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Julian Janus

A SINGULAR NONLINEAR BOUNDARY VALUE PROBLEM WITH NEUMANN CONDITIONS

Abstract. We study the existence of solutions for the equations $x'' \pm g(t,x) = h(t)$, $t \in (0,1)$ with Neumann boundary conditions, where $g\colon [0,1] \times (0,+\infty) \to [0,+\infty)$ and $h\colon [0,1] \to \mathbb{R}$ are continuous and $g(t,\cdot)$ is singular at 0 for each $t \in [0,1]$.

Keywords: singular nonlinear boundary value problem, Neumann boundary conditions, second order equations, maximal and minimal solutions.

Mathematics Subject Classification: 34K10.

1. INTRODUCTION

This paper is devoted to the study of the existence of solutions and the existence of maximal and minimal solutions for the following problem:

$$x'' + g(t,x) = h(t), \quad t \in (0,1) \tag{1.1}$$

$$x'(0) = a, \quad x'(1) = b \tag{1.2}$$

where the function $g \in C([0,1]\times(0,+\infty), [0,+\infty))$ is such that $\lim_{s\to 0^+} g(t,s) = +\infty$ for every $t \in [0,1]$, $h \in C([0,1],\mathbb{R})$ and $a,\ b \in \mathbb{R}$.

Apart from problem (1.1), (1.2), we shall also study the existence of solutions for the following one

$$x'' - g(t,x) = h(t), \quad t \in (0,1) \tag{1.3}$$

with boundary conditions (1.2).

By a solution of (1.1), (1.2) (resp. (1.3), (1.2)) we mean a function $x \in C^2([0,1],\mathbb{R})$ satisfying (1.1), (1.2) (resp. (1.3), (1.2)).

A model example of equation (1.1) is the generalized Emden–Fowler equation

$$x'' + \frac{f(t)}{x^\sigma} = h(t), \quad t \in (0,1). \tag{1.4}$$

Equation (1.4) with $h = 0$ and the boundary condition $x(0) = x(1) = 0$ has been studied by several authors. Nachman and Callegari [14] have proved the existence, uniqueness and analyticity of the solution of equation (1.4) with $f(t) = t$, $\sigma = 1$. Lunning and Perry [13], by using the Picard iteration method, have proved the existence of positive solutions of equation (1.4) with $0 < \sigma \leq 1$.

Necessary and sufficient conditions for the existence of positive solutions of equation (1.4) have been given by Talafiero [18, 19], he has used a shooting method. Bobisud *et al.* [1, 2], by means of the topological transversality arguments, have proved the existence of positive solutions of the equation (1.4).

Gatica *et al.* [8] by means of a fixed point theorem for cones, have proved the existence of positive solutions for the problem $x'' + f(t,x) = 0$, $\alpha x(0) + \beta x'(0) = 0$, $\gamma x(1) + \delta x'(1) = 0$, where α, β, γ, $\delta \geq 0$. For related results see [3–5, 15, 16, 20, 21].

In the case $h \neq 0$ equation (1.4) with the boundary condition $x(0) = a$, $\alpha x'(1) + {}$ ${}+\beta x(1) = c$ has been studied by J. Janus and J. Myjak [11], P. Habets and F. Zanolin [9, 10]. They have used sub and super solution arguments and truncation arguments.

In [6] H. Gacki and J. Janus gave a necessary and sufficient condition for the existence of a T-periodic solution of equations:

$$x''(t) + g(t, x(t) + \tau(t)) = h(t), \qquad t \in \mathbb{R}$$

and

$$x''(t) - g(t, x(t) + \tau(t)) = h(t), t \in \mathbb{R}$$

where $g \in C(\mathbb{R} \times (0,+\infty), (0,+\infty))$ and $\tau, h \in C(\mathbb{R},\mathbb{R})$ are T-periodic. They have used the continuation method and truncation arguments based on a priori upper and lower bounds of solutions.

Denote by $\mathcal{S}_1$ (resp. $\mathcal{S}_2$) the set of all solutions of (1.1), (1.2) (resp. (1.3), (1.2)).

By $\|\cdot\|_0$ and $\|\cdot\|_1$ we denote the norms in $C([0,1],\mathbb{R})$ defined by:

$$\|x\|_0 = \sup\{|x(t)| \mid t \in [0,1]\}, \qquad \|x\|_1 = \int_0^1 |x(t)|\, dt.$$

A function $x_* \in C^2([0,1],\mathbb{R})$ is called a lower solution of the problem (1.1), (1.2) if

$$x_*''(t) + g(t, x_*(t)) \geq h(t), \quad t \in (0,1)$$

and

$$x_*'(0) \geq a, \quad x_*'(1) \leq b.$$

Upper solution is defined by reversing the above inequalities signs.

