, \mu\}$ starting from an appropriate class of data. The theory developed in [3] being slightly modified enables us to avoid the restriction of absence of purely imaginary eigenvalues and to solve the direct problem (the description of the spectrum) and the inverse problem (the recovery of the set of parameters $\{T(s), p, \nu, \mu\}$).

Assume that $T(s) > 0$ for $s \in [0, l]$ and $T(s) \in W_2^2(0, l)$. It enables us to apply the Liouville transformation [5]:

$$x(s) = \int_0^s (T(s'))^{-1/2}\, ds', \tag{1.7}$$

$$y(\lambda, x) = (T[x])^{1/4}\, v[\lambda, x]. \tag{1.8}$$

Here $v[\lambda, x] = v(\lambda, s(x))$ and $T[x] = T(s(x))$.

Substituting (1.7), (1.8) into (1.4)–(1.6) we obtain:

$$y''(\lambda, x) + (\lambda^2 - i\,\lambda p - q(x))y(\lambda, x) = 0, \tag{1.9}$$

$$y(\lambda, 0) = 0, \tag{1.10}$$

$$y'(\lambda, a) + (-m\lambda^2 + i\alpha\lambda + \beta)y(\lambda, a) = 0, \tag{1.11}$$

where:

$$q(x) = (T[x])^{-1/4} \frac{d^2}{dx^2} (T[x])^{1/4}, \tag{1.12}$$

$$m = \mu\, (T[a])^{1/2}, \tag{1.13}$$

$$\alpha = \nu\, (T[a])^{1/2}, \tag{1.14}$$

$$\beta = -\frac{1}{4}\,(T[a])^{-1}\,\frac{dT[x]}{dx}\bigg|_{x=a}, \tag{1.15}$$

$$a = \int_0^l (T(s))^{-1/2}\,ds. \tag{1.16}$$

Let us introduce the following operator pencils:

1) the operator pencil

$$\widetilde{L}(\lambda) = \lambda^2\widetilde{M} - i\lambda\widetilde{K} - \widetilde{A}, \qquad \Big(D\big(\widetilde{L}(\lambda)\big) = D(\widetilde{A})\Big)$$

acting on $L_2(0,l) \oplus \mathbf{C}$, where:

$$D(\widetilde{A}) = \left\{\begin{pmatrix} v(s) \\ v(l) \end{pmatrix}:\ v(s) \in W_2^2(0,l), \quad v(0) = 0\right\},$$

$$\widetilde{A}Y = \widetilde{A}\begin{pmatrix} v(s) \\ v(l) \end{pmatrix} = \begin{pmatrix} -(T(s)v'(s))' \\ T(l)v'(l) \end{pmatrix}$$

$$\widetilde{K} = \begin{pmatrix} pI & 0 \\ 0 & T(l)\nu \end{pmatrix}, \qquad \widetilde{M} = \begin{pmatrix} I & 0 \\ 0 & T(l)\mu \end{pmatrix},$$

2) the operator pencil

$$\mathcal{L}(\lambda) = \lambda^2 M - i\lambda K - A \tag{1.17}$$

acting on $L_2(0,a) \oplus \mathbf{C}$, where:

$$D(\mathcal{L}) = D(A) = \left\{\begin{pmatrix} y(x) \\ y(a) \end{pmatrix}:\ y(x) \in W_2^2(0,a), \quad y(0) = 0\right\},$$

$$A\begin{pmatrix} y(x) \\ y(a) \end{pmatrix} = \begin{pmatrix} -y'' + q(x)y \\ y'(a) + \beta y(a) \end{pmatrix}, \tag{1.18}$$

and

$$K = \begin{pmatrix} pI & 0 \\ 0 & \alpha \end{pmatrix}, \qquad M = \begin{pmatrix} I & 0 \\ 0 & m \end{pmatrix},$$

## 2. BASIC PROPERTIES OF THE OPERATOR PENCILS

In this section we give some definitions and discuss basic properties of the spectra of the operator pencils defined in Sec. 1.

**Definition 2.1.** *Let $L(\lambda)$ be an operator pencil defined on a complex Hilbert space $\mathcal{H}$. The set of values $\lambda \in \mathbf{C}$ such that $L(\lambda)^{-1}$ exists as a bounded linear operator on $\mathcal{H}$ is called the resolvent set $\varrho(L)$ of the operator pencil $L(\lambda)$. We denote by $\sigma(L)$ the spectrum of $L(\lambda)$, i.e., the set $\sigma(L) = \mathbf{C}\backslash\varrho(L)$. The number $\lambda_0 \in \mathbf{C}$ is said to be* *an eigenvalue of* $L(\lambda)$ *if there exists a nonzero vector* $y_0$ *(called an eigenvector) such that* $L(\lambda_0)y_0 = 0$. *The vectors* $y_1, y_2, \cdots, y_{r-1}$ *are called corresponding associated eigenvectors if*

$$\sum_{s=0}^{n} \frac{1}{s!} \frac{d^s}{d\lambda^s} L(\lambda)\bigg|_{\lambda=\lambda_0} y_{n-s} = 0, \qquad n = 1, \cdots, r-1. \tag{2.1}$$

*The number* $r$ *is called the length of the chain composed of the eigenvector and its associated eigenvectors. The geometric multiplicity of an eigenvalue is defined to be the maximal number of corresponding linearly independent eigenvectors. Its algebraic multiplicity is defined as the maximal value of the sum of the lengths of chains corresponding to linearly independent eigenvectors. An eigenvalue is said to be isolated if it has a deleted neighbourhood contained in the resolvent set. An isolated eigenvalue* $\lambda_0$ *of finite algebraic multiplicity is said to be normal if the image* $ImL(\lambda_0)$ *is closed. We denote by* $\sigma_0(L)$ *the set of normal eigenvalues of* $L(\lambda)$.

We identify the spectrum of the problem (1.4)–(1.6), i. e. the spectrum of the problem (1.9)–(1.11), with the spectrum of the operator pencil $\mathcal{L}(\lambda)$ or what is the same with the spectrum of the operator pencil $\tilde{L}(\lambda)$.

