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CALCULATION OF DISTRIBUTION OF TEMPERATURE IN THREE-DIMENSIONAL SOLID CHANGING ITS SHAPE DURING THE PROCESS

Abstract. The present paper suplements and continues [2]. Galerkin method for the Fourier–Kirchhoff equation in the case when $\Omega(t)$ – equation domain, dependending on time t, is constructed. For special case $\Omega(t) \subset \mathbb{R}^2$ the computer program for above method is written. Binaries and sources of this program are available on `http://wms.mat.agh.edu.pl/~bozek`.

Keywords: parabolic partial differential equations, non-stationary distribution of heat, finite element method, Galerkin method.

Mathematics Subject Classification: 65M60, 65Y99.

1. INTRODUCTION

An inspiration to writing this paper were researches devoted to a simulation of the squeeze out of carbon electrodes in transient press. Results were published in [3] and [1]. In this paper the domain, in which the suitable differential equations describing the project under discussion, depends on time. This situation takes place also in many other technological processes such as continuous steel founding or producing of wire by squeezing out technology (see [5]). Adaptation of Galerkin method to the differential problem describing these matters runs into many difficulties. In this paper, in order to make the problem easy, the Fourier–Kirchhoff equation is considered in some domain $\Omega(t)$ of time and the Galerkin method for this problem is constructed. These results are generalizations of [2]. The main difficulty lies in the fact that the nodal points of triangulation which are used to discretization (12) of variational problem (5) are varying in time, and it implies that the problems of work out of time derivatives of basic functions $\varphi_{p(t)}$ arrive. These derivatives also appear in the discretization (12). In formula (15) where the time derivative $\frac{\partial}{\partial t}\varphi_{p(t)}$ is defined, the unknown field v of the drift of the mesh occurs. In the special case of $\Omega(t)$ defined by the formula (18), the construction of this field is shown.

2. INITIAL-BOUNDARY PROBLEM

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, and $v\colon [0,t^\star] \times \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field. We assume that field v is such that:

1) for every $x_0 \in \Omega$ the equation $x'(t) = v(t, x(t))$ with initial condition $x(t_0) = x_0$ has a unique solution $x(t, x_0)$ defined on the interval $[0, t^\star]$,
2) for every $t \in [0, t^\star]$ the set

$$\Omega(t) := \{x(t, x_0)\colon \quad x_0 \in \Omega\} \tag{1}$$

is a bounded Lipschitz domain.

We consider the following Dirichlet problem for the Fourier equation of heat conduction

$$\begin{cases} \dfrac{\partial T}{\partial t}(t) + \operatorname{div}(-\Theta_t \cdot \nabla T(t)) = f(t) & \text{in } \Omega(t) \\ T(t) = T_\partial & \text{on } \partial\Omega(t) \text{ for } t \in [0, t^\star] \\ T(0) = T_0 & \text{in } \Omega(0) \end{cases} \tag{2}$$

where:

$t^* \in\,]0, \infty]$ – time of observation of heat flow;
$\Omega(t) \subset \mathbb{R}^3$ – domain, in which heat is spread out at a moment $t \in [0, t^\star]$;
$T(t)\colon \Omega(t) \ni x \xrightarrow{df} T(t,x) \in \mathbb{R}$ – seeking distribution of temperature at t;
$T_\partial(t)\colon \partial\Omega(t) \ni \zeta \xrightarrow{df} T_\partial(t,\zeta) \in \mathbb{R}$ – given lateral temperature at t;
$T_0\colon \Omega(0) \to \mathbb{R}$ – given initial temperature;
$f(t)\colon \Omega(t) \ni x \xrightarrow{df} f(t,x) \in \mathbb{R}$ – given external heat sources (or absorbtions);
$\Theta\colon \mathbb{R} \to\,]0, \infty[$ – coefficient of heat conduction,

here we denote $\Theta_t : \Omega(t) \ni x \xrightarrow{df} \Theta(T(t,x)) \in\,]0,\infty[$, $\nabla := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$, $\operatorname{div}_x w = \operatorname{tr} d_x w = \sum_{i=1}^3 \frac{\partial w_i}{\partial x_i}$ for any differentiable vector field $w : \Omega(t) \to \mathbb{R}^3$ and for each $x \in \Omega$.

