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ON SOME APPLICATION OF BIORTHOGONAL SPLINE SYSTEMS TO INTEGRAL EQUATIONS

Abstract. We consider an operator $P_N\colon L_p(I) \to S_n(\Delta_N)$, such that $P_N f = f$ for $f \in S_n(\Delta_N)$, where $S_n(\Delta_N)$ is the space of splines of degree n with repect to a given partition Δ_N of the interval I. This operator is defined by means of a system of step functions biorthogonal to B-splines. Then we use this operator to approximation to the solution of the Fredholm integral equation of the second kind. Convergence rates for the aproximation of the solution of this equation are given.

Keywords: operator associated with step functions, B-splines, integral equation, approximation.

Mathematics Subject Classification: 41A15, 45B05, 45L10, 65R20.

1. INTRODUCTION

The purpose of the paper is to give some application of biorthogonal spline systems defined earlier by the author in [15] to the Fredholm integral equation of the second kind.

Let Δ be a given partition of the interval $I = [a, b]$ and let $\{N_j\}$ be a system of normalized B-splines of degree n with respect to Δ. We constructed a system of step functions $\{\lambda_j\}$ biorthogonal to the system $\{N_j\}$ such that $\operatorname{supp} \lambda_j \subset \operatorname{supp} N_j$ in [15]. Then we defined the following operator: $P_N\colon L_p(I) \to S_n(\Delta_N)$, such that $P_N f = f$ for $f \in S_n(\Delta_N)$, where $S_n(\Delta_N)$ is the space of splines of degree n with repect to a given partition Δ_N of the interval I.

$$P_{N,f}(x) = \sum_j (f, \lambda_j) N_j(x), \quad f \in L^2(I), \tag{1}$$

where $(f,g)=\int\limits_a^b f(t)\,\overline{g(t)}\,dt$ and we estimated the difference $f-P_f$ with respect to the modulus of smoothness of the function f in the space $L^p(I)$, $1\le p\le\infty$.

Consider the Fredholm integral equation of the second kind

$$y(x)=f(x)+\lambda\int\limits_a^b K(x,t)\,y(t)\,dt, \tag{2}$$

where $f\in C(I)$, $K\in C(I^2)$ and $\lambda\in\mathbb{R}$.

We may find basic facts on integral equations and main methods for finding the solutions of them in [1, 2, 9, 11]. A method of application of interpolating splines is given in [12]. Our method of approximation of the solution of the equation (2) is based on three methods of finding the solutions of integral equations: a change the kernel K by the degenerated kernel P_K, the method of the Bubnov–Galerkin and the method of iteration (cf. [1, 2, 9]).

We assume that

$$\lambda\max_{x\in I}\int\limits_a^b |K(x,t)|\,dt=\varrho<1.$$

Then we approximate the function f by the operators of the form (1) and the kernel K by the operators of the form

$$P_{N,K}(x,t)=\sum_{i,j}(K,\lambda_i\lambda_j)\,N_i(x)\,N_j(t),$$

where $(K,\lambda_i\,\lambda_j)=\int\limits_{I^2} K(x,t)\,\lambda_i(x)\,\lambda_j(t)\,dx\,dt$. For any $\varepsilon>0$ we can find an operator $P_{N,K}$ (see [15] and also [13, 14]) such that

$$|P_{N,K}(x,t)-P(x,t)|<\varepsilon\quad\text{for}\quad(x,t)\in I^2$$

and

$$\lambda\max_{x\in I}\int\limits_a^b |P_{N,K}(x,t)|\,dt<1.$$

Then we solve the following integral equation with the degenerate kernel $P_{N,K}$:

$$y(x)=P_{N,f}+\lambda\int\limits_a^b P_{N,K}(x,t)\,y(t)\,dt.$$

The solution of this equation is a spline. We find it using the method of iteration and we give the recurrence formula for it in the case of equidistant partitions.

At the end of the paper we consider the order of approximation of the solution of the equation (2) in the space $W_p^n(I)$ for $1\le p\le\infty$.

It seems that the simplicity and good properties of approximation of the algorithm may have some applications.