**Lemma 2.2.** *The operator pencil* $\tilde{L}(\lambda)$ *has the following properties:*

1) *The spectrum of* $\tilde{L}(\lambda)$ *consists only of normal eigenvalues.*

2) *All eigenvalues of* $\tilde{L}(\lambda)$ *have geometric multiplicity one.*

3) *The eigenvalues of* $\tilde{L}(\lambda)$ *are located in the open upper half-plane.*

The first part follows from the fact that the operator pencil $\widetilde{L}(\lambda)$ is compactly invertible for $\lambda = -i\gamma$ with $\gamma > 0$ large enough. The second statement follows from the existence of only one linearly independent solution of (1.9) which vanishes at $x = 0$. The third statement follows from Theorem 2.1 of [4].

It was shown in [2] that the spectra of the operator pencils coincide with the set of zeros of the function

$$\chi(\lambda) = S'[\lambda, a] + (-m\lambda^2 + i\alpha\lambda + \beta)S[\lambda, a], \tag{2.2}$$

where $S[\lambda, x]$ is the solution of (1.9) satisfying the conditions $S[\lambda, 0] = S'[\lambda, 0] - 1 = 0$. This solution can be presented (see [11], Corollary after Theorem 1.2.1) in the form

$$S[\lambda, x] = S(\tau(\lambda), x) = \frac{\sin \tau(\lambda)x}{\tau(\lambda)} + \int_0^x K(x,t) \frac{\sin \tau(\lambda)t}{\tau(\lambda)}\, dt, \tag{2.3}$$

where $\tau(\lambda) = \sqrt{\lambda^2 - i\lambda p}$.

## 3. MAIN RESULTS ON THE DIRECT PROBLEM

**Definition 3.1.** *A sequence of complex numbers $\{\lambda_k\}_{k\in\mathbf{Z}}$ or $\{\lambda_k\}_{0\neq k\in\mathbf{Z}}$ is said to be properly enumerated if:*

1) $Re\,\lambda_k \geq Re\,\lambda_p$ *for all* $k > p$;
2) $\lambda_{-k} = -\overline{\lambda_k}$ *for all* $\lambda_k$ *not purely imaginary;*
3) *a certain complex number appears in the sequence at most finitely many times.*

To state these results, let $\{-i\gamma_k\}_{k=1}^{\kappa}$ be the eigenvalues of $\widetilde{L}(\lambda)$ located in the closed lower half-plane and hence on the nonpositive imaginary axis, numbered such that $0 \leq \gamma_1 < \ldots < \gamma_{\kappa-1} < \gamma_\kappa$.

**Definition 3.2.** *Let $\kappa$ be a nonnegative integer. Then the properly enumerated sequence $\{\lambda_k\}_{0\neq k\in\mathbf{Z}}$ is said to have the $\mathcal{SHB}_\kappa^-$ property if:*

1) *All but $\kappa$ terms of the sequence lie in the open upper half-plane.*
2) *All terms in the closed lower half-plane are purely imaginary and occur only once. If $\kappa \geq 1$, we denote them as $\lambda_{-j} = -i\,|\lambda_{-j}|$ $(j = 1,\ldots,\kappa)$. We assume that $|\lambda_{-j}| < \left|\lambda_{-(j+1)}\right|$ $(j = 1,\ldots,\kappa-1)$.*
3) *If $\kappa \geq 1$, the numbers $i\,|\lambda_{-j}|$ $(j = 1,\ldots,\kappa)$ (with the exception of $\lambda_{-1}$ if it equals zero) are not terms of the sequence.*
4) *If $\kappa \geq 2$, then the number of terms in the intervals $(i\,|\lambda_{-j}|\,, i\left|\lambda_{-(j+1)}\right|)$ $(j = 1, \ldots,\kappa-1)$ is odd.*
5) *If $|\lambda_{-1}| > 0$, then the interval $(0, i\,|\lambda_{-1}|)$ contains a nonzero and even number of terms.*
6) *If $\kappa \geq 1$, then the interval $(i\,|\lambda_{-\kappa}|\,, i\infty)$ contains an odd number of terms.*
7) *If $\kappa = 0$, then the sequence has an even or zero number of positive imaginary terms.*

**Definition 3.3.** *Let us denote by $\mathcal{B}_\mp$ the class of sets $\{a, q, p, m, \alpha, \beta\}$ such that $a > 0$, $p > 0$, $m > 0$, $\pm(\alpha - pm) > 0$, $\beta \in \mathbf{R}$, and $q$ is a real function in $L_2(0, a)$ having the property that the selfadjoint operator $B$ defined by:*

$$Bf = -f'' + qf,$$

$$D(B) = \left\{f \in W_2^2(0,a)\colon f'(a) + \beta f(a) = 0,\ f(0) = 0\right\},$$

*is strictly positive. In addition, if $q$ belongs to the Sobolev space $W_2^2(0,a)$, then we shall say that the set $\{a, q, p, m, \alpha, \beta\}$ belongs to $\mathcal{B}_\mp^0$. Furthermore, if $\{a, q, p, m, \alpha, \beta\}$ belongs to $\mathcal{B}_+$* (resp., *$\mathcal{B}_+^0$) and $\alpha > 0$, then we shall say that $\{a, q, p, m, \alpha, \beta\}$ belongs to $\hat{\mathcal{B}}_+$* (resp., *$\hat{\mathcal{B}}_+^0$).*

By $B_1$ we denote the selfadjoint operator acting in $L_2(0,a)$ according to the formulae:

$$B_1 f = -f'' + \left(q - \frac{p^2}{4}\right) f,$$

$$D(B_1) = \left\{ f \in W_2^2(0,a)\colon f'(a) + \left(\beta - \frac{\alpha p}{2} + \frac{mp^2}{4}\right) f(a) = 0,\ f(0) = 0 \right\}.$$

**Theorem 3.4.** *Let $\{a, q, m, p, \alpha, \beta\} \in \mathcal{B}_-^0$. Then the spectrum of problem* (1.9)–(1.11) *being properly enumerated satisfies the following conditions:*