Let a, $b \in \mathbb{R}$ and let

$$h_0 = \int_0^1 h(t)\, dt.$$

Let G_* and G^* be continuous functions defined by

$$G_*(s) = \inf\{g(t,s) \mid t \in [0,1]\} \tag{1.5}$$

and

$$G^*(s) = \sup\{g(t,s) \mid t \in [0,1]\}. \tag{1.6}$$

We say that the functions g and h satisfy condition (A) if

$$\limsup_{s\to+\infty} G^*(s) < |h_0 + a - b|\,. \tag{1.7}$$

Moreover we use the following assumptions on g:

(H_1) $\quad \lim\limits_{s\to 0^+} G_*(s) = +\infty$

(H_2) $\quad \int_0^1 G_*(s)\, ds = +\infty$

for each $s_2 > s_1 > 0$ there is a constant $\omega = \omega(s_1, s_2) > 0$ such that

(H_3) $\quad g(t,\xi) - g(t,\eta) \le \omega(\xi - \eta)$

where $s_1 \le \eta < \xi \le s_2$ and $t \in [0,1]$.

Theorem 1.1. *Suppose that conditions (A), (H_1) and (H_2) are fulfilled. Then problem* (1.1), (1.2) *(resp.* (1.3), (1.2)*) has at least one positive solution provided $b - a < h_0$ (resp. $b - a > h_0$).*

Theorem 1.2. *Suppose that conditions (A) and (H_1) are fulfilled. Let $b - a < h_0$ and $a, b \ge 0$. Then problem* (1.1), (1.2) *has at least one positive solution.*

Remark 1.1. *The conclusions of Theorem 1.1 and Theorem 1.2 for problem* (1.1), (1.2) *hold if in the place of (A) we assume that there exists an upper solution x^* of* (1.1), (1.2).

Theorem 1.3. *Let the hypotheses of Theorem 1.1 or Theorem 1.2 be satisfied and let $b - a < h_0$. Then problem* (1.1), (1.2) *has a minimal and a maximal solution. In addition, suppose that condition (H_3) is fulfilled. Then the minimal and maximal solutions can be computed iteratively.*

In the last section we will show that the assumptions (H_1) and (H_2) are, in a meaning, optimal for these results.

2. PRELIMINARY RESULTS

Lemma 2.1. *Suppose that condition (A) is fulfilled. Then there is $\lambda > 0$ such that for every $x \in C([0,1],\mathbb{R})$ with $\inf\{x(t) \mid t \in [0,1]\} \geq \lambda$ we have*

$$\int_0^1 g(t,x(t))\,dt < |h_0 + a - b| \tag{2.1}$$

Proof. By (1.6) there is $\lambda > 0$ such that

$$g(t,s) < |h_0 + a - b|$$

for every $s \geq \lambda$ and $t \in [0,1]$. Let $x \in C([0,1],\mathbb{R})$ be such that $x(t) \geq \lambda$ for $t \in [0,1]$. Clearly $g(t,x(t)) < |h_0 + a - b|$ for $t \in [0,1]$. Integrating the last inequality from 0 to 1 we obtain (2.1). Since $x \in C([0,1],\mathbb{R})$ is any, the proof of Lemma 2.1 is completed. □

Lemma 2.2. *Suppose that condition (A) is fulfilled and that $h_0 > b - a$ (resp. $h_0 < b - a$). Let $x \in \mathcal{S}_1$ (resp. $x \in \mathcal{S}_2$). Then for every $t \in [0,1]$ we have:*

(i) $|x'(t)| \leq \tau$ *where* $\tau = \|h\|_1 + \max\{|a|, |b|\}$,

(ii) $|x(t)| \leq \tau + \lambda$ *where λ is given as in Lemma 2.1.*

Proof. We prove only the case $x \in \mathcal{S}_1$, in the other case the proof is similar.

(i) Integrating (1.1) from 0 to t we get

$$x'(t) = x'(0) - \int_0^t g(s,x(s))\,ds + \int_0^t h(s)\,ds \leq \tau \tag{2.2}$$

and integrating (1.1) from t to 1 gives

$$x'(t) = x'(1) + \int_t^1 g(s,x(s))\,ds - \int_t^1 h(s)\,ds \geq -\tau. \tag{2.3}$$

So (i) follows.