2. A SYSTEM OF STEP FUNCTIONS BIORTHOGONAL TO B-SPLINES AND APPROXIMATION BY SPLINES

For the simplicity we confine to the equidistant partitions of the interval $I = [a, b]$. Let

$$\Delta_N = \{a = t_{-n} = \ldots = t_0 < t_1 < \ldots < t_N = \ldots = t_{N+n} = b\}\,, \tag{3}$$

where $t_j = a + j\,h_N$, $h_N = \frac{b-a}{N}$, $j = 1, \ldots, N$. Setting $x_+^k := (\max\{0, x\})^k$, the B-spline of degree n with respect to Δ_N is defined as follows: (see [8] or [4, 5, 6, 7])

$$M_{i,n}(s) = M_{i,n}(x_i, \ldots, x_{i+n+1}; s) = [x_i, \ldots, x_{i+n+1} : (x-s)_+^n],$$

where $[x_i, \ldots, x_{i+n+1} : f]$ is the $(n+1)^{th}$ order divided difference of f at $x_i, \ldots, x_{i+n+1}$. The normalize B-spline $N_{i,n}$ is defined as follows

$$N_{i,n}(x) = \frac{x_{i+n+1} - x_i}{n+1} M_{i,n}(x).$$

Further we need the following properties of B-splines:

$$\begin{aligned} &\operatorname{supp} M_{i,n+1} = \operatorname{supp} N_{i,n+1} = [x_i, x_{i+n+1}],\ N_{i,n+1} \in C^{n-1}(I), \\ &\int\limits_I M_{i,n+1}(x)\,dx = 1, \\ &\sum_{i=-n}^{N-1} N_{i,n+1}(x) = 1 \quad \text{for} \quad x \in I, \\ &N_{i,n+1}(x) \geq 0 \quad \text{for} \quad x \in I. \end{aligned}$$

Theorem 2.1 (cf. [4, 7]). *Let* $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda_i \in L_q(I) = L_p^*(I)$, $i = -n, \ldots, N-1$. *Then for any integer* $j = -n, \ldots, N-1$ $\lambda_i(N_{j,n+1}) = \delta_{i,j}$ *if and only if* $\lambda_i = D^{n+1} f$ *for some* f *such that* $f|_{\Delta_N} = \psi_{i,n+1}^+|_{\Delta_N}$, *where*

$$\psi_{i,n+1}^+(x) = (x - t_{i+1})_+ \cdot (x - t_{i+2}) \cdot \ldots \cdot (x - t_{i+n})/n!$$

Using this theorem we construct a system of step functions $\{\lambda_i\}_{i=-n}^{N-1}$ biorthonormal to the system of B-splines $\{N_{j,n}\}_{j=-n}^{N-1}$ as in [15] such that $\operatorname{supp} \lambda_i \subset [t_{i+k}, t_{i+k+1}]$ for $i = -k, -k+1, \ldots, N-k-1$ with $n = 2k$ or $n = 2k+1$ and for $i = -n, \ldots, -k-1$ or $i = N-k, \ldots, N-1$ $\operatorname{supp} \lambda_i \subset [t_0, t_1]$ or $\operatorname{supp} \lambda_i \subset [t_{N-1}, t_N]$ respectively.

Let $\operatorname{supp} \lambda_i \subset [t_m, t_{m+1}]$ and $\lambda_i = \sum_{j=0}^n A_{i,j}\,\chi_{[\tau_j, \tau_{j+1})}(x)$, where $\tau_j = t_m + jh$ for $j = 0, \ldots, n$, $h = (t_{m+1} - t_m)/(n+1)$ and $\chi_{[\tau_j, \tau_{j+1})}$ is the characteristic function of the interval $[\tau_j, \tau_{j+1})$. To obtain the function λ_i it suffices to solve the following Cramer system of $n+1$ equations with $n+1$ unknowns $A_{i,j}$

$$\int\limits_{t_m}^{t_{m+1}} \lambda_i(x)\, N_{j,n}(x)\,dx = \delta_{i,j}, \quad j = m-n, \ldots, m.$$

