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A NOTE ON GEODESIC AND ALMOST GEODESIC MAPPINGS OF HOMOGENEOUS RIEMANNIAN MANIFOLDS

Abstract. Let M be a differentiable manifold and denote by ∇ and $\widetilde{\nabla}$ two linear connections on M. ∇ and $\widetilde{\nabla}$ are said to be geodesically equivalent if and only if they have the same geodesics. A Riemannian manifold (M, g) is a naturally reductive homogeneous manifold if and only if ∇ and $\widetilde{\nabla} = \nabla - T$ are geodesically equivalent, where T is a homogeneous structure on (M, g) ([7]). In the present paper we prove that if it is possible to map geodesically a homogeneous Riemannian manifold (M, g) onto $(M, \widetilde{\nabla})$, then the map is affine. If a naturally reductive manifold (M, g) admits a nontrivial geodesic mapping onto a Riemannian manifold $(\overline{M}, \overline{g})$ then both manifolds are of constant cutvature. We also give some results for almost geodesic mappings $(M, g) \to (M, \widetilde{\nabla})$.

Keywords: homogeneous Riemannian manifold, geodesic, almost geodesic, geodesic mapping, almost geodesic mapping..

Mathematics Subject Classification: 53B30, 53C25.

1. INTRODUCTION

Let (M, g) be an n-dimensional Riemannian manifold of class C^∞. Let $\mathfrak{F}(M)$ be the ring of differentiable functions and $\mathfrak{X}(M)$ the $\mathfrak{F}(M)-$module of differentiable vector fields on M. A complete and simply connected manifold (M, g) is homogeneous if there exists a transitive and effective group G of isometries of M. Ambrose and Singer proved (see [7]) that a complete and simply connected Riemannian manifold (M, g) is homogeneous if and only if there exists a tensor field T of type $(1, 2)$ such that:

$$\begin{aligned} &\text{(i)} \quad g(T_X Y, Z) + g(Y, T_X Z) = 0, \\ &\text{(ii)} \quad (\nabla_X R)_{YZ} = [T_X, R_{YZ}] - R_{T_X YZ} - R_{Y T_X Z}, \qquad (1.1) \\ &\text{(iii)} \quad (\nabla_X T)_Y = [T_X, T_Y] - T_{T_X Y}, \end{aligned}$$

for $X, Y, Z \in \mathfrak{X}(M)$. Here ∇ and R denote the Levi–Civita connection and the Riemannian tensor field, respectively. A tensor field T satisfying the conditions (1.1) on M is called a homogeneous structure on (M, g). It is easy to see that the conditions (1.1) are equivalent to

$$\left(\widetilde{\nabla}_X g\right)(Y, Z) = 0, \quad \left(\widetilde{\nabla}_X R\right)(Y, Z) = 0, \quad \left(\widetilde{\nabla}_X T\right)(Y, Z) = 0 \tag{1.2}$$

where $\widetilde{\nabla}$ is the connection determined by

$$\widetilde{\nabla}_X Y = \nabla_X Y - T(X, Y). \tag{1.3}$$

where $T(X, Y) = T_X Y$, $X, Y, Z \in \mathfrak{X}(M)$.

In [7] F. Tricerri and L. Vanhecke studied the decomposition of the space of all the tensors T satisfying the conditions (1.1) into the irreducible components under the action of orthogonal group. As is well-known, a geodesic in a Riemannian manifold M is a curve of $c : I \to M$ whose tangent vector field $\dot{c}$ is parallel along c (I is an open interval in the real line R^1). A curve c is almost geodesic in a Riemannian manifold M if there exists a 2− dimensional distribution E^2 complanar along c, to which the tangent vector $\dot{c}$ of this curve belongs at every point. Let $(\overline{M}, \overline{\nabla})$ be a differentiable manifold with a linear symmetric connection $\overline{\nabla}$. A mapping $f\colon (M, g) \to (\overline{M}, \overline{\nabla})$ is called geodesic or projective if f carries geodesics in M to geodesics in $\overline{M}$. The mapping f is an almost geodesic mapping if, as a result of f, every geodesic in the manifold M passes into an almost geodesic curve in the manifold $\overline{M}$. If $\overline{M}$ coincides with M and f is a diffeomorphism, f is called a geodesic or an almost geodesic transformation of M.