(ii) Integrating (1.1) from 0 to 1 we have

$$b - a + \int_0^1 g(t,x(t))\,dt = \int_0^1 h(t)\,dt.$$

From this and Lemma 2.1, it follows that $x(t_0) < \lambda$ for some $t_0 \in [0,1]$. By the last inequality and (i), for every $t \in [0,1]$ we have

$$x(t) = x(t_0) + \int_{t_0}^t x'(s)\,ds < \lambda + \tau.$$

This completes the proof of Lemma 2.2. □

Remark 2.1. *Suppose that conditions* (H_1) *and* (H_2)*, are fulfilled. It is easy to see that there are* ξ *and* η*,* $0 < \eta < \xi$*, such that*

$$G_*(s) > 2\tau + \|h\|_1 \text{ for } 0 < s < \xi \tag{2.4}$$

and

$$\int_\eta^\xi G_*(s)\,ds > \tau(2\tau + \|h\|_1), \tag{2.5}$$

where τ *is as in Lemma 2.2.*

Lemma 2.3. *Suppose that* g *satisfies conditions* (H_1) *and* (H_2)*. Then for every* $x \in \mathcal{S}_1$ *(resp.* $x \in \mathcal{S}_2$*) we have* $x(t) \geq \eta$*,* $t \in [0,1]$ *where* η *is given by Remark 2.1.*

Proof. Let $x \in \mathcal{S}_1$, for $x \in \mathcal{S}_2$ the proof is similar.

Claim. There is $t_0 \in [0,1]$ such that $x(t_0) > \xi$ where ξ is given by Remark 2.1.

Indeed, suppose on the contrary that $x(t) \leq \xi$ for each $t \in [0,1]$. Integrating (1.1) from 0 to 1 then using the last inequality (2.4), (1.5), (2.5) and Remark 2.1 we have

$$x'(1) - x'(0) = -\int_0^1 g(t, x(t))\,dt + \int_0^1 h(t)\,dt < -2\tau.$$

On the other hand, by Lemma 2.2 (i), we have $x'(1) - x'(0) \geq -2\tau$, a contradiction. This proves the Claim.

By Lemma 2.2 (i) for every $t \in [0,1]$ we have

$$\int_{t_0}^t G_*(x(s))x'(s)\,ds \geq \tau \operatorname{sgn}(t_0 - t) \int_{t_0}^t G_*(x(s))\,ds. \tag{2.6}$$

On the other hand, integrating (1.1) from t_0 to t, by virtue of (1.5) we have

$$\operatorname{sgn}(t - t_0)\left(\int_{t_0}^t h(s)\,ds - \int_{t_0}^t G_*(x(s))\,ds - x'(t) + x'(t_0)\right) \geq 0. \tag{2.7}$$

It is routine to see, using (2.6), (2.7), (1.5), and Lemma 2.2 (i) that

$$\int_{x(t)}^{x(t_0)} G_*(s)\,ds \leq \tau(2\tau + \|h\|_1) \quad (t \in [0,1]). \tag{2.8}$$

Now, to prove the statement of Lemma 2.3, suppose on the contrary that there is $t_1 \in [0,1]$ such that $x(t_1) < \eta$. By the Claim there is $t_0 \in [0,1]$ such that $x(t_0) > \xi$.

Clearly

$$\int_{\eta}^{\xi} G_*(s)\,ds \le \int_{x(t_1)}^{x(t_0)} G_*(s)\,ds.$$

The last inequality together with (2.8) and (2.5) furnishes a contradiction. This completes the proof. □

Lemma 2.4. *Suppose that g satisfies (H_1) and $a, b \ge 0$. Then there is $\varepsilon > 0$ such that for each $x \in \mathcal{S}_1$, $x(t) \ge \varepsilon$, $t \in [0,1]$.*