Assumptions 1. *The following* regularity *of data will be assumed:*

1) $\Omega(t) \subset \mathbb{R}^3$ – *bounded Lipschitz domain for* $t \in [0, t^\star]$;

2) $\Theta\colon\ \mathbb{R} \to]0,\infty[$ – *locally Lipschitz function;*
3) $T_0 \in L^2(\Omega(0))$;
4) $\tilde{f} \in L^2\left((0,t^*;\left(H_0^1(\Omega(t))\right)'\right)$ *for* $t \in [0,t^\star]$ *(describing, instead of the classical "f", generalized outward effects);*
5) *for every moment* $t \in [0,t^\star]$ *the boundary temperature* $T_\partial(t)\colon\ \partial\Omega(t) \to \mathbb{R}$ *can be extended on the whole* $\Omega(t)$ *to some function* $\overline{T}_\partial(t)\colon\ \Omega(t) \to \mathbb{R}$ *in this way:*
   – $\overline{T}_\partial \in L^2\left(0,t^*;H^1\left(\Omega(t)\right)\right)$,
   – *the evolution* $\overline{T}_\partial\colon\ [0,t^*] \to L^2\left(\Omega(t)\right)$ *is absolutely continuous and*

$$\frac{d}{dt}\overline{T}_\partial \in L^2\left(0,t^*;L^{\frac{6}{5}}\left(\Omega(t)\right)\right).$$

3. VARIATIONAL SOLUTION OF THE DIRICHLET PROBLEM

Definition 1. *We say that a continuous curve* $T\colon\ [0,t^\star] \to L^2\left(\Omega(t)\right)$ *is a solution of (2), if and only if* $\int_0^{t^\star} \|\nabla T(t)\|^2_{L^2(\Omega(t))}\, dt < \infty$, $T^{-1}\left(H^1\left(\Omega(t)\right)\right)$ *has the full measure in* $[0,t^\star]$ *and:*

1) $\forall!\, t \in [0,t^\star] : T(t)_{|\partial\Omega(t)} = T_\partial(t)$ *(in the sense of the trace theory);*
2) $\forall\, (t,\varphi) \in [0,t^*] \times H_0^1(\Omega(t))$:

$$\int\limits_{\Omega(t)} \varphi(t)\, \frac{\partial T}{\partial t}(t)\, dm = \widetilde{f}(t)\varphi - \int\limits_{\Omega(t)} \Theta_t\left(\nabla\varphi(t), \nabla T(t)\right)\, dm \tag{3}$$

where m *stands for the volume Lebesgue measure in* $\mathbb{R}^3$*, while "*$\forall!$*" denotes "for almost everywhere…" in the sense of the standard Lebesgue measure. Here,* $(\cdot,\cdot)$ *stands for the inner product in* $\mathbb{R}^3$*, and* $\|\cdot\|_{L^2\left(\Omega(t)\right)}$ *– is the usual norm in the Hilbert space* $L^2\left(\Omega(t),\mathbb{R}^3\right)$ *of all square summable vector fields on* $\Omega(t)$*. Finally,* $(\cdot,\cdot)_{L^2\left(\Omega(t)\right)}$ *denotes the scalar product in the Hilbert space* $L^2\left(\Omega(t)\right)$ *of square summable scalar functions.*

4. HOMOGENIZATION OF THE BOUNDARY PROBLEM

The temperature T in the equation (3) may be decomposed to the sum of two functions, i.e.

$$T(t) = \overline{T}_\partial(t) + T^\circ(t), \tag{4}$$

where $\overline{T}_\partial(t)$ is an arbitrary chosen extension of $T_\partial(t)$ satisfying the regularity e) of Assumptions 1 given above, i.e.

$$\overline{T}_\partial(t)\colon\ \overline{\Omega(t)} \to \mathbb{R}, \quad \underset{t}{\forall}\, \overline{T}_\partial(t)_{|\partial\Omega(t)} = T_\partial(t), \quad \overline{T}_\partial(t) \in H^1\left(\Omega(t)\right)$$

while

$$\underset{t}{\forall}\, T^{\circ}(t)_{|\partial\Omega(t)} \equiv 0, \quad \text{or more precisely} \quad T^{\circ}(t) \in H_0^1\big(\Omega(t)\big).$$

Using decomposition (4) we can rewrite the equation (3) as follows

$$\begin{aligned}\int\limits_{\Omega(t)} \varphi(t)\, \frac{\partial T^{\circ}}{\partial t}(t)\, dm = \widetilde{f}(t)\varphi - \int\limits_{\Omega(t)} \Theta_t\left(\nabla\varphi(t), \nabla T^{\circ}(t)\right)\, dm + \\ - \int\limits_{\Omega(t)} \Theta_t\left(\nabla\varphi(t), \nabla\overline{T}_{\partial}(t)\right)\, dm + \int\limits_{\Omega(t)} \varphi(t)\, \frac{\partial \overline{T}_{\partial}}{\partial t}(t)\, dm\end{aligned} \tag{5}$$

for a.e. $t \in [0, t^{\star}]$.