Example 2.1. *Let* $\Delta_N = \{t_i\}_{i=-1}^{N+1}$, $t_{-1} = 0$, $t_j = j$ *for* $j = 0, \ldots, N$ *and* $t_{N+1} = N$. *Then for* $n = 1$

$$N_{i,1}(x) = \begin{cases} x - i & \text{for } i < x \leq i+1, \\ i + 2 - x & \text{for } i+1 < x \leq i+2,\ i = 0, \ldots, N-2, \\ 0 & \text{otherwise,} \end{cases}$$

$$N_{-1,1}(x) = \begin{cases} 1 - x & \text{for } 0 < x \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$N_{N-1,1}(x) = \begin{cases} x - N + 1 & \text{for } N-1 < x \leq N, \\ 0 & \text{otherwise,} \end{cases}$$

$$\lambda_i(x) = \begin{cases} -1 & \text{for } i < x \leq i + \frac{1}{2},\ i = 0, ..., N-1, \\ 3 & \text{for } i + \frac{1}{2} < x \leq i+1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\lambda_{-1}(x) = \begin{cases} 3 & \text{for } 0 < x \leq \frac{1}{2}, \\ -1 & \text{for } \frac{1}{2} < x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

For the partition (3) *we have*

$$N_{-1,h}(x) = N_{-1,1}\left(\frac{x - t_0}{h}\right),\ N_{i,h}(x) = N_{0,1}\left(\frac{x - t_i}{h}\right),\ i = 0, \ldots, N-2,\ h = \frac{b-a}{N},$$

$$N_{N-1,h}(x) = \begin{cases} N_{0,1}\left(\frac{x - t_{N-1}}{h}\right) & \text{for } t_{N-1} < x \leq t_N, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\lambda_{-1,h}(x) = h^{-1}\lambda_{-1}\left(\frac{x - t_0}{h}\right),\ \lambda_{i,h}(x) = h^{-1}\lambda_0\left(\frac{x - t_i}{h}\right),\ i = 0, \ldots, N-2,$$

$$\lambda_{N-1,h}(x) = \begin{cases} h^{-1}\lambda_0\left(\frac{x - t_0}{h}\right) & \text{for } t_{N-1} < x \leq t_N, \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.2. *Let* $\Delta_N = \{t_i\}_{i=-1}^{N+1}$, $t_{-2} = t_{-1} = 0$, $t_j = j$ *for* $j = 0, \ldots, N$ *and* $t_{N+1} = t_{N+2} = N$. *Then for* $n = 2$

$$N_{i,2}(x) = \begin{cases} \frac{(x-i)^2}{2} & \text{for } i < x \leq i+1, \\ -(x-i)^2 + 3(x-i) - \frac{3}{2} & \text{for } i+1 < x \leq i+3,\ i = 0, \ldots, N-3, \\ 0 & \text{otherwise,} \end{cases}$$

$$N_{-1,2}(x) = \begin{cases} 2x - \frac{3}{2}x^2 & \text{for } 0 < x \leq 1, \\ \frac{(2-x)^2}{2} & \text{for } 1 < x \leq 2, \\ 0 & \text{otherwise,} \end{cases}$$

$$N_{-2,2}(x) = \begin{cases} (x-1)^2 & \textit{for } 0 < x \le 1, \\ 0 & \textit{otherwise,} \end{cases}$$

$$N_{N-2,2}(x) = N_{-1,2}(N-x) = \begin{cases} \frac{1}{2}(x-N)(3N-4-x) & \textit{for } N-2 < x \le N-1, \\ \frac{(x-N-2)^2}{2} & \textit{for } N-1 < x \le N, \\ 0 & \textit{otherwise,} \end{cases}$$

$$N_{N-1,2}(x) = N_{-2,2}(N-x) = \begin{cases} (x-N-1)^2 & \textit{for } N-1 < x \le N, \\ 0 & \textit{otherwise,} \end{cases}$$

$$\lambda_i(x) = \begin{cases} -\frac{7}{2} & \textit{for } i+1 < x \le i+\frac{4}{3}, \\ 10 & \textit{for } i+\frac{4}{3} \le i+\frac{5}{3},\ i=-1,0,\dots,N-2, \\ -\frac{7}{2} & \textit{for } i+\frac{5}{3} < x \le i+2, \\ 0 & \textit{otherwise,} \end{cases}$$

$$\lambda_{-2}(x) = \begin{cases} \frac{11}{2} & \textit{for } 0 < x \le \frac{1}{3}, \\ -\frac{7}{2} & \textit{for } \frac{1}{3} < x \le \frac{2}{3}, \\ 1 & \textit{for } \frac{2}{3} < x \le 1, \\ 0 & \textit{otherwise,} \end{cases}$$

$$\lambda_{N-1}(x) = \lambda_{-2}(N-x) = \begin{cases} 1 & \textit{for } N-1 < x \le N-\frac{2}{3}, \\ -\frac{7}{2} & \textit{for } N-\frac{2}{3} < x \le N-\frac{1}{3}, \\ \frac{11}{2} & \textit{for } N-\frac{1}{3} < x \le N, \\ 0 & \textit{otherwise.} \end{cases}$$