Proof. Let $\varepsilon > 0$ be such that $g(t,s) > \|h\|_0$ for $(t,s) \in I \times (0,\varepsilon]$ (such ε exists by (H_1)). Let $x \in \mathcal{S}_1$. We claim that $x(t) \ge \varepsilon$ for $t \in [0,1]$. Indeed, suppose on the contrary that $\{t \mid x(t) < \varepsilon\} \ne \emptyset$. Let $t_0 = inf\{t \mid x(t) < \varepsilon\}$. Obviously $x'(t_0) \le 0$. By this and the fact that $x''(t_0) = -g(t_0, x(t_0)) + h(t_0) < 0$, there exists $\delta > 0$ such that $x'(t) < 0$ for $t \in (t_0, t_0 + \delta)$. By means of a continuation argument one obtains the last inequality throughout $(t_0, 1]$. Since $x'(1) \ge 0$ the contradiction is achieved. The ε does not depend on x, so the statement of Lemma 2.4 holds. □

Let X, Z be normed vector spaces, $L\colon \operatorname{dom} L \subset X \to Z$ a linear mapping, and $N\colon X \to Z$ a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if:

(i) $\dim\ker L = \operatorname{codim}\operatorname{im} L < +\infty$,

(ii) $\operatorname{im} L$ is closed in Z.

If L is a Fredholm mapping of index 0 then there exist continuous projectors $P\colon X \to X$ and $Q\colon Z \to Z$ such that:

$$\operatorname{im} P = \ker L :$$
$$\operatorname{im} L = \ker Q = \operatorname{im}(\mathrm{id} - Q).$$

It follows that $L_{|\operatorname{dom} L \cap \ker P}\colon (id - P)X \to \operatorname{im} L$ is invertible. We denote the inverse of that map by K_P. If Ω is an open subset of X, the mapping N will be called L-compact on $\bar\Omega$ if $QN(\bar\Omega)$ is bounded and $K_P(\mathrm{id} - Q)N\colon \bar\Omega \to X$ is compact. Since $\operatorname{im} Q$ is isomorphic to $\ker L$, there exist isomorphisms $J\colon \operatorname{im} Q \to \ker L$.

We recall the following "Continuation Theorem".

Theorem 2.1 ([7], p. 40). *Let X, Z be normed vector spaces, $L\colon X \to Z$ linear Fredholm mapping of index zero and $N\colon X \to Z$ continuous function. Assume that there exists an open subset Ω of X such that the following conditions hold:*

(i) *N is L-compact on $\bar\Omega$;*

(ii) *for each $\lambda \in (0,1)$, every solution x of $Lx = \lambda N x$ is such that $x \notin \partial\Omega$;*

(iii) *for each $x \in \ker L \cap \partial\Omega$ one has $QNx \ne 0$, where $Q\colon Z \to Z$ is a continuous projector such that $\operatorname{im} L = \ker Q$;*

(iv) $d(JQN_{|\ker L}, \Omega \cap \ker L, 0) \neq 0$, *where* d *denotes the Brouwer degree and* $J\colon \operatorname{im} Q \to \ker L$ *is any isomorphism. Then the equation* $Lx = Nx$ *has at least one solution in* $\operatorname{dom} L \cap \bar{\Omega}$.

3. PROOF OF RESULTS

Proof of Theorem 1.1. Suppose that hypotheses of Theorem 1.1 hold. Since the arguments for problems (1.1), (1.2) and (1.3), (1.2) are similar, we consider only problem (1.1), (1.2).

First of all using Lemma 2.3 we will show that there is a function $\tilde{g}\colon [0,1] \times \mathbb{R} \to (0, +\infty)$ such that the set of all solutions of (1.1), (1.2) with $\tilde{g}$ in place of g coincides with $\mathcal{S}_1$.

Let $\delta > 0$ be such that

$$g(0, \delta) = h_0 + a - b.$$

Let $0 < \sigma_0 < \delta$ be such that

$$g(t, s) > h_0 + a - b, \quad (t, s) \in [0,1] \times (0, \delta - \sigma_0]. \tag{3.1}$$

We define a continuous function $\tilde{g}\colon [0,1] \times \mathbb{R} \to (0, +\infty)$ by

$$\tilde{g}(t, s) = \begin{cases} g(t, s), & \text{if } s \geq s_0 \\ g(t, s_0), & \text{if } s < s_0 \end{cases} \tag{3.2}$$

where $s_0 = \min\{\delta - \sigma_0, \eta\}$ (η is given by Remark 2.1).

Consider the problem

$$x''(t) + \tilde{g}(t, x(t)) = h(t), \quad t \in (0,1) \tag{3.3}$$

$$x'(0) = a, x'(1) = b. \tag{3.4}$$

By the same argument as in the proof of Lemma 2.3 one can show that for every solution x of problem (3.3), (3.4) we have $x(t) \geq \eta$, $t \in [0,1]$. Hence from the definition of $\tilde{g}$ it follows that the set of all solutions of problem (1.1), (1.2) coincides with the set of all solutions of problem (3.3), (3.4).

