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A NOTE ON INDUCTIVE LIMIT MODEL OF BARGMANN SPACE OF INFINITE ORDER

Abstract. It is shown that the generalized creation and annihilation operators on Bargmann space of infinite order in a direction $a = (a_1, a_2, \dots) \in l^2$ are inductive limits of the creation and annihilation operator acting on Bargmann space of n-th order.

Keywords: Hilbert space, Bargmann space, creation operator, annihilation operator, inductive limit.

Mathematics Subject Classification: 47B38, Secondary 47B37, 47A80.

1. INTRODUCTION

An interesting connection between some particular Bergman-type space of analytic functions [5] and quantum mechanics was uncovered by V. Bargmann and I. Segal in the 1960s [1, 18]. The model of the Fock space [9] as an L^2-space of entire functions developed by Bargmann [1] provides a convenient and precise methods for studying free Bose fields. On these spaces, which we denote by B_n and which contain functions on $\mathbf{C}^n$, the Fock boson creation operators are represented as multiplications by linear functions of independent complex variables z_j, $j = 1, \dots, n$. In [1] Bargmann defines a Hilbert space B_n of all complex holomorphic functions on $\mathbf{C}^n$, square integrable with respect to the Gaussian measure. Toeplitz operators on these spaces appeared very useful in describing certain physical observables [1, 3, 4, 11, 12, 15]. Bargmann defines in [2] a Hilbert space B_∞ of all complex holomorphic functions on l^2 such, that its bases is an amalganation of all bases of B_n over all natural "n". B_∞ was suggested as a convenient functional model for implementing the ideas of Fock, Dirac, Friedrichs, Cook and Segal [6, 8, 18, 19]. An attempt to extend Berger–Coburn's program [3, 4] for infinitely many variables (degrees of freedom) proposed by Janas and Rudol in [13, 14], met serious difficulties. One would like to have an appropriate Gaussian measure μ on l^2 such, that space B_∞ can be regarded as a space of all complex holomorphic and square integrable functions on l^2. But for such a measure we have $\mu(l^2) = 0$. Trying to overcome this problem the authors have considered B_∞ as the closure in $L^2(m)$ of the set of all continuous on E_- complex polynomials, where m is a measure on E_- given by Milnos–Sozonov Theorem [16]. Here E_- is a Fréchet space from the triple of complex spaces $E_+ \subset l^2 \subset E_-$ with continuous inclusions such, that $E_+ = (E_-)^*$ and for $\phi \in E_+$, $z \in l^2$ we have $\phi(z) =< z, \phi >_{l^2}$. The main disadvantage of this "measure–theoretic" model of B_∞ is the non-existence of non-zero compact Toeplitz operators in spite of the situation for the Toeplitz operators on B_n [13].

In [14] Janas and Rudol studied an inductive model of B_∞ and they obtained there more encouraging results for Toeplitz operators. In [2] Bargmann defined the generalized creation and annihilation operators in a direction $a \in l^2$ and he pointed out that these operators correspond to those introduced by Friedrichs [8][1)]. In this paper it is shown that the generalized creation and annihilation operators in a direction $a = (a_1, a_2, \ldots) \in l^2$ are inductive limits of the creation and annihilation operator dealing in B_n in direction $(a_1, \ldots, a_n) \in \mathbf{C}^n$. This provides yet another motivation to using the inductive model of B_∞ to develop the ideas of Fock, Dirac and others in the case of infinitely many degrees of freedom.

2. THE BARGMANN'S HILBERT SPACES

In this Section we recall Bargmann's definition of the Hilbert space B_∞ and some properties of this space useful in the next parts of this paper.

We denote by $\mathbf{T}$ the set of all sequences of nonnegative integers with only a finite number of nonzero entries. In the sequel the set $\mathbf{Z}_+^n (resp.\mathbf{C}^n)$ will be interpreted as a subset of the set $\mathbf{T}(resp.l^2)$, where l^2 denotes the set of all square-summable, complex sequences and $\mathbf{C}^n(resp.\mathbf{Z}_+^n)$ denotes the cartesian product of n- copies of the complex number field $\mathbf{C}$ (resp. of the set of all nonnegative integers $\mathbf{Z}_+$). For $z := (z_1, \ldots, z_n, z_{n+1}, \ldots) \in l^2$ and $\alpha := (\alpha_1, \ldots, \alpha_n, 0, \ldots) \in \mathbf{T}$, we use following standard notation

$$\|z\|^2 := \sum_{i=1}^{\infty} |z_i|^2, z^\alpha = z_1^{\alpha_1} \cdot \ldots \cdot z_n^{\alpha_n},\ \alpha! = \alpha_1! \cdot \ldots \cdot \alpha_n!,$$

$$\alpha + \beta := (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n \ldots),\ |\alpha| = \sum_{i=1}^{\infty} \alpha_i,\ e_\alpha(z) = \frac{z^\alpha}{\sqrt{\alpha!}}.$$

Now we recall Bargmann's definition of his Hilbert space B_∞ of an infinite order [2]. Denote by $l^2(\mathbf{T})$ the set of all square-summable sequences indexed by the set $\mathbf{T}$.