For the partition (3) *we have*

$$N_{-2,h}(x) = N_{-2,2}\left(\frac{x-t_0}{h}\right),\ N_{i,h}(x) = N_{0,2}\left(\frac{x-t_i}{h}\right),$$
$$i=-1,0,\dots,N-2,\ h = \frac{b-a}{N},$$
$$N_{N-2,h}(x) = N_{-1,2}\left(\frac{t_N-x}{h}\right),\ N_{N-1,h}(x) = N_{-2,2}\left(\frac{t_N-x}{h}\right)$$

and

$$\lambda_{i,h}(x) = h^{-1}\lambda_0\left(\frac{x-t_i}{h}\right),\ i=-1,0,\dots,N-2,$$
$$\lambda_{-2,h}(x) = h^{-1}\lambda_{-2}\left(\frac{x-t_0}{h}\right),\ \lambda_{N-1,h}(x) = h^{-1}\lambda_{-2}\left(\frac{t_N-x}{h}\right).$$

We may write the operator (1) in the form

$$P_N(x) = P_{N,f}(x) = \sum_{j=-n}^{N-1} (\lambda_j, f)\, N_{j,n}(x) \tag{4}$$

and for the norm of the operator $P_N\colon L_p(I) \longrightarrow L_p(I)$, for $1 \le p \le \infty$ we have the estimate

$$\|P_N\|_{L_p(I)} \le \sum_{i=-n}^{N-1} \int_a^b |\lambda_{i,n}(t)|\, dt,$$

where $\lambda_{i,n}$ is defined for the partition (3) (see [15]).

Further we need the following

Theorem 2.2 (see [15] and also [6, 7, 13, 14])**.** *There exist constants $C_{k,n,p}$ depending only on k, n and p such that for $1 \le p \le \infty$, $f \in W_p^r(I)$, $0 \le k \le r \le n+1$, $k \le n$*

$$\left\| f^{(k)} - P_{N,f}^{(k)} \right\|_{L_p(I)} \le C_{k,n,p}\, h_N^{r-k}\, \omega_{n+1-r}^{(p)}(f^{(r)}, h_N) \text{ for } k = 0,\dots,r,\ r = 0,\dots,n+1,$$

where $h_N = \frac{b-a}{N}$ and

$$\omega_{n+1}^{(p)}(f,\delta) = \sup_{0<h\le\delta} \left\| \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} f(x+ih) \right\|_{L_p([a,b-nh])}, \quad 1 \le p \le \infty$$

is the $(n+1)^{th}$ modulus of smoothness of the function f in the space $L_p(I)$.

3. NUMERICAL SOLUTION OF THE FREDHOLM INTEGRAL EQUATION OF THE SECOND KIND

Let Δ_N be the partition of the interval $I = [a,b]$ defined by (3). Consider the following integral equation:

$$y(x) = P_{N,f}(x) + \lambda \int_a^b P_{N,K}(x,t)\, y(t)\, dt, \tag{5}$$

where $P_{N,f}$ is defined by (4) and

$$P_{N,K}(x,t) = \sum_{i=-n}^{N-1} \sum_{j=-n}^{N-1} \left(\int_{I^2} K(\xi,\tau)\, \lambda_{N,i}(\xi)\, \lambda_{N,j}(\tau)\, d\xi d\tau \right) N_{i,n}(x)\, N_{j,n}(t).$$

Now $\lambda_{N,i}$ is defined as follows: Let $\Delta'_N = \{x_i\}_{i=-n}^{N+n}$, $x_{-n} = \ldots = x_{-1} = 0$, $x_k = k$, $k = 0,\dots,N$, $x_{N+1} = \ldots = x_{N+n} = N$.

$$\lambda_{N,i}(x) = \begin{cases} \dfrac{1}{h}\, \lambda_0\left(\frac{x-x_i}{h}\right) & \text{for } i = 0,\dots,N-n, \\ \dfrac{1}{h}\, \lambda_i\left(\frac{x-x_0}{h}\right) & \text{for } i = -n,\dots,-1, \\ \dfrac{1}{h}\, \lambda_{i-N}\left(\frac{X_N-x}{h}\right) & \text{for } i = N-n+1,\dots,N-1, \end{cases}$$

where λ_i is defined for the partition Δ'_N by means of Theorem 1 (see [15]).