Now applying Continuation Theorem we will show that problem (3.3), (3.4) has at least one solution.

Let us define a continuous function $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ by

$$f(t, s) = h(t) - \tilde{g}(t, s). \tag{3.5}$$

If we define

$$X = C^2([0,1], \mathbb{R}), Z = C([0,1], \mathbb{R}) \times \mathbb{R}^2$$

$$L\colon X \to Z, \ x \to \big(x''(\cdot), (x'(0), x'(1))\big)$$

$$N\colon\ X \to Z,\ x \to \big(f(\cdot, x(\cdot)), (a,b)\big)$$

then problem (3.3), (3.4) is equivalent to the operator equation

$$Lx = Nx.$$

Let us define

$$P\colon\ X \to X,\ x \to x(0),$$

$$Q\colon\ Z \to Z,\ \big(w,(c,d)\big) \to \left(\int_0^1 w(s)\,ds + c - d, (0,0)\right),$$

$$K_P\colon\ Z \to X,\ \big(w,(c,d)\big) \to \int_0^1 G(\cdot, s) w(s)\,ds,$$

where

$$G(t,s) = \begin{cases} (t-1)s, & \text{if } 0 \le s \le t \le 1; \\ (s-1)t, & \text{if } 0 \le t \le s \le 1. \end{cases}$$

Let $m > 0$ be such that $|f(t,s)| < m$, $(t,s) \in [0,1] \times \mathbb{R}$. By (1.7) there is $\zeta > 0$:

$$\int_0^1 f(t, x(t))\,dt + a - b > 0 \tag{3.6}$$

for all $x \in C^2([0,1],\mathbb{R})$ such that $\inf\{x(t) \mid t \in [0,1]\} \ge \zeta$. From (3.1) and (3.2) it follows that

$$\int_0^1 f(t, x(t))\,dt + a - b < 0 \tag{3.7}$$

for all $x \in C^2([0,1],\mathbb{R})$ such that $\sup\{x(t) \mid t \in [0,1]\} \le s_0$. Let $\varrho > \tau + \zeta$ where $\tau = \|h\|_1 + \max\{|a|, |b|\}$.

We claim that L, N, Q and $\Omega = B(0, \varrho)$ satisfy the assumption of Continuation Theorem. Indeed, it is easy to see that

$$\ker L = \{\, x \mid x = c, c \in \mathbb{R}\}$$

$$\operatorname{im} L = \left\{\, (w,(c,d)) \in Z \mid\ d - c = \int_0^1 w(t)\,dt \right\}$$

$$\dim \ker L = \operatorname{codim} \operatorname{im} L = 1.$$

Since $\operatorname{im} L$ is closed in Z, L is a Fredholm mapping of index 0.

(i) Since

$$QN(x) = \left(\int_0^1 f(s, x(s))\, ds + a - b, (0,0) \right) \tag{3.8}$$

and

$$K_P(id - Q)N(x) =$$
$$= \int_0^1 G(\cdot, s) f(s, x(s))\, ds - \left(\int_0^1 f(s, x(s))\, ds - (a+b) \right) \int_0^1 G(\cdot, s)\, ds,$$

we see that $QN(\overline{B(0,\varrho)})$ is bounded and $K_P(id - Q)N\colon \overline{B(0,\varrho)} \to X$ is compact. Hence N is L-compact on $B(0,\varrho)$.

(ii) Let $\lambda \in (0,1)$ and x be any possible solution of

$$Lx = \lambda N x.$$

Then

$$x'' = \lambda f(t,x), \quad t \in [0,1], \tag{3.9}$$

and

$$x'(0) = \lambda a, \quad x'(1) = \lambda b, \tag{3.10}$$

which implies that

$$b - a = \int_0^1 f\big(t, x(t)\big)\, dt. \tag{3.11}$$

Using the same arguments as in the proof of Lemma 2.2 (i) one can show that

$$|x'(t)| < \lambda \tau. \tag{3.12}$$

On the other hand it follows from (3.11) and (3.6) that there exists $t_0 \in [0,1]$ such that $|x(t_0)| < \zeta$. Therefore, for all $t \in [0,1]$, we have

$$|x(t)| = \left| x(t_0) + \int_{t_0}^{t} x'(s)\, ds \right| < \zeta + \lambda \tau. \tag{3.13}$$

Hence $x \notin \partial B(0,\varrho)$ and the proof of (ii) is complete.