1) $\{\lambda_k\}_{0 \neq k \in \mathbf{Z}} \in \mathcal{SHB}_0^-$;
2) $\{\lambda_k - (ip/2)\}_{0 \neq k \in \mathbf{Z}} \in \mathcal{SHB}_\kappa^-$, *where $\kappa$ is the number of nonpositive eigenvalues of the operator $B_1$;*
3) *we have the following equation:*

$$\lambda_k \overset{k \to \pm\infty}{=} \frac{\pi(k-1)}{a} + \frac{ip}{2} + \frac{p_0}{k-1} + \frac{i\,p_1}{(k-1)^2} + \frac{p_2}{(k-1)^3} + \frac{b_k}{k^3}, \tag{3.1}$$

*where $p_0, p_2 \in \mathbf{R}$, $p_1 > 0$, and $\sum_{0 \neq k \in \mathbf{Z}} |b_k|^2 < \infty$.*

*Proof.* Statements 1 and 3 were proved in [2]. In order to prove statement 2 let us transform the spectral parameter: $z = \lambda - \frac{ip}{2}$. Performing this transformation in (1.17) we obtain

$$\tilde{\mathcal{L}}(z) =: \mathcal{L}(z + \frac{ip}{2}) = z^2 M - izK_1 - A_1 \tag{3.2}$$

acting in $L_2(0,a) \oplus \mathbf{C}$, where:

$$D(\mathcal{L}) = D(A_1) = D(A) = \left\{ \begin{pmatrix} y(x) \\ y(a) \end{pmatrix} \colon\ y(x) \in W_2^2(0,a), \quad y(0) = 0 \right\}$$

$$A_1 = A - \begin{pmatrix} \frac{p^2}{4} I & 0 \\ 0 & \frac{\alpha p}{2} - \frac{mp^2}{4} \end{pmatrix} \tag{3.3}$$

and

$$K_1 = \begin{pmatrix} 0 & 0 \\ 0 & \alpha - pm \end{pmatrix} =: (\alpha - pm) P_1. \tag{3.4}$$

According to Definition 3.3, $\alpha > pm$ in our case and therefore the operator $K_1$ is nonnegative. Thus we can apply Theorem 3.1 of [4] (taking into account the difference in enumeration) and obtain statement 2) with $\kappa$ equal to the number of nonpositive eigenvalues (which are all simple) of the operator $A_1$. It is obvious that the spectrum of the operator $B_1$ coincides with the spectrum of the linear pencil $\lambda P - A_1$, where $P = I - P_1$. The number of nonpositive eigenvalues of $\lambda P - A_1$ is the same as that of the operator $A_1$. $\square$

The spectrum of problem (1.9)–(1.11) coincides with the set of zeros of the entire function

$$\chi(\lambda) = S'(\tau(\lambda), a) + \big(-m\lambda^2 + i\lambda\alpha + \beta\big) S(\tau(\lambda), a), \tag{3.5}$$

where the prime denotes the derivative with respect to $x$. We need the following proposition.

**Proposition 3.5.** *If $\chi(i\gamma) = 0$ for some $\gamma \in \big(0, \frac{p}{2}\big)$, then $\chi(ip - i\gamma) \neq 0$.*

*Proof of Proposition 3.5.* Since $\tau^2(i\gamma) = \tau^2(ip - i\gamma)$ the two functions $S'(\tau(\lambda), a)$ and $S(\tau(\lambda), a)$ being even functions of $\tau$ satisfy

$$S'(\tau(i\gamma), a) = S'(\tau(ip - i\gamma), a), \tag{3.6}$$

$$S(\tau(i\gamma), a) = S(\tau(ip - i\gamma), a). \tag{3.7}$$

Let $\chi(i\gamma) = 0$ for some $\gamma \in (0, \frac{p}{2})$. Then due to (3.5) we obtain

$$S'(\tau(i\gamma), a) + (m\gamma^2 - \gamma\alpha + \beta) S(\tau(i\gamma), a) = 0. \tag{3.8}$$

Substituting (3.6)–(3.7) into (3.8), we obtain

$$S'(\tau(ip - i\gamma), a) + (m\gamma^2 - \gamma\alpha + \beta) S(\tau(ip - i\gamma), a) = 0. \tag{3.9}$$

Suppose now that $\chi(ip - i\gamma) = 0$. Then

$$S'(\tau(ip - i\gamma), a) + (m(p-\gamma)^2 - (p-\gamma)\alpha + \beta) S(\tau(ip - i\gamma), a) = 0. \tag{3.10}$$

Combining (3.9) and (3.10) we obtain

$$(mp - \alpha)(p - 2\gamma) S(\tau(ip - i\gamma)) = 0$$

and

$$S(\tau(ip - i\gamma), a) = 0. \tag{3.11}$$

Substituting (3.11) into (3.9) we obtain

$$S(\tau(ip - i\gamma), a) = S'(\tau(ip - i\gamma), a) = 0,$$

which implies that $S(\tau(ip - i\gamma), x) \equiv 0$, a contradiction. □

To continue the proof, let us consider the operator pencil

$$\tilde{\mathcal{L}}(z, \eta) = z^2 M - iz\eta P_1 - A_1$$

in which $\eta$ occurs as a parameter, keeping $z$ as the spectral parameter. The spectrum of $\tilde{\mathcal{L}}(z, 0) = z^2 I - A_1$ is symmetric with respect to both the real and the imaginary axis. Evidently, $\tilde{\mathcal{L}}(z, \alpha - mp) = \tilde{\mathcal{L}}(z)$. The eigenvalues of $\tilde{\mathcal{L}}(z, \eta)$ are piecewise analytic functions of $\eta$ which may loose their analyticity only when they collide [6, 7].

Differentiating the identity $(\tilde{\mathcal{L}}(z_j,\eta)y_j(\eta),y_j(\eta))=0$ with respect to $\eta$, where $z_j(\eta)$ is a purely imaginary eigenvalue of $\tilde{\mathcal{L}}(z,\eta)$ and $y_j(\eta)$ is the corresponding eigenvector, and taking into account Rellich's theorem [8] (Theorem XII.3) we obtain the following identity (cf., e.g., [4]):

$$z_j'(\eta)=\frac{i\,z_j(\eta)(P_1y_j(\eta),y_j(\eta))}{2z_j(\eta)(My_j(\eta),y_j(\eta))-i\eta(P_1y_j(\eta),y_j(\eta))}, \tag{3.12}$$

where $P_1$ is defined by (3.4) and the prime indicates the derivative with respect to $\eta$. Of course, (3.12) has sense only if the denominator is not zero.