Theorem 1. *Under Assumptions 1 the problem* (5) *has the unique solution* $T^{\circ}(t) \in H_0^1\big(\Omega(t)\big)$ *for* $t \in [0, t^{\star}]$.

5. DISCRETIZATION $\Omega(t)$

First we triangulate $\Omega(0)$ in manner described in paper [2], or any other one, for instance the Delaunay algorithm. In a sequel, for every nodal point of our triangulation we construct the sets of its neighbour nodal points and simplexes.

By W we denote the set of all nodal points of our triangulation. By W_{∂} and W° we denote respectively the sets of boundary nodal points and inner nodal points of triangulation. Of course we have

$$W = W_{\partial} \cup W^{\circ}, \qquad W_{\partial} \cap W^{\circ} = \emptyset.$$

6. GALERKIN METHOD

Now we consider regular source $\tilde{f}(t)\varphi := \int_{\Omega(t)} f(t)\varphi\, dm$. We look for approximate solution of the problem (5) in the following form

$$T^{\circ}(t) = \sum_{p \in W^{\circ}} \lambda_p(t) \cdot \varphi_{p(t)}, \tag{6}$$

where $\varphi_{p(t)}$ is an unique function, which is affinity on every simplex of cubic triangulation, moreover in a nodal point $p(t)$ it has a value 1 and in other nodal points has a value 0.

Arbitrary chosen extension of $\overline{T}_{\partial}(t)$ is approximated by the sum

$$\overline{T}_{\partial}(t) \approx \sum_{p \in W_{\partial}} T_{\partial}(t, p(t)) \cdot \varphi_{p(t)}. \tag{7}$$

We denote

$$T(t,p) := \begin{cases} \lambda_p(t) & \text{for } p \in W^{\circ} \\ T_{\partial}(t, p(t)) & \text{for } p \in W_{\partial}. \end{cases} \tag{8}$$

Coefficient of a heat convection is approximated by the sum

$$\Theta_t \approx \sum_{p \in W} \Theta\big(T\big(t, p(t)\big)\big) \cdot \varphi_{p(t)}. \tag{9}$$

We approximate cubic heat sources by the following

$$f(t) \approx \sum_{p \in W} f(t, p(t)) \cdot \varphi_{p(t)}. \tag{10}$$

Homogeneous part of initial temperature is approximated by the formula

$$T_0^\circ \approx \sum_{p \in W^\circ} T_0^\circ(p(0)) \cdot \varphi_{p(0)}. \tag{11}$$

As a result we get the following system of differential equations indexed by the inner nodal points $w \in W^\circ$

$$\begin{aligned} &\sum_{p \in W^\circ} I(w,p)\dot{\lambda}_p(t) = \\ &\quad = -\sum_{p \in W_\partial} \left(\frac{d}{dt}\left(T_\partial(t,p(t))\right) I(w,p) + T_\partial(t,p(t)) I\left(w, \frac{p}{t}\right)\right) + \\ &\quad - \sum_{p \in W^\circ} \lambda_p(t) I\left(w, \frac{p}{t}\right) + \sum_{p \in W} f(t,p(t)) I(w,p) + \\ &\quad - \sum_{\substack{q \in W \\ p \in W^\circ}} \Theta\left(\overline{T}_\partial(t,q(t)) + \lambda_q(t)\right) \lambda_p(t) I\left(q, (\nabla w | \nabla p)\right) + \\ &\quad - \sum_{\substack{q \in W \\ p \in W_\partial}} \Theta\left(\overline{T}_\partial(t,q(t)) + \lambda_q(t)\right) T_\partial(p(t)) I\left(q, (\nabla w | \nabla p)\right), \end{aligned} \tag{12}$$

where:

$$\begin{aligned} I(w,p) &= \int_{\Omega(t)} \varphi_{w(t)} \varphi_{p(t)} dm, \\ I\left(w, \frac{p}{t}\right) &:= \int_{\Omega(t)} \varphi_{w(t)} \frac{d}{dt}\left(\varphi_{p(t)}\right) dm, \\ I\big(w, (\nabla w | \nabla p)\big) &:= \int_{\Omega(t)} \varphi_{w(t)} \left(\nabla \varphi_{q(t)} | \nabla \varphi_{p(t)}\right) dm, \end{aligned}$$

with the initial conditions

$$\lambda_w(0) = T_0^\circ(w(0)) \quad (w \in W^\circ), \tag{13}$$

In order to obtain (12) we use following formula

$$\frac{d}{dt}\int\limits_{\Omega(t)} \psi(t,x)\,dx = \int\limits_{\Omega(t)} \left(\frac{d\psi}{dt}(t,x) + \operatorname*{div}_x(\psi\cdot v)(t,x)\right)dx \tag{14}$$

which is consequence of Liouville's theorem. Let us notice, that for field $\psi\colon\ \psi_{|\partial\Omega} = 0$ second component of right side of (14) disappears.

From theorem concerning implicite functions one can lead out the following identity

$$\frac{d}{dt}\varphi_{w(t)}(x) = -\sum_{j=0}^{3} \frac{\det\left(q_1(t)-q_0(t), q_2(t)-q_0(t), v\left(t, q_j(t)\right)\right)}{\det\left(q_1(t)-q_0(t), q_2(t)-q_0(t), q_3(t)-q_0(t)\right)}\varphi_{q_j(t)}(x) \tag{15}$$

for $x \in S(t) := \operatorname{conv}\left(q_0(t), q_1(t), q_2(t), q_3(t)\right)$, where $q_j(t)$ $(j = 0,1,2,3)$ are adjoining nodes of triangulation $W(t)$ and $q_3(t) = w(t)$.

Theorem 2. *Let* $\overline{T}_\partial(t) \in H^1\big(\Omega(t)\big)$, $\big(\overline{T}_\partial(t)\big)_{|\partial\Omega(t)} = T_\partial(t)$. *Let* $(V_N(t))_{N=1}^{\infty}$ *be an approximation of Hilbert space* $H_0^1(\Omega(t))$. *Then:*

1) *for every* $N \in \mathbb{N}$ *the variational problem* (5) *has exactly one solution* $T_N^\circ(t) \in V_N(t)$,
2) $T_N^\circ(t) + \overline{T}_\partial(t) \overset{N\to\infty}{\longrightarrow} T(t)$ *in* $H^1(\Omega(t))$, *where* $T(t)$ *is the only solution of the problem* (3).

In our case we have

$$V_N(t) := \operatorname{span}\left\{\varphi_{p(t)}\colon\ p \in W^\circ\right\}, \qquad N := \#W^\circ.$$

7. SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

Let us introduce the following notations

$$\Lambda(t) := \left(\lambda_v(t)\right)_{v\in W^\circ}, \qquad \Lambda_i := \Lambda(t_i).$$

Here we solve the system of ordinary differential equations which takes form:

$$G(t)\dot{\Lambda}(t) = F\left(t, \Lambda(t)\right) \tag{16}$$

with initial condition determined by initial condition for equation (13), where $G(t)$ denotes a Gram matrix of spline functions vanishing on $\partial\Omega(t)$. The system (16) can be rewritten in the following equivalent form

$$\dot{\Lambda}(t) = G(t)^{-1}F(t,\Lambda(t)) =: \widetilde{F}(t,\Lambda(t)).$$

For consecutive time points t_i $(i = 0,1,\ldots)$ we consider consecutive values $\Lambda_i := \Lambda(t_i)$ $(i = 1,2,\ldots)$ of temperature in points of the set W°.

In order to solve this last equation, we can apply one of the multi-step methods e.g. Adams–Bashforth of third degree

$$\Lambda_{i+1} - \Lambda_i = \frac{h}{12}\left(23\widetilde{F}\left(t_i, \Lambda_i\right) - 16\widetilde{F}\left(t_{i-1}, \Lambda_{i-1}\right) + 5\widetilde{F}\left(t_{i-2}, \Lambda_{i-2}\right)\right).$$

From above we have

$$G\left(t_i\right) \cdot \left(\Lambda_{i+1} - \Lambda_i\right) = \frac{h}{12}\left(23F\left(t_i, \Lambda_i\right) - 16F\left(t_{i-1}, \Lambda_{i-1}\right) + 5F\left(t_{i-2}, \Lambda_{i-2}\right)\right). \quad (17)$$

Let us put R_i as a solution of the system (17) (linear equations), then we can write

$$\Lambda_{i+1} = \Lambda_i + R_i.$$

8. EXAMPLE OF VECTOR FIELD v

Now we consider situation, when domain $\Omega(t)$ is known for every t, and we have to construct vector field v induced by formula (1). We examine one special case of definition $\Omega(t)$ and method of construction of the field $v(t,x)$ in this case.