(iii) By (3.6), (3.7) and (3.8) we have that

$$QN(\varrho) > 0 \tag{3.14}$$

and

$$QN(-\varrho) < 0 \tag{3.15}$$

so (iii) holds.

(iv) Observe that $\operatorname{im} Q$ and $\ker L$ can be naturally identified with $\mathbb{R}$. Hence any isomorphism $J\colon \operatorname{im} Q \to \ker L$ is of the form $Jx = kx$ where $k \in \mathbb{R}\setminus\{0\}$. From this and (3.14), (3.15) it follows that $JQN_{|\ker L}$ is homotopic with id when $k > 0$ and with $-id$ when $k < 0$. Therefore, $d(JQN_{|\ker L}, B(0,\varrho)\cap \ker L, 0) = \pm 1$. Hence condition (iv) is satisfied, which proves that problem (3.3), (3.4) has at least one solution. This completes the proof of Theorem 1.1 □

The proof of Theorem 1.2 is the same as the proof of Theorem 1.1. □

Proof of Remark 1.1. The proof follows the argument considered for the "classical" upper and lower solutions (see e.g. [17, p. 276]). Let $\tilde{g}$ be given by (3.2) with $s_0 = \min\big\{\inf_{t\in[0,1]} x^*(t), \delta - \sigma_0, \eta\big\}$. Now we show that there is a lower solution x_* of (3.3), (3.4). To this end, consider the following problem

$$w''(t) = h(t) - h_0 - a + b, \quad t \in [0,1] \tag{3.16}$$

$$w'(0) = a, \quad w'(1) = b. \tag{3.17}$$

Let w be a solution of (3.16), (3.17). Choose $\lambda > 0$ such that $-\lambda + w(t) \le s_0$ for each $t \in [0,1]$. Set $x_* = -\lambda + w$. By (3.1) we have

$$x_*''(t) + \tilde{g}(t, x_*(t)) \ge h(t), t \in [0,1].$$

Thus x_* is a lower solution of (3.2), (3.3) and the proof is complete. □

Proof of Theorem 1.3. The proof follows the argument considered for the "classical" upper and lower solutions (see e.g. [17, p. 279–280]). Let $\sigma > 0$ be such that $g(t,s) \le h_0 + a - b$ for $(t,s) \in [0,1]\times[\delta+\sigma, +\infty)$ where $\delta > 0\colon\ g(0,\delta) = h_0 + a - b$. Let $\tilde{g}$ be given by (3.2) and let w be a solution of problem (3.16), (3.17). Choose $\lambda > 0$ such that $-\lambda + w(t) \le s_0$ (s_0 as in (3.2)) and $\lambda + w(t) \ge \max\big\{\delta + \sigma, \eta\big\}$ (η is given as in Remark 2.1). Set $x_* = -\lambda + w$ and $x^* = \lambda + w$. It is easy to see that x_* (resp. x^*) is a lower (resp. upper) solution of problem (3.3), (3.4). Thus, problem (1.1), (1.2) has a minimal and a maximal solution.

Moreover, if (H_3) is fulfilled, then using arguments as in [17, p. 280], $x_{\min}$ and $x_{\max}$ can be computed iteratively by means of the following iteration scheme:

$$x_{i+1}'' - \omega x_{i+1} = -g(t, x_i(t)) - \omega x_i(t), \quad t \in (0,1) \tag{3.18}$$

$$x_{i+1}'(0) = a, \quad x_{i+1}'(1) = b. \tag{3.19}$$

If $x_1 = x_*$ (resp. $x_1 = x^*$) then the sequence $\{x_i\}_{i\in\mathbb{N}}$ is increasing (resp. decreasing) and converges in $C^2([0,1],\mathbb{R})$ to $x_{\min}$ (resp. $x_{\max}$). This completes the proof of Theorem 1.3. □

4. CONCLUDING REMARKS

Remark 4.1. *Theorem 1.3 fails for problem* (1.1), (1.2) *if in the place of* (H_1) *we assume that*

$$\lim_{x\to 0^+} g(t,s) = +\infty \quad \textit{for each } t\in[0,1]. \tag{4.1}$$

Indeed, let $g\colon [0,1]\times(0,+\infty)\to[0,+\infty)$ be a continuous function defined by

$$g(t,s)=\begin{cases} \dfrac{t^2}{s}, & \text{if } t\in(0,1],\ 0<s\le t^2;\\ \dfrac{1}{s}+\dfrac{t^2}{s}\left(1-\dfrac{1}{s}\right), & \text{if } 0\le t\le\sqrt{s},\ 0\le s\le 1;\\ \dfrac{1}{s}, & \text{otherwise}\end{cases} \tag{4.2}$$

It is easy to see that g satisfies (4.1). We claim that for $\alpha>2$ the problem

$$x''+g(t,x)=3,\quad t\in(0,1), \tag{4.3}$$

$$x'(0)=0,\quad x'(1)=\alpha \tag{4.4}$$

has no solutions.