Consequently,

$$z_j'(0)=\frac{i\,(P_1y_j(0),y_j(0))}{2\,(My_j(0),y_j(0))}. \tag{3.13}$$

In the next proposition it is proved that the numerator and denominator of (3.13) are positive numbers.

**Proposition 3.6.** *For any purely imaginary $z_j(0)$ we have $(P_1y_j(0),y_j(0))>0$. Hence,*

$$Re\,z_j'(0)=0, \qquad Im\,z_j'(0)>0.$$

*Proof.* Since $P_1\geq 0$, the identity $(P_1y_j(0),y_j(0))=0$ implies that $P_1y_j(0)=0$ and consequently $y_j(\eta=0,a)=0$ and hence $y_j(\eta=0)=0$. On the other hand, $P_1(0)=0$ implies

$$z_j(0)^2My_j(0)-A_1y_j(0)=0,$$

which means that

$$\begin{aligned} &z_j(\eta=0)^2y_j(\eta=0,x)+\frac{d^2}{dx^2}y_j(\eta=0,x)-qy_j(\eta=0,x)=0,\\ &y_j(\eta=0,a)=\frac{d}{dx}y_j(\eta=0,x)\Big|_{x=a}=0. \end{aligned}$$

Hence $y_j(\eta=0,x)\equiv 0$, which is a contradiction. □

Let us continue the proof of Theorem 3.4. Taking into account the symmetry of the problem on reflection with respect to the imaginary line, we have $z_{-k}(\eta)=-\overline{z_k(\eta)}$ for all not purely imaginary $z_{-k}(\eta)$ with $\eta\geq 0$, and hence new eigenvalues can appear on the imaginary axis only in pairs, which implies statements 1)–3) of the theorem.

**Proposition 3.7.** *There exists a constant $C>0$ such that all the eigenvalues of $\tilde{\mathcal{L}}(\lambda,\eta)$ lie in a horizontal strip $|\mathrm{Im} z_j(\eta)|\leq C$ for all $\eta$: $0\leq\eta\leq\alpha-mp$.*

This proposition is a consequence of Lemmas A.1 and A.2 of [2].

Due to the symmetry of the problem on reflection with respect to the imaginary line, new eigenvalues can appear on the imaginary axis only in pairs. This means that the number of purely imaginary eigenvalues (with multiplicities taken into account) is even. Then assertion 5) follows. Assertion 4) now follows if we take into account assertion 3).

**Theorem 3.8.** *Let* $\{a, q, m, p, \alpha, \beta\} \in \mathcal{B}^0_+$. *Then:*

1) *statement* 3) *of Theorem 3.4 is true with* $p_1 < 0$*;*
2) $\{(ip/2) - \lambda_k\}_{0\neq k\in\mathbf{Z}} \in \mathcal{SHB}^-_\kappa$, *where* $\kappa$ *is the number of nonpositive eigenvalues of* $B_1$*;*
3) *if* $\{a, q, m, p, \alpha, \beta\} \in \hat{\mathcal{B}}^0_+$, *then statement* 1) *of Theorem 3.4 is also satisfied.*

*Proof.* Assertion 1) has been proved in [2]. To prove assertion 2) let us consider the operator pencil

$$\tilde{\mathcal{L}}_1(z) \overset{def}{=} \tilde{\mathcal{L}}(-z) = \mathcal{L}(-z + \frac{ip}{2}) = z^2M + izK_1 - A_1.$$

Under the conditions of our theorem $pm - \alpha > 0$ and consequently $-K_1 \geq 0$. According to Theorem 3.4 the spectrum $\{z_k\}_{0\neq k\in Z}$ of $\tilde{\mathcal{L}}_1(z)$ satisfies the condition $\{z_k\}_{0\neq k\in\mathbf{Z}} \in \mathcal{SHB}^-_\kappa$, where $\kappa$ is the same as in Theorem 3.4. That means the spectrum $\{\lambda_k\}_{0\neq k\in Z}$ of $\mathcal{L}(\lambda)$ obtained from the spectrum $\{z_k\}_{0\neq k\in Z}$ via transformation $\lambda = = \frac{ip}{2} - z$ satisfies the condition $\{(ip/2) - \lambda_k\}_{0\neq k\in\mathbf{Z}} \in \mathcal{SHB}^-_\kappa$. Assertion 3) follows from Theorem 2.1 of [4]. □

## 4. INVERSE PROBLEM

**Definition 4.1.** *An entire function* $\omega(\lambda)$ *is said to belong to the Hermite–Biehler class* $\mathcal{HB}$ [9] *if it has no zeros in the closed lower half-plane and*

$$\left|\omega(\lambda)/\overline{\omega(\overline{\lambda})}\right| < 1, \qquad Im\,\lambda > 0. \tag{4.1}$$

**Definition 4.2.** *The function* $\omega \in \mathcal{HB}$ *is said to belong to the symmetric Hermite-Biehler class* $\mathcal{SHB}$ *if*

$$\omega(-\lambda) = \overline{\omega(\overline{\lambda})}, \qquad \lambda \in \mathbf{C}. \tag{4.2}$$

Any function $\omega \in \mathcal{HB}$ can be presented in the form [9]

$$\omega(\lambda) = P(\lambda) + iQ(\lambda),$$

where $P(\lambda)$ and $Q(\lambda)$ are real entire functions (i.e., they are real on the real line). Moreover, if $\omega \in \mathcal{SHB}$, then (4.2) implies that

$$P(-\lambda) + iQ(-\lambda) = P(\lambda) - iQ(\lambda), \qquad \lambda \in \mathbf{R}.$$

This means that for $\omega \in \mathcal{SHB}$ the functions $P(\lambda)$ and $\hat{Q}(\lambda) = \lambda^{-1}Q(\lambda)$ are even entire functions satisfying

$$P(\lambda) = \frac{\omega(\lambda) + \omega(-\lambda)}{2}, \tag{4.3}$$

$$\hat{Q}(\lambda) = \frac{\omega(\lambda) - \omega(-\lambda)}{2i\lambda}. \tag{4.4}$$

Notice that the set of zeros of a function $\omega \in \mathcal{SHB}$ belongs to one of the two disjoint sets of properly enumerated sequences $\mathcal{SHB}_0^+$ and $\mathcal{SHB}_0^-$.