Let for every $t \in I$ $K(\cdot,t) \in W_p^r(I)$ and for every $x \in I$ $K(x,\cdot) \in W_p^r(I)$, where $K(\cdot,t)$ denotes the function $K(x,t)$ of x at fixed t. Since $P_{N,K}(x,t) = P_{N,P_{N,K(\cdot,t)}}(x)$, then by Theorem 2 we obtain

$$\begin{aligned}
&\|K(x,t) - P_{N,K}(x,t)\|_{L_p(I)} \le \\
&\quad \le \left\|K(x,t) - P_{N,K(\cdot,t)}(x)\right\|_{L_p(I)} + \left\|P_{N,K(\cdot,t)}(x) - P_{N,K}(x,t)\right\|_{L_p(I)} \le \\
&\quad \le C_{k,n,p}\, h_N^k\, \tilde{\omega}_{n+1-k}^{(p)}\left(\frac{\partial^k}{\partial x^k} K, h_N\right) + C_{k,n,p}\, h_N^k\, \tilde{\omega}_{n+1-k}^{(p)}\left(\frac{\partial^k}{\partial t^k} K, h_N\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad k = 0, \dots, r,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\omega}_m^{(p)}\left(\frac{\partial^k}{\partial x^k} K, h\right) &= \sup_{t\in I} \omega_m^{(p)}\left(\frac{\partial^k}{\partial x^k} K(\cdot,t), h\right), \\
\tilde{\omega}_m^{(p)}\left(\frac{\partial^k}{\partial t^k} K, h\right) &= \sup_{x\in I} \omega_m^{(p)}\left(\frac{\partial^k}{\partial t^k} K(x,\cdot), h\right).
\end{aligned}$$

Let

$$\begin{aligned}
&\varrho = \lambda \sup_{x\in I} \int_a^b |K(x,t)|\, dt < 1 \quad \text{for} \quad p = \infty, \\
&\varrho = \lambda \left[\int_a^b \left(\int_a^b |K(x,t)|^p\, dx\right)^{\frac{q}{p}} dt\right]^{\frac{1}{q}} \quad \text{for} \quad 1 < p < \infty, \qquad (6) \\
&\varrho = \lambda \sup_{t\in I} \int_a^b |K(x,t)|\, dx < 1 \quad \text{for} \quad p = 1
\end{aligned}$$

and ϱ_N denotes the above quantities for the kernel $K_N(x,t)$.

Hence there exists N_0 such that for $N > N_0$

$$\varrho_N < \varrho_0 = \frac{1+\varrho}{2} < 1. \qquad (7)$$

The kernel $P_{N,K}$ is degenerated and because of (7) the solution of the integral equation (5) is a spline of degree n with respect to the partition Δ_N. Denote it by s_N.

We have the following

Theorem 3.1. *Let for every $t \in I$, $K(\cdot.t) \in W_p^r(I)$ and for every $x \in I$, $K(x,\cdot) \in W_p^r(I)$ and ϱ satisfies (6). Then there exist constants $A_{k,n,p}$ and $B_{k,n,p}$ depending only on k, n and p such that*

$$\left\|s_N^{(k)} - y^{(k)}\right\|_{L_p(I)} \le$$

$$\leq A_{k,n,p}\, h_N^{r-k}\, \omega_{n+1-r}^{(p)}\,(y^{(r)}, h_N)+$$
$$+\, B_{k,n,p}\, \|P_N\|_{C(I)} \|y\|_{L^p(I)}\, h_N^{r-k}\, \tilde{\omega}_{n+1-r}^{(q)} \left(\frac{\partial^r}{\partial\, t^r}\, K, h_N \right),$$

where y *and* s_N *are the solutions of the equations* (2) *and* (5) *respectively and* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let $p = \infty$ and $N > N_0$. Applying the operator P_N to the solution y of the equation (2) we obtain