Indeed, suppose on the contrary that (4.3), (4.4) has a solution x. We consider the following three cases.

Case 1. Assume that $x(t)\le 1$, $t\in[0,1]$. Observe that for $\alpha=2$ problem (4.3), (4.4) has the solution $x_0(t)=t^2$, $t\in[0,1]$. From (4.2) it follows that

$$g(t,x(t))\ge g(t,x_0(t)),\quad t\in[0,1].$$

By this, the fact that x (resp. x_0) is a solution of problem (4.3), (4.4) (resp. (4.3), (4.4) with $\alpha=2$) we have $(x_0-x)''(t)\ge 0$, $t\in(0,1)$. Therefore $(x_0-x)'(\cdot)$ is nondecreasing, so

$$0\le x_0'(1)-x'(1)-x_0'(0)+x'(0)=2-\alpha<0,$$

which is impossible.

Case 2. Assume that $x(0)>1$. By (4.2) and (4.3) follows that $x(t)>1$ for each $t\in[0,1]$. Substituting x in (4.3) and integrating from 0 to 1 we get

$$x'(1)-x'(0)=3-\int_0^1 g\bigl(t,x(t)\bigr)\,dt. \tag{4.5}$$

The left-hand side of (4.5) is greater then 2 and by (4.2) the right-hand side of (4.5) is equals 2, so we have a contradiction.

Case 3. Assume that $x(0)<1$ and $\max\bigl\{x(t)\mid t\in[0,1]\bigr\}>1$. Set $\tilde t=\min\bigl\{t\in[0,1]\mid x(t)=1\bigr\}$, $\bar t=\max\bigl\{t\in[0,1]\mid x'(t)=0 \text{ and } x(z)\le 1 \text{ for } z\in[0,t]\bigr\}$.

Substituting x in (4.3) and integrating from $\bar{t}$ to 1 and from $\tilde{t}$ to $\bar{t}$ we get

$$x'(1) - x'(\bar{t}) < 2(1 - \bar{t}) \tag{4.6}$$

and

$$x'(\bar{t}) \leq 2(\bar{t} - \tilde{t}). \tag{4.7}$$

By (4.6) and (4.7) we have $x'(1) \leq 2 - 2\tilde{t} < 2$, which gives a contradiction. Hence for $\alpha > 2$ problem (4.3), (4.4) has no solutions.

It is easy to see that $g, h(t) = 3$ and $\alpha\colon\ 2 < \alpha < 3$ satisfy assumption (A). This completes the proof of Remark 4.1.

Remark 4.2. *Theorem 1.1 fails for problem* (1.1), (1.2) *if in the place of* (H_2) *we assume that*

$$\int_0^1 g(t,s)\,ds = +\infty \quad \textit{for each}\, t \in [0,1]. \tag{4.8}$$

Indeed, for any given $\alpha \geq 1$ consider a function $\varphi\colon\ [0,1] \to [0, \alpha - 2/3]$ defined by

$$\varphi(t) = -\frac{2}{3}t^{\frac{3}{2}} + \alpha t, \quad t \in [0,1].$$

Let $g\colon\ [0,1] \times (0,+\infty) \to (0,+\infty)$ be a continuous function defined by

$$g(t,s) = \begin{cases} \dfrac{\varphi(t)}{2s\sqrt{t}}, & \text{if } 0 < t \leq 1,\ 0 < s \leq \varphi(t); \\ \dfrac{1}{s}\left(1 - \dfrac{t}{\varphi^{-1}(s)}\right) + \dfrac{\sqrt{t}}{2\varphi^{-1}(s)}, & \text{if } 0 \leq t \leq \varphi^{-1}(s),\ 0 < s \leq \alpha - \dfrac{2}{3}; \\ \left(\dfrac{1-t}{\alpha - \frac{2}{3}} + \dfrac{\sqrt{t}}{2}\right)\dfrac{1}{s + \frac{5}{3} - \alpha} & \text{otherwise} \end{cases} \tag{4.9}$$