**Theorem 4.3.** *Let the properly enumerated sequence $\{\lambda_k\}_{0\neq k\in\mathbf{Z}}$ of complex numbers satisfy the following conditions:*

1) $\{\lambda_k\}_{0\neq k\in\mathbf{Z}} \in \mathcal{SHB}_0^-$,
2) $\{\lambda_k - (ip/2)\}_{0\neq k\in\mathbf{Z}} \in \mathcal{SHB}_\kappa^-$ *for some $\kappa \geq 0$, and some $p > 0$,*
3) *formula* (3.1) *is valid with $p_1 > 0$, $p_0 \in \mathbf{R}$, $p_2 \in \mathbf{R}$ and $\sum_{0\neq k\in\mathbf{Z}} |b_k|^2 < \infty$.*

*Then there exists a unique set $\{a, q, p, m, \alpha, \beta\} \in \mathcal{B}_-$ such that $\{\lambda_k\}_{0\neq k\in\mathbf{Z}}$ is the spectrum of problem* (1.9)–(1.11) *generated by the set $\{a, q, p, m, \alpha, \beta\}$.*

*Proof.* Put

$$a = \lim_{k\to\infty} (\pi k/\lambda_k). \tag{4.5}$$

By (3.1), this limit exists. As will be shown shortly, this limit $a$ will turn out to be the length of the interval on which (1.9) is valid.

Consider the auxiliary function

$$\varphi_0(\lambda) = \left(\lambda - \frac{\pi}{4} - i\epsilon\right)\left(\lambda + \frac{\pi}{4} - i\epsilon\right)\frac{\sin\big((\lambda - i\epsilon)a\big)}{\lambda - i\epsilon}, \quad \epsilon > 0. \tag{4.6}$$

Then $\phi_0 \in \mathcal{SHB}$. Let us denote its properly enumerated sequence of zeros by $\{\lambda_k^{(0)}\}_{0\neq k\in\mathbf{Z}}$, where $\lambda_{\pm 1}^{(0)} = \pm\frac{\pi}{4} + i\epsilon$ and $\lambda_k^{(0)} = \frac{\pi}{a}\mathrm{sign}(k)(|k|-1) + i\epsilon$ for $k = \pm 2, \pm 3, \ldots$.

Put

$$P_0(\lambda) = \frac{\varphi_0(\lambda) + \varphi_0(-\lambda)}{2},$$

$$\hat{Q}_0(\lambda) = \frac{\varphi_0(\lambda) - \varphi_0(-\lambda)}{2i\lambda}.$$

Then the functions $P_0(\lambda)$ and $\hat{Q}_0(\lambda)$ are both even functions. Let us introduce the real entire functions

$$\tilde{P}_0(\lambda) = P_0(\sqrt{\lambda}),$$

$$\tilde{\hat{Q}}_0(\lambda) = \hat{Q}_0(\sqrt{\lambda}).$$

Due to Theorem 3, p. 311 [9], the zeros $\{\zeta_k^{(0)}\}_{0\neq k\in\mathbf{Z}}$ of $P_0(\lambda)$ and the zeros $\{\xi_k^{(0)}\}_{0\neq k\in\mathbf{Z}}$ of $\hat{Q}_0(\lambda)$ interlace, i.e., we have

$$\ldots < \xi_{-2}^{(0)} < \zeta_{-2}^{(0)} < \xi_{-1}^{(0)} < \zeta_{-1}^{(0)} < 0 < \zeta_1^{(0)} < \xi_1^{(0)} < \zeta_2^{(0)} < \xi_2^{(0)} < \ldots . \tag{4.7}$$

□

**Proposition 4.4.** *There exists a sequence of continuous and piecewise analytic functions* $\{\lambda_k(t)\}_{0\neq k\in\mathbf{Z}}$ *such that* $\lambda_k(0) = \lambda_k^{(0)}$ *and* $\lambda_k(1) = \lambda_k - i\frac{p}{2}$ $(0 \neq k \in \mathbf{Z})$ *and*

$$\{\lambda_k(t)\}_{0\neq k\in\mathbf{Z}} \in \mathcal{SHB}^-_{\kappa(t)}$$

*for any fixed* $t \in [0,1]$.

The proof of this proposition can be found in [3] (Proposition 4.8 there).

Continuing the proof of Theorem 4.3, let us construct the function

$$\varphi(\lambda,t) \stackrel{\text{def}}{=} \big(\lambda - \lambda_1(t)\big)\big(\lambda - \lambda_{-1}(t)\big)\prod_2^\infty \frac{(\lambda - \lambda_k(t))(\lambda - \lambda_{-k}(t))a^2}{\pi^2(k-1)^2}. \tag{4.8}$$

Then

$$\varphi(\lambda,0) = C\phi_0(\lambda), \quad C \neq 0,$$

$$\varphi(\lambda) \stackrel{\text{def}}{=} \varphi(\lambda,1) = \big(\lambda - \lambda_1(1)\big)\big(\lambda - \lambda_{-1}(1)\big)\prod_2^\infty \frac{(\lambda - \lambda_k(1))(\lambda - \lambda_{-k}(1))a}{\pi^2(k-1)^2}. \tag{4.9}$$

Put

$$P(\lambda,t) = \frac{\varphi(\lambda,t) + \varphi(-\lambda,t)}{2}, \tag{4.10}$$

$$\hat{Q}(\lambda,t) = \frac{\varphi(\lambda,t) - \varphi(-\lambda,t)}{2i\lambda}, \tag{4.11}$$

and then define $P(\lambda) = P(\lambda,1)$ and $\hat{Q}(\lambda) = \hat{Q}(\lambda,1)$.

Denote by $\{\zeta_k(t)\}_{0\neq k\in\mathbf{Z}}$ the set of zeros of $P(\lambda,t)$ and by $\{\xi_k(t)\}_{0\neq k\in\mathbf{Z}}$ the set of zeros of $\hat{Q}(\lambda,t)$. Then $\{\zeta_k(t)^2\}_{k=1}^\infty$ are the zeros of $\tilde{P}(\lambda,t)$ and $\{\xi_k(t)^2\}_{k=1}^\infty$ are the zeros of $\tilde{\hat{Q}}(\lambda,t)$.