Let us consider an arbitrary domain $\Xi \subset \mathbb{R}^3$ with regular boundary such that

$$\forall \zeta \in \partial\Xi \colon\ n(\zeta) \nparallel e_3$$

and

$$\mathbb{R}\, e_3 \subset \Xi,$$

where $n(\zeta)$ denotes normal vector to $\partial\Xi$ in the point ζ and e_3 denotes axis versor. There are given functions $h_{\min}$, $h_{\max}$, such that $0 \le h_{\min}(t) < h_{\max}(t)$.

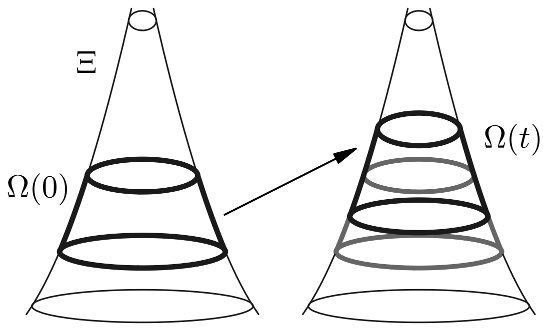

Fig. 1. Example of domain $\Omega(t)$ defined by formula (18)

We define (see Fig. 1):

$$\Omega(t) := \left\{x \in \Xi \colon\ h_{\min}(t) < x_3 < h_{\max}(t)\right\}. \quad (18)$$

If Ξ is rotational set with rotate axis $\mathbb{R}\, e_3$ we define the field v of drift by the following formula

$$v(t,x) = \frac{(h_{\max}(t) - x_3)\, \dot{h}_{\min}(t) + (x_3 - h_{\min}(t))\, \dot{h}_{\max}(t)}{h_{\max}(t) - h_{\min}(t)} u(x). \tag{19}$$

Horizontal part u of the field v we construct as below.

Let

$$R\colon\ \mathbb{R} \ni h \to R(h) \in \mathbb{R}_+$$

be a given function. $R(h)$ stand for radius of the rotational solid Ξ on the „height” h. We consider the function

$$\psi(h,r) := r - R(h). \tag{20}$$

It is easy to see that $\psi(h,r) = 0$ for $r = R(h)$ and

$$\nabla_{(h,r)}\psi = \big(-\dot{R}(h), 1\big), \qquad \big|\nabla_{(h,r)}\psi\big| = \sqrt{\dot{R}(h)^2 + 1}, \tag{21}$$

where $|\cdot|$ denote euclidean norm in $\mathbb{R}^3$, so the normal takes form

$$n\big(h, R(h)\big) = \frac{\nabla_{(h,r)}\psi}{\big|\nabla_{(h,r)}\psi\big|} = \frac{\big(-\dot{R}(h), 1\big)}{\sqrt{\dot{R}(h)^2 + 1}}. \tag{22}$$

Let us assume $h = (x \mid e_3)$, $x = x' + he_3$, where $(\cdot|\cdot)$ stands for the inner product in $\mathbb{R}^3$. In this situation for point x lying on cone's side surface:

$$n(x) = \frac{1}{\sqrt{\dot{R}(h)^2 + 1}} \left(-\dot{R}(h)e_3 + \frac{x'}{|x'|} \right). \tag{23}$$

Example 1. *Cylinder: $R(h) = const$. In this case $n(x) = \frac{x'}{|x'|}$.*

Example 2. *Cone: $R(h) = c \cdot h$. In this case $n(x) = \frac{1}{\sqrt{c^2+1}} \left(-ce_3 + \frac{x'}{|x'|} \right)$.*

We assume, that nodal point $w = w(0)$ on the lavel 0 is moving up together with net of triangulation and on the high h takes position $x = w(h)$.