It is well known, that the identity transformation is geodesic if and only the connection deformation tensor $P(X, Y) = \overline{\nabla}_X Y - \nabla_X Y$ has the form ([1, 2, 3, 5])

$$P(X, Y) = \psi(X)Y + \psi(Y)X, \tag{1.4}$$

where ψ is a certain 1-form and ∇ denotes the Levi–Civita connection of (M, g), $X, Y \in \mathfrak{X}(M)$.

In this case, $\overline{\nabla}$ and ∇ are said to be geodesically (or projectively) equivalent or geodesically (projectively) related. Two such connections define the same system of geodesics. Obviously $\sim$ is an equivalence relation and an equivalence class $[\nabla]$ containing ∇ is called a projective structure on M.

Sinyukow [5] defined three kinds of almost geodesic mappings, namely π_1, π_2 and π_3 which are characterized, respectively, by the conditions:

$$\pi_1\colon \underset{X,Y,Z}{\mathfrak{S}} \left[(\nabla_X P)(Y, Z) + P\left(P(X, Y), Z\right) - a(X, Y)Z - P(X, Y)b(Z)\right] = 0 \tag{1.5}$$

($\mathfrak{S}$ is cyclic sum),

$$\begin{aligned}\pi_2\colon\ P(X,Y) &= \psi(X)Y + \psi(Y)X + F(X)\varphi(Y) + F(Y)\varphi(X),\\ &(\nabla_X F)(Y) + (\nabla_Y F)(X) + F(F(X))\varphi(Y) + F(F(Y))\varphi(X) =\\ &= \mu(X)F(Y) + \mu(Y)F(X) + \rho(X)Y + \rho(Y)X;\end{aligned} \tag{1.6}$$

$$\begin{aligned}\pi_3\colon\ P(X,Y) &= \psi(X)Y + \psi(Y)X + a(X,Y)\nu,\\ \nabla_X\nu &= \theta(X)\nu + \lambda X, \qquad \lambda \in \mathfrak{F}(M);\ X,Y,Z \in \mathfrak{X}(M),\end{aligned} \tag{1.7}$$

where $P(X,Y) = \overline{\nabla}_X Y - \nabla_X Y$ is the connection deformation tensor and φ, ψ, b, θ, ρ, ν, a, F are tensors of the corresponding types.

In the present paper we shall study a geodesic and an almost geodesic related connections ∇ and $\widetilde{\nabla} = \nabla - T$, where T is a homogeneous structure on (M,g).

2. GEODESIC MAPPINGS OF HOMOGENEOUS RIEMANNIAN MANIFOLDS

By [7], Theorem 6.8, a complete and simply connected Riemannian manifold (M,g) is naturally reductive homogeneous manifold if and only if there exists a tensor field T of type $(1,2)$ satisfying the conditions (1.1) and such that $\widetilde{\nabla}$ and ∇ are geodesically equivalent.

Now we shall prove

Lemma 2.1. *If it is possible to map geodesically a homogeneous Riemannian manifold (M,g) onto a manifold $(M,\widetilde{\nabla})$, then the map is affine.*

Proof. The connections ∇ and $\widetilde{\nabla}$ are geodesically equivalent if and only if the connection deformation D have the form

$$D(X,Y) = -T(X,Y) = \psi(X)Y + \psi(Y)X + S(X,Y) \tag{2.1}$$

where ψ is a 1-form and the tensor field S satisfies

$$S(X,Y) + S(Y,X) = 0.$$

We put

$$\overset{0}{P}(X,Y) = \frac{1}{2}\left(T(X,Y) + T(Y,X)\right)$$

and

$$\overset{0}{P}(X,Y,Z) = g(\overset{0}{P}(X,Y),Z).$$

From (1.1 (i)) we obtain

$$\underset{X,Y,Z}{\mathfrak{S}}\overset{0}{P}(X,Y,Z) = 0.$$

Hence and from (2.1) we have

$$\psi(X)g(Y,Z) + \psi(Y)g(X,Z) + \psi(Z)g(X,Y) = 0.$$

Therefore $\psi(X) = 0$ for all $X \in \mathfrak{X}(M)$. This completes the proof. □

If $T = 0$, then (1.1) implies that (M, g) is a symmetric manifold. In view of the Sinyukov theorem we obtain: if it is possible to map geodesically a complete and simply connected Riemannian manifold with the homogeneous structure $T = 0$ into a manifold $(\overline{M}, \overline{g})$ then both manifolds are of constant sectional curvature.