**Proposition 4.5.** *For any fixed* $t \in [0,1]$ *the sets of squared zeros* $\{\zeta_k(t)^2\}_{k=1}^\infty$ *and* $\{\xi_k(t)^2\}_{k=1}^\infty$ *interlace, i.e.,*

$$-\infty < \zeta_1(t)^2 < \xi_1(t)^2 < \zeta_2(t)^2 < \xi_2(t)^2 < \ldots . \tag{4.12}$$

The proof of this proposition is the same as the proof of Proposition 4.9 in [3], although the function $\varphi_0(\lambda)$ given by (4.6) is different from the one in [3].

Denote

$$\tilde{P}(\lambda, s) = P(\lambda) - s\left(-m\lambda^2 - \frac{\alpha p}{2} + \frac{mp^2}{4} + \beta\right)(\alpha - mp)^{-1}\hat{Q}(\lambda) \tag{4.13}$$

and denote $\{\hat{\zeta}_k(s)\}_{0\neq k\in\mathbf{Z}}$ the set of zeros of $\hat{P}(\lambda, s)$.

**Proposition 4.6.** *For any fixed $s \in [0,1]$ the sets of squared zeros $\{\hat{\zeta}_k(s)^2\}_{k=1}^{\infty}$ and $\{\xi_k(1)^2\}_{k=1}^{\infty}$ interlace, i.e.,*

$$-\infty < \hat{\zeta}_1(s)^2 < \xi_1(1)^2 < \hat{\zeta}_2(s)^2 < \xi_2(1)^2 < \dots \tag{4.14}$$

*Proof of Proposition 4.6.* Due to Proposition 4.5, this statement is true for $s = 0$. The function $\hat{P}(\lambda, s)$ is an entire function of $\lambda$ for every $s \in [0,1]$ and a continuous function of $t \in [0,1]$ for every $\lambda \in \mathbf{C}$. This means that the inequalities (4.14) can only be violated if for some $s_1 \in [0,1]$ we have $\xi_k(1)^2 = \hat{\zeta}_k(s_1)^2$ or $\xi_k(1)^2 = \hat{\zeta}_{k+1}(s_1)^2$. But either identity implies

$$P(\xi_k(1), s_1) = \hat{Q}(\xi_k(1)) = 0,$$

$$P(-\xi_k(1), s_1) = \hat{Q}(-\xi_k(1)) = 0,$$

because $P(\lambda, s)$ and $\tilde{Q}(\lambda, s)$ are even functions and, consequently,

$$P(\xi_k(1)) = \hat{Q}(\xi_k(1)) = 0,$$

which contradicts Proposition 4.5. □

Due to (3.1), (4.8) and condition 1) of Theorem 4.3 the function $\varphi(\lambda)$ satisfies the conditions of Theorem 6 (Chap. VII.3) in [9] and therefore belongs to $\mathcal{HB}$. It belongs to $\mathcal{SHB}$ due to the symmetry of the sequence $\{\lambda_k\}_{0\neq k\in\mathbf{Z}}$.

We now need the following definition.

**Definition 4.7.** *An entire function $\omega(\lambda)$ of exponential type $\sigma > 0$ is said to be a function of sine-type* [10] *if:*

1) *There exists $h > 0$ such that the zeros of $\omega(\lambda)$ are lying in the strip $|Im\,\lambda| < h$.*
2) *There exists $h_1 \in \mathbf{R}$ such that $0 < m \le |\omega(\lambda)| \le M < \infty$ for all $\lambda$ with $Im\,\lambda = h_1$.*
3) *The exponential type of $\omega(\lambda)$ in the lower half-plane coincides with its exponential type in the upper half-plane.*

We make use of the following lemma (the proof can be found in [2], Lemma 4.1).

**Lemma 4.8.** *Let $\{\lambda_k\}_{0\neq k\in\mathbf{Z}}$ be a sequence satisfying the conditions of Theorem 4.3 and having the asymptotics* (3.1) *with $a > 0$ and $p > 0$. Then the entire function $\varphi(\lambda - \frac{ip}{2})$ can be presented in the form*

$$\chi(\lambda) \stackrel{\text{def}}{=} \varphi\left(\lambda - \frac{ip}{2}\right) =$$

$$= E_0\left(\left(\tau + iE_1 + E_2\tau^{-1} + iE_3\tau^{-2}\right)\sin\tau a + \left(F_1 + iF_2\tau^{-1} + F_3\tau^{-2}\right)\cos\tau a\right) + \\ + \Psi_1(\tau)\delta_1(\tau)\tau^{-2} + \Psi_2(\tau)\delta_2(\tau)\tau^{-2}, \quad (4.15)$$

*where:* $\tau = \sqrt{\lambda^2 - ip\lambda}$, $p \in \mathbf{R}$, $F_k \in \mathbf{R}$ $(k = 1,2,3)$, $E_k \in \mathbf{R}$ $(k = 0,1,2,3)$, $F_1 \neq 0$, $E_0 \neq 0$, $\Psi_1(\tau) = \int_0^a e^{i\tau x} f_1(x)dx$, $\Psi_2(\tau) = \int_0^a e^{-i\tau x} f_2(x)dx$, $f_k \in L_2(0,a)$, $\delta_k(\tau)$ $(k = 1,2)$ *are bounded functions,*

$$a = \lim_{n\to\infty} \frac{\pi n}{\lambda_n}, \quad (4.16)$$

$$p = -2i \lim_{n\to\infty}\left(\lambda_n - \frac{\pi n}{a}\right). \quad (4.17)$$

Let the function $\varphi(\lambda) = \varphi(\lambda, 1)$ be defined by (4.8) where $\lambda_k(1) = \lambda_k - \frac{ip}{2}$ and put $\chi(\lambda) = \varphi\left(\lambda - \frac{ip}{2}\right)$. Then, according to Lemma 4.8, $\chi(\lambda)$ is of the form (4.15). Hence, we can find the constants involved.