$$P_{N,y}(x) = P_{N,f}(x) + \lambda \int_a^b P_{N,K(\cdot,t)}(x)\, y(t)\, dt.$$

Hence

$$s_N(x) - P_{N,y}(x) = \lambda \int_a^b P_{N,K}(x,t)\, [s_N(t) - y(t)]\, dt +$$
$$+\, \lambda \int_a^b [P_{N,K}(x,t) - P_{N,K(\cdot,t)}(x)]\, y(t)\, dt$$

and by (7)

$$\|s_N - y\|_{C(I)} \leq \|s_N - P_{N,y}\|_{C(I)} + \|P_{N,y} - y\|_{C(I)} \leq$$
$$\leq \varrho_0\, \|s_N - y\|_{C(I)} + \lambda \int_a^b \left|P_{N,K}(x,t) - P_{N,K(\cdot,t)}(x)\right| |y(t)|\, dt + \|P_{N,y} - y\|_{C(I)}.$$

Using the properties of B-splines we obtain

$$\int_a^b \left|P_{N,K}(x,t) - P_{N,K(\cdot,t)}(x)\right| |y(t)|\, dt =$$
$$= \int_a^b \left| \sum_{i=-n}^{N-1} \left\{ \int_a^b \left[P_{N,K(\xi,\cdot)}(t) - K(\xi,t)\right] \lambda_{N,i}(\xi)\, d\xi \right\} N_{i,n}(x) \right| |y(t)|\, dt \leq$$
$$\leq \|P_N\|_{C(I)}\, \|y\|_{C(I)} \sup_{\xi \in I} \int_a^b \left|P_{N,K(\xi,\cdot)}(t) - K(\xi,t)\right|\, dt.$$

Hence by Theorem 2.2 we obtain

$$\|s_N - y\|_{C(I)} \leq \frac{1}{1 - \varrho_0}\, \|P_{N,y} - y\|_{C(I)} +$$

$$
\begin{aligned}
&+\frac{\lambda}{1-\varrho_0}\,\|P_N\|_{C(I)}\,\|y\|_{C(I)}\,\sup_{x\in I}\int_a^b \left|P_{N,K}(x,t)-P_{N,K(\cdot,t)}(x)\right|\,dt \le\\
&\le \frac{A_{0,n,\infty}}{1-\varrho_0}\,h_N^r\,\omega_{n+1-r}(y^{(r)},h_N)+\\
&+\frac{\lambda}{1-\varrho_0}\,\|P_N\|_{C(I)}\,\|y\|_{C(I)}\,C_{0,n,\infty}\,h_N^r\,\tilde\omega_{n+1-r}\left(\frac{\partial^r}{\partial\,t^r}\,K,h_N\right).
\end{aligned}
$$

Now using the Markov inequality and Theorem 2.2 we obtain

$$
\begin{aligned}
&\|s_N'-y'\|_{C(I)}\le\|s_N'-P_{N,y}'\|_{C(I)}+\|P_{N,y}'-y\|_{C(I)}\le\\
&\quad\le\lambda\sup_{x\in I}\left|\int_a^b\frac{\partial}{\partial x}P_{N,K}(x,t)[s_N(t)-y(t)]\,dt\right|+\\
&\quad+\lambda\sup_{x\in I}\left|\int_I\left[\frac{\partial}{\partial x}P_{N,K}(x,t)-\frac{\partial}{\partial x}P_{N,K(\cdot,t)}(x)\right]\,y(t)\,dt\right|+\|P_{N,y}'-y'\|_{C(I)}\le\\
&\quad\le\frac{M\,\lambda}{h_N}\sup_{x\in I}\left|\int_a^b\,P_{N,K}(x,t)\,[s_N(t)-y(t)]\,dt\right|+\\
&\quad+\frac{M\,\lambda}{h_N}\left|\int_a^b\,\left[P_{N,K}(x,t)-P_{N,K(\cdot,t)}(x)\right]y(t)\,dt\right|+\\
&\quad+\|P_{N,y}'-y'\|_{C(I)}\le\frac{M\,\lambda}{h_N}\sup_{x\in I}\int_a^b|P_{N,K}(x,t)|\,dt\,\|s_N-y\|_{C(I)}+\\
&\quad+\frac{M\,\lambda}{h_N}\int_a^b\left|P_{N,K}(x,t)-P_{N,K(\cdot,t)}(x)\right|\,|y(t)|\,dt+\|P_{N,y}'-y'\|_{C(I)}\le\\
&\quad\le M\varrho_0A_{0,n,\infty}\,h_N^{r-1}\,\omega_{n+1-r}(y^{(r)},h_N)+\\
&\quad+M\lambda\,B_{0,n,\infty}\,\|y\|_{C(I)}\,h_N^{r-1}\,\tilde\omega_{n+1-r}\left(\frac{\partial}{\partial t^r}K,h_N\right)\le\\
&\quad\le C_{1,n,\infty}\,h_N^{r-1}\,\omega_{n+1-r}(y^{(r)}h_N),
\end{aligned}
$$