Let (M, g) be a connected Riemannian manifold and suppose M admits a non-trivial homogeneous structure T by

$$T(X, Y, Z) + T(Y, X, Z) = 0, \tag{2.2}$$

where $X, Y, Z \in \mathfrak{X}(M)$.

From (1.1) and (2.2) we get easily.

Lemma 2.2. *Let* (M, g) *be a connected Riemannian manifold with the homogeneous structure of type* (2.2)*. Then Ricci tensor on* M *satisfies*

$$\underset{X,Y,Z}{\mathfrak{S}}(\nabla_X Ric)(Y, Z) = 0. \tag{2.3}$$

Now we shall prove

Theorem 2.1. *If it is possible to map geodesically on* (M, g) *satysfying* (2.3) *onto a manifold* $(\overline{M}, \overline{g})$*, then both manifolds are of constant curvature.*

Proof. As is well-known a manifold (M, g) admits a geodesic mapping if and only if there exists a function $\varphi \in \mathfrak{F}(M)$ and a symmetric non-singular bilinear form a on M satisfying

$$(\nabla_X a)(Y, Z) = (Y\varphi) g(X, Z) + (Z\varphi) g(X, Y) \tag{2.4}$$

for all $X, Y, Z \in \mathfrak{X}(M)$ ([5]).

Let $p \in M$ be such that $d\varphi \neq 0$ and (2.4) hold at p. Choose a local coordinate system (U, x) so that $p \in U$. By R^l_{ijk}, R_{ik}, g_{ik}, a_{ik}, φ_{ik} we denote the components of the tensors R, Ric, g, a and the Hessian $H\varphi$ of φ in this coordinate system. Differentiating covariantly (2.4) and applying the Ricci identity we get

$$a_{it}R^t_{jkl} + a_{tj}R^t_{ikl} = \varphi_{li}g_{jk} + \varphi_{lj}g_{ik} - \varphi_{ki}g_{jl} - \varphi_{kj}g_{il}. \tag{2.5}$$

Differentiating covariantly (2.5) with respect to x^m, contracting with g^{lm} and applying the Ricci identity, by (2.4) and (2.3), we obtain

$$4\varphi_t R^t_{jki} = 3R^t_k\varphi_t g_{ji} - 4\varphi_k R_{ji} + 4\psi_i R_{jk} - 3g_{jk}R^t_i\varphi_t + a_j g_{ik} - a_k g_{ji}, \tag{2.6}$$

where $a_i = \nabla_s\varphi_{it}g^{ts}$. Transvecting (2.6) with g^{jk} we get $R^t_i\varphi_t = \rho\varphi_i$, $\rho \in \mathfrak{F}(U)$. Following considerations made in [6] we get

$$a^t_i\varphi_t = \tau\varphi_i, \qquad \varphi^t_i\varphi_t = \lambda\varphi_i, \qquad \tau, \lambda \in \mathfrak{F}(U),$$

and finally we obtain

$$H_\varphi(X, Y) = \Phi(\varphi)g(X, Y) \tag{2.7}$$

where H_φ is the Hessian of φ and $\Phi \in \mathfrak{F}(M)$.

By [7] if a complete and simply connected manifold with homogeneous structure T admits condition (2.7), then the manifold (M, g) is of constant curvature. This completes the proof. □

From Lemmas 2.1 i 2.2 and Theorem 2.1 we obtain

Theorem 2.2. *On a homogeneous manifold the geodesic of* ∇ *and* $\widetilde{\nabla} = \nabla - T$ *are the same if and only if* M *is naturally reductive. The geodesic mapping* $(M, g) \to (M, \widetilde{\nabla})$ *is affine. If a naturally reductive manifold* (M, g) *admits a non-trivial geodesic mapping onto a Riemannian manifold* $(\overline{M}, \overline{g})$*, then both manifolds are of constant curvature.*

3. ALMOST GEODESIC MAPPINGS OF HOMOGENEOUS MANIFOLDS

On the basis [7] the most general form of the structure tensor T is following

$$T(X,Y) = g(X,Y)\Phi - g(\Phi,Y)X + \overset{2}{T}(X,Y) \tag{3.1}$$

where Φ is a given vector field on (M, g) and $\overset{2}{T}$ is a tensor field such that

$$\begin{gathered} g(\overset{2}{T}(X,Y),Z) + g(Y,\overset{2}{T}(X,Z)) = 0, \\ \widetilde{\nabla}\overset{2}{T} = 0, \\ C_{12}(\overset{2}{T}) = \sum_{i=1}^{n} \overset{2}{T}(X_i, X_i) = 0, \end{gathered} \tag{3.2}$$

where X_i is the base vector of the natural frame.