Set

$$\theta_n = \frac{ip}{2} + \sqrt{\left(\frac{\frac{\pi}{2} + 2\pi n}{a}\right)^2 - \frac{p^2}{4}},$$

where $\operatorname{Re}\theta_n > 0$ for $n$ large enough. Then using (4.15) we obtain

$$E_0 = \lim_{n\to\infty}\left(\chi(\theta_n)\frac{a}{2\pi n}\right), \quad (4.18)$$

$$E_1 = -i\lim_{n\to\infty}\left(E_0^{-1}\chi(\theta_n) - \frac{1}{a}\left(2\pi n + \frac{\pi}{2}\right)\right), \quad (4.19)$$

$$E_2 = \lim_{n\to\infty}\frac{2\pi n}{a}\left(E_0^{-1}\chi(\theta_n) - \frac{1}{a}\left(2\pi n + \frac{\pi}{2}\right) - iE_1\right), \quad (4.20)$$

$$E_3 = -i\lim_{n\to\infty}\left(\frac{2\pi n}{a}\right)^2\left(E_0^{-1}\chi(\theta_n) - \frac{1}{a}\left(2\pi n + \frac{\pi}{2}\right) - iE_1 - \frac{aE_2}{2\pi n + \frac{\pi}{2}}\right). \quad (4.21)$$

Set

$$\xi_n = \frac{ip}{2} + \sqrt{\left(\frac{2\pi n}{a}\right)^2 - \frac{p^2}{4}},$$

where $\operatorname{Re}\xi_n > 0$ for $n$ large enough. Then

$$F_1 = E_0^{-1}\lim_{n\to\infty}\chi(\xi_n), \quad (4.22)$$

$$F_2 = -i\lim_{n\to\infty}\frac{2\pi n}{a}\left(E_0^{-1}\chi(\xi_n) - F_1\right), \quad (4.23)$$

$$F_3 = \lim_{n\to\infty}\left(\frac{2\pi n}{a}\right)^2\left(E_0^{-1}\chi(\xi_n) - F_1 - \frac{iaF_2}{2\pi n}\right). \quad (4.24)$$

**Lemma 4.9.** *Under the assumptions of Theorem 4.3 the constants given by* (4.18), (4.19) *satisfy the inequalities* $E_0 < 0$, $E_1 < 0$.

Lemmas 4.9 and 4.10 as well as Corollaries 4.11 and 4.12 and Lemma 4.13 have been proved in [2] under the additional assumption of absence of purely imaginary $\lambda_k$. But under the assumptions of Theorem 4.3 the proof does not require any changes. It should be mentioned that Lemma 4.2 of [2] contains a misprint. Namely, it should be $B_1 < 0$ instead of $B_1 > 0$ and vice versa in the case of conditions in Theorem 4.16 (see below).

**Lemma 4.10.** *The assumptions of Theorem 4.3 imply* $F_1 E_1 > F_2$, *where* $F_1$, $F_2$, $E_1$ *are defined by* (4.22), (4.23), (4.19).

Set:

$$m = E_1 \left(F_2 - F_1 E_1\right)^{-1}, \tag{4.25}$$

$$\alpha = m\left(p - E_1\right), \tag{4.26}$$

$$\beta = E_2 m + 8^{-1}p^2 m + 2^{-1}p(\alpha - mp) + F_2 E_1^{-1} - mE_3 E_1^{-1}. \tag{4.27}$$

Lemmas 4.9, 4.10 imply the following corollaries.

**Corollary 4.11.** *The inequality* $m > 0$ *holds.*

**Corollary 4.12.** *The inequality* $\alpha > m\,p$ *holds.*

Set:

$$g_1(\tau) \stackrel{\text{def}}{=} \frac{-m\left(\chi\left(\frac{ip}{2} + \sqrt{\tau^2 - \frac{p^2}{4}}\right) - \chi\left(\frac{ip}{2} - \sqrt{\tau^2 - \frac{p^2}{4}}\right)\right)}{2E_0 i(\alpha - mp)\sqrt{\tau^2 - \frac{p^2}{4}}} \tag{4.28}$$

$$\begin{aligned} g_2(\tau) \stackrel{\text{def}}{=} -\frac{m}{2E_0}\left(\chi\left(\frac{ip}{2} + \sqrt{\tau^2 - \frac{p^2}{4}}\right) + \chi\left(\frac{ip}{2} - \sqrt{\tau^2 - \frac{ip}{2}}\right)\right) + \\ + \left(m\left(\tau^2 - \frac{p^2}{2}\right) + \frac{\alpha p}{2} - \beta\right) g_1(\tau). \end{aligned} \tag{4.29}$$

**Lemma 4.13.** $g_1(\tau)$ *and* $g_2(\tau)$ *are entire functions of* $\tau$ *and admit the representations*

$$g_1(\tau) = \frac{\sin \tau a}{\tau} + \frac{F_2}{E_1}\frac{\cos \tau a}{\tau^2} + \frac{E_3}{E_1}\frac{\sin \tau a}{\tau^3} + \frac{\Psi_1(\tau)}{\tau^3}, \tag{4.30}$$

$$g_2(\tau) = \cos \tau a - \frac{F_2}{E_1}\frac{\sin \tau a}{\tau} + \frac{\Psi_2(\tau)}{\tau}, \tag{4.31}$$

*where* $\Psi_k(\tau)$ $(k = 1, 2)$ *are entire functions of exponential type* $\leq a$ *belonging to* $L_2(-\infty, \infty)$.

