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Stanisław Białas

A NECESSARY AND SUFFICIENT CONDITION FOR σ-HURWITZ STABILITY OF THE CONVEX COMBINATION OF THE POLYNOMIALS

Abstract. In the paper are given a necessary and sufficient condition for σ-Hurwitz stability of the convex combination of the polynomials.

Keywords: Convex sets of polynomials, stability of polynomial, Hurwitz stability, σ--stability.

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1. INTRODUCTION

We will consider the set of real polynomials

$$F(x,Q) = \left\{ a_n(q)x^n + a_{n-1}(q)x^{n-1} + \cdots + a_1(q)x + a_0(q) \right\},$$

where $q = (q_1, q_2, \ldots, q_k) \in Q \subset R^k$, Q is a compact set, $a_i(q)\colon Q \to R$ $(i = 0, 1, \ldots, n)$, $a_n(q) \neq 0$ for each $q \in Q$.

Let $\sigma \in R$ and $\sigma > 0$.

Definition 1. *We shall say that the real polynomial*

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = a_n(x - x_1)(x - x_2)\cdots(x - x_n) \qquad (1)$$

where $a_n \neq 0$, is Hurwitz stable if $Re(x_i) < 0$ $(i = 1, 2, \ldots, n)$. The polynomial (1) *is called σ-Hurwitz stable if $Re(x_i) < -\sigma$ $(i = 1, 2, \ldots, n)$.*

Definition 2. *The set of the polynomials $F(x,Q)$ is called σ-Hurwitz stable if each polynomial $g(x) \in F(x,Q)$ is σ-Hurwitz stable.*

Consider the interval polynomial

$$G(x) = [\underline{a}_n, \overline{a}_n]x^n + [\underline{a}_{n-1}, \overline{a}_{n-1}]x^{n-1} + \cdots + [\underline{a}_1, \overline{a}_1]x + [\underline{a}_0, \overline{a}_0],$$

and the set of the polynomials

$$W(x) = \{f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \colon\ a_i \in \{\underline{a}_i, \overline{a}_i\}\ (i = 0, 1, \ldots, n)\}.$$

The following theorem is true

Theorem 1 (Bhattacharyya, Chapellat, Keel [2]). *The interval real polynomial*

$$G(x) = [\underline{a}_n, \overline{a}_n]x^n + [\underline{a}_{n-1}, \overline{a}_{n-1}]x^{n-1} + \ldots + [\underline{a}_1, \overline{a}_1]x + [\underline{a}_0, \overline{a}_0],$$

where $0 \notin [\underline{a}_n, \overline{a}_n]$*, is* σ*-Hurwitz stable if and only if the set of the polynomials* $W(x)$ *is* σ*-Hurwitz stable.*

Let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = a_n(x - x_1)(x - x_2)\cdots(x - x_n),$$

where $a_n \neq 0$.

Denote by $H(f)$ the Hurwitz matrix for the polynomial $f(x)$, i.e.

$$H(f) = \begin{bmatrix} a_{n-1} & a_n & 0 & 0 & 0 & \ldots & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} & a_n & 0 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & \ldots & 0 & a_0 \end{bmatrix}.$$

It is easy to see that $H(f) \in R^{n\times n}$.

Consider the real polynomials

$$f_j(x) = a_n^{(j)}x^n + a_{n-1}^{(j)}x^{n-1} + \cdots + a_1^{(j)}x + a_0^{(j)} \tag{2}$$

for $j = 1, 2, \ldots, m$, where $a_n^{(j)} \neq 0$ $(j = 1, 2, \ldots, m)$, and the convex combinations of these polynomials

$$C(f_1, f_2, \ldots, f_m) = \{\alpha_1 f_1(x) + \alpha_2 f_2(x) + \cdots + \alpha_m f_m(x)\colon \\ \alpha_j \geq 0 \quad (j = 1, 2, \ldots, m),\ \alpha_1 + \alpha_2 + \cdots + \alpha_m = 1\}.$$

In this paper we give the necessary and sufficient condition for σ-Hurwitz stability of the convex combination $C(f_1, f_2, \ldots, f_m)$.