where $M=M(n)$ is a constant depending only on n taken from the Markov inequality and $A_{0,n,\infty}$, $B_{0,n,\infty}$ and $C_{0,n,\infty}$ are taken from Theorem 2.2.

The remaining inequalities we prove similarly.

The proof for $p\neq\infty$ is analogous. □

The solution of the equation (5) we may obtain using the method of iteration. We proceed as follows: Let

$$P_{N,f}(x)=\sum_{j=-n}^{N-1}c_j\,N_j(x),\quad P_{N,K}(x,t)=\sum_{i=-n}^{N-1}\sum_{j=-n}^{N-1}a_{i,j}\,N_i(x)\,N_j(t),$$

$$d_{i,j}=\int\limits_a^b N_i(x)\,N_j(x)\,dx$$

and we put

$$s_{N,m+1}(x)=P_{N,f}(x)+\lambda\int\limits_a^b P_{N,K}(x,t)\,s_{N,m}(t)\,dt, \tag{8}$$

where $s_{N,m}$ is a spline of the m^{th} step of iteration.

Putting

$$s_{N,m}(x)=\sum_{k=-n}^{N-1}b_{k,m}\,N_k(x)$$

in (8) and comparing the coefficients b_k at N_k we obtain

$$b_{k,m+1}=c_k+\lambda\sum_{i=-n}^{N-1}a_{k,i}\sum_{j=-n}^{N-1}b_{j,m}\,d_{i,j},\quad m=1,2,\ldots,\ k=-n,\ldots,N-1. \tag{9}$$

Remark 3.1. *We can also use this method for the numerical solution of the Voltera integral equation of the second kind*

$$y(x)=f(x)+\lambda\int\limits_a^x K_0(x,t)\,y(t)\,dt.$$

Let $D=\{(x,t)\colon a\le x\le b,\ a\le t\le x$. *Putting*

$$K(x,t)=\begin{cases}K_0(x,t) & \text{for } (x,t)\in D,\\ 0 & \text{for } (x,t)\notin D,\end{cases}$$

we obtain the Fredholm integral equation (2)*. Now* $\operatorname{supp}P_{N,K}\subset D_N=\{(x,t)\colon a\le\le x\le b,\ a\le t\le\min[x+(n+1)h_N,b]\}$*, where* $h_N=(b-a)/N$*. If the first condition from* (6) *is satisfied, then we may solve this equation as above. Unfortunately the function* $s_{N,m+1}$ *from the recurrence relations* (8) *is not a spline of degree* n *with respect to the partition* Δ_N*. Hence we cannot apply* (9)*.*

If $\varrho\ge 1$, then we are looking the solution of the equation (5) in the following form

$$y(x)=\sum_{k=0}^{\infty}\lambda^k\phi_k(x), \tag{10}$$

where

$$\phi_0(x) = P_{N,K}(x), \quad \phi_{k+1}(x) = \int\limits_a^{x_n} P_{N,K}(x,t)\,\phi_k(t)\,dt,$$

where $x_n = \min[x + (n+1)h_N, b]$, $k = 0, 1, \ldots$ As in [3, 11] we may prove that

$$|\phi_k(x)| < \frac{\lambda^k M^k \|P_{N,f}\|_{C(I)}[b - a + k(n+1)h_N]}{k!},$$

$k = 1, 2, \ldots$, where $M = \sup\limits_{(x,t)\in D_N} |P_{N,K}(x,t)|$. Hence for $N > (b-a)M\lambda e$ the series (10) is convergent uniformly to the solution of the equation (5) on the interval $[a, b]$.

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