**Corollary 4.14.** *The zeros of $g_1(\tau)$ and $g_2(\tau)$ behave asymptotically as follows:*

$$\nu_n = \frac{\pi n}{a} - \frac{F_2}{\pi E_1 n} + \frac{b_{1n}}{n},$$

$$\mu_n = \frac{\pi(n-\frac{1}{2})}{a} - \frac{F_2}{\pi E_1 n} + \frac{b_{2n}}{n},$$

*where $n \in \mathbf{N}$, $\{b_{kn}\}_{n=1}^{\infty} \in l_2$, $k = 1, 2$.*

*Proof.* Applying Lemma 3.4.2 of [11, p. 225] adapted to the interval $(0, a)$ to the functions $g_1(\tau)$ and $g_2(\tau)$ we get Corollary 4.14. □

**Lemma 4.15.** *The squares of the zeros $\{\nu_k\}$ of $g_1(\tau)$ and $\{\mu_k\}$ of $g_2(\tau)$ interlace:*

$$\mu_1^2 < \nu_1^2 < \mu_2^2 < \nu_2^2 < \dots$$

*Proof.* The definitions (4.28), (4.29), (4.15), (4.10), (4.11), (4.3) and (4.4) imply that

$$g_2(\tau) = -\frac{m}{E_0}\tilde{P}\left(\sqrt{\tau^2 - \frac{p^2}{4}}, 1\right),$$

$$g_1(\tau) = -\frac{m}{E_0(\alpha - mp)}\hat{Q}\left(\sqrt{\tau^2 - \frac{p^2}{4}}, 1\right).$$

Now it is clear that

$$\mu_k^2 - \frac{p^2}{4} = \zeta_k(1)^2,$$

$$\nu_k^2 - \frac{p^2}{4} = \xi_k(1)^2.$$

Thus, the statement of Lemma 4.15 follows from Proposition 4.6. □

Due to Corollary 4.14 and Lemma 4.15 the sequences $\{\mu_k\}_{0 \neq k \in \mathbf{Z}}$ and $\{\nu_k\}_{0 \neq k \in \mathbf{Z}}$ satisfy the conditions of Theorem 3.4.1 of [11] adapted to the interval $(0, a)$. Therefore they are the spectra of the Dirichlet–Neumann and Dirichlet–Dirichlet problems, respectively, generated by a potential which can be recovered as follows. Without loss of generality let us assume that $\mu_1^2 > 0$, otherwise we can apply a shift. The function

$$e(\tau) = e^{-i\tau a}\left(g_2(\tau) + i\tau g_1(\tau)\right)$$

is the so-called Jost function of the corresponding prolonged Sturm–Liouville problem on the semiaxis. In our case this function has no zeros in the closed lower half-plane. Introduce the so-called $S$-function

$$S(\tau) = \frac{e(\tau)}{e(-\tau)}$$

and the function

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 - S(\tau)) e^{i\tau x} d\tau.$$

Then the Marchenko integral equation

$$K(x,t) + F(x+t) + \int_{-\infty}^{\infty} K(x,s) F(s+t) ds = 0$$

has a unique solution $K(x,t)$ and

$$\widetilde{q}(x) = -2 \frac{dK(x,x)}{dx}$$

is the potential of the prolonged Sturm–Liouville problem on the semiaxis. We have to prove now that the restriction $q(x) = \widetilde{q}(x)$ $(x \in [0,a])$ is the unknown function (potential) we are looking for.

The proof that this potential together with $a$, $p$, $m$, $\alpha$, $\beta$ found via (4.16), (4.17), (4.25)–(4.27) generates problem (1.9)–(1.11) with the spectrum $\{\lambda_k\}_{0\neq k\in\mathbf{Z}}$ is given in [2]. Using Theorem 2.1 of [4] we conclude that the number of nonpositive eigenvalues of the operator $A$ or what is the same that of the operator $B$ is equal to the number of $\lambda_k$ in the closed lower half-plane, i.e. equal to zero. This means the operator $B$ is strictly positive.

**Theorem 4.16.** *1. Let the properly enumerated sequence $\{\lambda_k\}_{0\neq k\in\mathbf{Z}}$ of complex numbers satisfy the following conditions:*

1) $\{(ip/2) - \lambda_k\}_{0\neq k\in\mathbf{Z}} \in \mathcal{SHB}_\kappa^-$ *for some* $\kappa \geq 0$, *and some* $p > 0$,
2) *formula* (3.1) *is valid* (*with* $p_1 < 0$).

*Then there exists a unique set $\{a, q, p, m, \alpha, \beta\}$ such that $a > 0$, $m > 0$, $\alpha \in \mathbf{R}$, $\beta \in \mathbf{R}$, $q \in L_2(0,a)$ is real-valued and $\{\lambda_k\}_{0\neq k\in\mathbf{Z}}$ is the spectrum of problem* (1.9)–(1.11) *generated by the set $\{a, q, p, m, \alpha, \beta\}$.*

*2. If in addition $\{\lambda_k\}_{0\neq k\in\mathbf{Z}} \in \mathcal{SHB}$ and $E_1 < p$, where $E_1$ is given by* (4.19), *$\chi(\lambda)$ is defined by* (4.15) *and $\varphi(\lambda) = \varphi(\lambda, 1)$ by* (4.8), *then we have $\{a, q, p, m, \alpha, \beta\} \in \mathcal{B}_-$.*

*Proof.* The proof of existence of a real-valued $q(x) \in L_2(0,a)$ generating together with the constants $\{p, m, \alpha, \beta\}$ the constants the prescribed spectrum is the same as that in Theorem 4.3 with $-\lambda$ instead of $\lambda$. The proof of the inequalities $m > 0$, $\alpha > 0$ is given in [2]. The proof of the fact that the corresponding operator $B$ is strictly positive has been given while proving Theorem 4.3. □

Now consider the problem of recovering the parameter set $\{T(s), p, \mu, \nu\}$ of the string from given $\{\lambda_k\}_{0\neq k\in\mathbf{Z}}$ and $l > 0$. Denote by $\mathcal{T}_l^+$, $(\mathcal{T}_l^-)$ the class of sets $\{T(s), p, \mu, \nu\}$ such that $T(s) \in W_2^2(0,l)$, $T(s) > 0$ for $s \in [0,l]$, $p > 0$, $\mu > 0$, $\nu > p\mu$ $(0 < \nu < p\mu)$. Now we conclude that for our string admitting purely imaginary eigenvalues the following theorem proven in [2] remains true.

**Theorem 4.17.** *Let $\{\lambda_k\}_{0\neq k\in\mathbf{Z}}$ be a set of complex numbers satisfying the conditions of Theorem 4.3 (conditions 1 and 2 of Theorem 4.16). Then for any $l > 0$ there exists a unique set $\{T(s), p, \mu, \nu\}$ from $\mathcal{T}_l^+$ (from $\mathcal{T}_l^-$) such that the spectrum of the corresponding problem (1.4)–(1.6) coincides with $\{\lambda_k\}_{0\neq k\in\mathbf{Z}}$.*

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