From proportion

$$\frac{|x'|}{R(h)} = \frac{|w|}{R(0)}$$

we have

$$|x'| = \frac{R(h)}{R(0)}\, |w|\,, \qquad x' = \frac{R(h)}{R(0)} w.$$

So (cf. Fig. 2)

$$w(h) = x = x' + he_3 = \frac{R(h)}{R(0)} w + he_3 \tag{24}$$

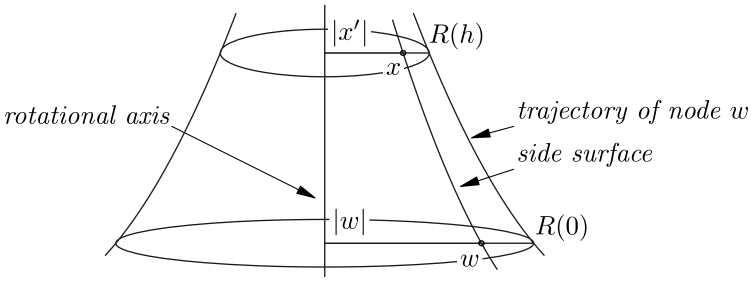

Fig. 2. Illustration to the formula (24)

and in consequence

$$\dot{w}(h) = \frac{\dot{R}(h)}{R(0)} w + e_3 =: u(w(h)),$$

or, what is the same,

$$u(x) = u(w(h)) = \frac{\dot{R}(h)}{R(0)} w + e_3. \tag{25}$$

Since $h = (x \mid e_3)$ and

$$w = \frac{x - he_3}{R(h)} R(0) = \frac{x - (x \mid e_3)\, e_3}{R((x \mid e_3))} R(0)$$

it follows finally that

$$u(x) = \frac{\dot{R}}{R}\big((x \mid e_3)\big)\big(x - (x \mid e_3)\, e_3\big) + e_3. \tag{26}$$

We can rewrite this formula in the form

$$u\,(x_1, x_2, x_3) = \left(\frac{\dot{R}\,(x_3)}{R\,(x_3)} x_1, \frac{\dot{R}\,(x_3)}{R\,(x_3)} x_2, 1\right). \tag{27}$$

Assume that $|x'| = R(h)$, which means that the point x lies on side surface. Then

$$\sqrt{\dot{R}(h)^2 + 1}\Big(n(x)|u(x)\Big) = \left(-\dot{R}(h)e_3 + \frac{x'}{R(h)}|\frac{\dot{R}}{R}(h)\big(x - he_3\big) + e_3\right) =$$

$$= \left(\frac{x'}{R(h)} - \dot{R}(h)e_3|\frac{\dot{R}(h)}{R(h)}x' + e_3\right) = \frac{|x'|^2\,\dot{R}(h)}{R^2(h)} - \dot{R}(h) =$$

$$= \frac{R^2(h)\dot{R}(h)}{R^2(h)} - \dot{R}(h) = 0,$$

so, for $x = x' + he_3$ such that $|x'| = R(h)$ we have

$$\big(n(x)|u(x)\big) = 0,$$

what means, that the field u is tangent to side surface.

On the other hand, if $|x'| = 0$ e.g. $x_1 = x_2 = 0$, then by (26) (or (27)) we have $u(x) = u(0, 0, x_3) = (0, 0, 1)$ what means, that the vector field v is parallel to versor e_3 for all points x from rotational axis.

In the situation in which Ξ is not rotational, but it can be approximated by a rotational set, we can approximate the field u by the formula

$$u(x) = \left(1 - \frac{|x'|}{R(h)}\right) e_3 + \frac{|x'|}{R(h)} \cdot \frac{s}{(s|e_3)}, \tag{28}$$

for $x = x' + he_3$ and $|x'| > 0$, where vector $s \neq 0$ tangent to $\partial\Xi$ lies in the plane which is determine by the rotational axis and the point x. For points x such that $|x'| = 0$ we define $u(x) = 0$.

9. NUMERICAL REALIZATION OF PRESENTED METHOD

Computer program which realizes algorithm (12) in two-dimensional case e.g. $n = 2$ and for $\Omega(t)$ defined by (18) was written in C. Source code and binaries of this program are available on `http://wms.mat.agh.edu.pl/~bozek`. Please consult the `Read.me` file in this distribution for instructions how to use this program. To generate the triangulation of domain $\Omega(0)$ we made use of source code of program EasyMesh v.1.4 written by Boyan Niceno, taken from `http://www-dinma.univ.trieste.it/~nirftc/research/easymesh/`. To solve algebraic linear systems of equations the function `linbcg` from Numerical Recipes library has been used.

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