We assume that the polynomials (2) are Hurwitz stable. Hence, follows that there exists the inverse matrix $H^{-1}(f_j)$ $(j = 1, 2, \ldots, m)$.

Let

$$\lambda_k\left(H^{-1}(f_j)H(f_i)\right) \quad (k = 1, 2, \ldots, n;\ i, j = 1, 2, \ldots, m;\ j < i)$$

denote the eigenvalues of the matrix $H^{-1}(f_j)H(f_i)$.

The following theorems are true:

Theorem 2 (Białas [3]). *If the real polynomials*

$$f_1(x) = a_n^{(1)} x^n + a_{n-1}^{(1)} x^{n-1} + \cdots + a_1^{(1)} x + a_0^{(1)},$$
$$f_2(x) = a_n^{(2)} x^n + a_{n-1}^{(2)} x^{n-1} + \cdots + a_1^{(2)} x + a_0^{(2)},$$

where $a_n^{(1} \neq 0$, $a_n^{(2)} \neq 0$, *are Hurwitz stable, then the convex combination*

$$C(f_1, f_2) = \{\alpha_1 f_1(x) + \alpha_2 f_2(x) \colon \alpha_1 \geq 0,\ \alpha_2 \geq 0,\ \alpha_1 + \alpha_2 = 1\}$$

is Hurwitz stable if and only if

$$\lambda_k(H^{-1}(f_1)H(f_2)) \notin (-\infty, 0\rangle \quad (k = 1, 2, \ldots, n).$$

Theorem 3 (Bartlett, Hollot, Huang [1]). *If the polynomials* (2) *are Hurwitz stable, then the convex combination*

$$C(f_1, f_2, \ldots, f_m) = \{\alpha_1 f_1(x) + \alpha_2 f_2(x) + \ldots + \alpha_m f_m(x) \colon \\ \alpha_j \geq 0 \quad (j = 1, 2, \ldots, m),\ \alpha_1 + \alpha_2 + \cdots + \alpha_m = 1\} \tag{3}$$

is Hurwitz stable if and only if the convex combinations $C(f_i, f_j)$ *are Hurwitz stable for each* $i, j = 1, 2, \ldots, m$; $i < j$.

2. MAIN RESULT

Let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_n \neq 0$.

It is easy to note that for $\alpha \in R$ we have

$$g(s) = f(s + \alpha) = b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0,$$

where

$$b_0 = f(\alpha),$$
$$b_i = \frac{1}{i!} \frac{d^i f(x)}{dx^i} \Big|_{x=\alpha} \qquad (i = 1, 2, \ldots, n).$$

As it is easy to see, we have the following result.

Lemma 1. *The real polynomial*

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_n \neq 0$*, is* σ*-Hurwitz stable if and only if the polynomial*

$$g(s) = f(s - \sigma)$$

is Hurwitz stable.

Now, we will prove

Theorem 4. *If the polynomials* (2) *are* σ*-Hurwitz stable, then the convex combination*

$$C(f_1, f_2, \ldots, f_m) = \{\alpha_1 f_1(x) + \alpha_2 f_2(x) + \cdots + \alpha_m f_m(x) : \alpha_j \geq 0 \quad (j = 1, 2, \ldots, m),\ \alpha_1 + \alpha_2 + \cdots + \alpha_m = 1\}$$

is σ*-Hurwitz stable if and only if*

$$\lambda_k(H^{-1}(g_i)H(g_j)) \notin (-\infty, 0\rangle \quad (k = 1, 2, \ldots, n) \tag{4}$$

for $i, j = 1, 2, \ldots, m;\ \ i < j$*, where* $g_i(s) = f_i(s - \delta), g_j(s) = f_j(s - \delta)$.

Proof. From Lemma1, it follows that the convex combination $C(f_i, f_j)$ is σ-Hurwitz stable if and only if the convex combination $C(g_i, g_j)$ is Hurwitz stable.

However, from Theorem 2 and 3 follows that the set $C(g_i, g_j)$ is Hurwitz stable if and only if the conditions (4) holds. This completes the proof of Theorem 4. □

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Stanisław Białas
sbialas@uci.agh.edu.pl

AGH University of Science and Technology
Faculty of Applied Mathematics
al. Mickiewicza 30, 30-059 Cracow, Poland

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