We put

$$\begin{aligned} \overset{1}{P}(X,Y) &= \frac{1}{2}\left(\psi(X)Y + \psi(Y)X\right) - g(X,Y), \\ \overset{1}{S}(X,Y) &= \frac{1}{2}\left(\psi(X)Y - \psi(Y)X\right), \\ \overset{2}{P}(X,Y) &= -\frac{1}{2}\left(\overset{2}{T}(X,Y) + \overset{2}{T}(Y,X)\right), \\ \overset{2}{S}(X,Y) &= -\frac{1}{2}\left(\overset{2}{T}(X,Y) - \overset{2}{T}(Y,X)\right), \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} P(X,Y) &= \overset{1}{P}(X,Y) + \overset{2}{P}(X,Y) \\ S(X,Y) &= \overset{1}{S}(X,Y) + \overset{2}{S}(X,Y) \end{aligned}$$

where $\psi(X) = g(X, \Phi)$.

Here P denotes the symmetric part of the tensor field T and S – the skew-symmetric one.

Then we have

$$\widetilde{\nabla}_X Y = \nabla_X Y + P\,(X, Y) + S(X, Y). \tag{3.4}$$

and the connection deformation tensor D have the form

$$D(X, Y) = P(X, Y) + S(X, Y) \tag{3.5}$$

for all $X, Y \in \mathfrak{X}(M)$.

We shall prove

Theorem 3.1. *On the homogeneous Riemannian manifold the connections* ∇ *and* $\widetilde{\nabla}$ *defined by* (3.3) *and* (3.4) *are almost geodesically related if and only if the tensor fields* $\overset{2}{P}$ *and* $\overset{2}{S}$ *satisfy the relations*

$$\begin{aligned}\underset{X,Y,Z}{\mathfrak{S}}\Bigg[&(\nabla_X \overset{2}{P})(Y, Z) + \overset{2}{P}(\overset{2}{P}(X, Y), Z) - \overset{2}{P}(X, Y)b(Z) - \overset{2}{P}(X, \psi)g(Y, Z)+\\ &- \overset{2}{P}(\overset{2}{S}(X, Y), Z) - \overset{2}{S}(X, \psi)g(Y, Z)+\\ &- h(X, Y)\nabla_Z \psi + k(X, Y, Z)\psi + q(X, Y)Z\Bigg] = 0,\end{aligned} \tag{3.6}$$

where: b, d, h, k, q are tensors of the corresponding types.

Proof. By [5] the mapping $\nabla \to \widetilde{\nabla}$ is almost geodesic if and only if the connection deformation tensor D satisfies the relations

$$\left(\nabla_\gamma D^h_{\alpha\beta} + D^h_{\delta\alpha} D^\delta_{\beta\gamma}\right) \lambda^\alpha \lambda^\beta \lambda^\gamma = b D^h_{\alpha\beta} \lambda^\alpha \lambda^\beta + a\lambda^h \tag{3.7}$$

where $\lambda^i = \frac{dc^i}{dt}$ denotes the vector tangent to the geodesic $c(t) = \left(c^i(t)\right)$. We conclude from (3.3), (3.4), (3.5), (3.7) that (3.6) holds. This proves the theorem. □

Corollary 3.1. *If* $\overset{2}{P} = 0$ *and* $\overset{2}{S} = 0$ *then the almost geodesic mapping is of the kind* (1.7).

Corollary 3.2. *If* $\overset{2}{P} = 0$ *and* $\overset{2}{S} = 0$ *then a homogeneous Riemannian manifold is a manifold of constant curvature (see [7]).*

Corollary 3.3. *If* $g(X, Y)\nabla_Z \psi + g(X, Y)\overset{2}{P}(Z, \psi) + g(X, Y)\overset{2}{S}(Z, \psi) + k(X, Y, Z)\psi = 0$ *then the almost geodesic mapping* (3.6) *is of the kind* (1.5).

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