It is easy to see that g satisfies assumptions (4.8) and (H_1). Observe that the function φ is a solution of the problem

$$x'' + g(t,x) = 0, \quad t \in (0,1), \tag{4.10}$$

$$x'(0) = \alpha, \quad x'(1) = \alpha - 1. \tag{4.11}$$

We claim that for $a > \alpha$ and $b \leq \alpha - 1$ the problem

$$x'' + g(t,x) = 0, \quad t \in (0,1), \tag{4.12}$$

$$x'(0) = a, \quad x'(1) = b. \tag{4.13}$$

has no solutions.

Indeed, suppose on the contrary that (4.12), (4.13) has a solution x. Since for each $t \in [0,1]$, $g(t,\cdot)$ is strictly decreasing and $x(0) \geq \varphi(0)$, $x'(0) > \varphi'(0)$, there exists a neighborhood U of 0 such that $(x-\varphi)''(t) > 0$ for $t \in U \cap (0,1)$. By means of a continuation argument one obtains the last inequality throughout $(0,1)$. Hence $(x-\varphi)'(\cdot)$ is strictly increasing on $[0,1]$. Therefore $a = x'(1) > \varphi'(1) = \alpha - 1$, which is impossible. It is easy to see that $g, h = 0$, $a = \alpha$, $b = \alpha - 1$ satisfy (A). This completes the proof of Remark 4.2. □

Remark 4.3. *Theorem 1.1 fails for problem* (1.3), (1.4) *if in place of* (H_2) *we assume* (4.8).

Indeed, let $\psi(t) = (2/3)t^{3/2}$, $t \in [0,1]$ and let $g\colon [0,1]\times(0,+\infty) \to (0,+\infty)$ be a continuous function defined by

$$g(t,s) = \begin{cases} \dfrac{\psi(t)}{2s\sqrt{t}}, & \text{if } 0 < t \leq 1,\ 0 < s \leq \psi(t); \\ \dfrac{1}{s}\left(1 - \dfrac{t}{\psi^{-1}(s)}\right) + \dfrac{\sqrt{(t)}}{2\psi^{-1}(s)}, & \text{if } 0 \leq t \leq \psi^{-1}(s),\ 0 < s \leq \frac{2}{3}; \\ \left(\dfrac{3(1-t)}{2} + \dfrac{\sqrt{t}}{2}\right)\dfrac{1}{s+\frac{1}{3}} & \text{otherwise.} \end{cases} \tag{4.14}$$

It is easy to see that g satisfies assumption (4.8) and (H_1). Observe that the function ψ is a solution of the problem

$$x'' = g(t,x), \quad t \in (0,1) \tag{4.15}$$

$$x'(0) = 0, \quad x'(1) = 1. \tag{4.16}$$

We claim that for $\beta > 1$ the problem

$$x'' = g(t,x), \quad t \in (0,1), \tag{4.17}$$

$$x'(0) = 1+\beta, \quad x'(1) = 1 + 2\beta \tag{4.18}$$

has no solutions. Indeed, suppose on the contrary that (4.17), (4.18) has a solution x. Substituting x in (4.17) and integrating from 0 to t we get

$$x'(t) = x'(0) + \int_0^t g(s,x(s))\,ds > 1 + \beta. \tag{4.19}$$

Since $0 \leq \psi'(t) \leq 1$, $t \in [0,1]$ and $x(0) \geq \psi(0)$, by (4.19) we have $x(t) \geq \psi(t)$, $t \in [0,1]$. From this, the fact that ψ and x satisfies equation (4.1) and the monotonity of g with respect to the second variable we get $(x-\psi)''(t) \leq 0$, $t \in (0,1)$. Consequently

$$1 + 2\beta - 1 = x'(1) - \psi'(1) \leq x'(0) - \psi'(0) = 1 + \beta,$$

which is impossible. Hence the equation (4.15) with a boundary condition has no solution.

It is easy to see that $g, h = 0$, $a = 1+\beta$, $b = 1 + 2\beta$ satisfy (A). This complete the proof of Remark 4.3.

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Julian Janus
janus@uci.agh.edu.pl

AGH University of Science and Technology
Faculty of Applied Mathematics
al. Mickiewicza 30, 30-059 Cracow, Poland

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