[1)] The detailed proof of these facts can be found in [22].

For any sequence $(f) := (f_\alpha)_{\alpha\in\mathbf{T}} \in l^2(\mathbf{T})$ we define the function $f\colon l^2 \longrightarrow \mathbf{C}$ as follows:

$$f(z) := \sum_{\alpha\in T} f_\alpha e_\alpha(z), \quad z \in l^2 \tag{1}$$

This definition is correct [21] and the correspondence $(f) \to f$ is linear injection and we may denote its image by B_∞. Namely

$$B_\infty := \{f\colon (f) \in l^2(\mathbf{T})\}.$$

On B_∞ we transfer the Hilbert space structure from $l^2(\mathbf{T})$ defining

$$\langle f, g\rangle_\infty \overset{df}{=} \langle (f), (g)\rangle_{l^2(T)}.$$

Then the map $l^2(T) \ni (f) \longrightarrow f \in B_\infty$ is isometric and in consequence $(B_\infty, \langle\cdot,\cdot\rangle_\infty)$, is a Hilbert space. The set $\{e_\alpha\colon \alpha \in \mathbf{T}\}$ is an orthonormal basis of B_∞ [21]. All functions from B_∞ are entire as functions from l^2 to $\mathbf{C}$, [21]. B_∞ is a Hilbert space with reproducing kernel

$$K(w, z) := \exp\langle w, z\rangle \quad \text{for} \quad w, z \in l^2 \tag{2}$$

and the following reproducing formula is fulfilled:

$$f(z) = \langle f, K(\cdot, z)\rangle, \quad f \in B_\infty. \tag{3}$$

For the others properties of the Hilbert space B_∞ we refer the readers in [20, 21, 22, 23]. Concepts related to infinite-dimensional holomorphy can be found in [7].

3. GENERALIZED CREATION AND ANNIHILATION OPERATORS IN B_n, B_∞

In this Section we recall the definitions and fundamental properties of generalized creation and annihilation operators in the direction $a \in l^2$ [21] as follows:

$$D(A_a^+) := \{f \in B_\infty\colon \langle\cdot, a\rangle f(\cdot) \in B_\infty\} \tag{4}$$

$$D(A_a^-) := \{f \in B_\infty\colon \left(z \longrightarrow \frac{d}{d\lambda} f(z+\lambda a)\Big|_{\lambda=0}\right) \in B_\infty\} \tag{5}$$

$$(A_a^+ f)z := \langle z, a\rangle f(z), \quad f \in D(A_a^+), z \in l^2 \tag{6}$$

$$(A_a^- f)z := \frac{d}{d\lambda}\, f(z+\lambda a)|_{\lambda=0}, \quad f \in D(A_a^-), z \in l^2. \tag{7}$$

Using the orthonormal basis $\{e_\beta\colon \beta \in \mathbf{T}\}$ and the coefficients f_β of (f), we obtain the following characterizations:

$$\left.\begin{array}{l} f \in D(A_a^+) \text{ if and only if } \sum\limits_{\beta\in T}\left|\sum\limits_{i\in\mathbf{N}} f_{\beta-\delta_i}\overline{a_i}\cdot\sqrt{\beta_i}\right|^2 < \infty \\ \text{where } \delta_i = (0,\ldots,0,1,0,0,\ldots) \in \mathbf{T} \text{ and } a = (a_1, a_2, \ldots) \\ \text{and } f_{\beta-\delta_i} := 0 \text{ if } \beta_i = 0. \\ \text{If } f \in D(A_a^+), \text{ then } A_a^+ f = \sum\limits_{\beta\in T}\left(\sum\limits_{i\in\mathbf{N}} f_{\beta-\delta_i}\overline{a_i}\sqrt{\beta_i}\right) e_\beta. \end{array}\right] \tag{8}$$

$$\left.\begin{array}{l} f \in D(A_a^-) \text{ if and only if } \sum\limits_{\beta\in T}\left|\sum\limits_{i\in\mathbf{N}} f_{\beta+\sigma_i} a_i\sqrt{\beta_i+1}\right|^2 < \infty \\ \text{If } f\in D(A_a^-), \text{ then } A_a^- f = \sum\limits_{\beta\in T}\left(\sum\limits_{i\in\mathbf{N}} f_{\beta+\sigma_i} a_i \cdot \sqrt{b_i+1}\right) e_\beta. \end{array}\right] \tag{9}$$

For other properties of the generalized creation and annihilation operators we refer the readers to [21] and [23].

These are natural extensions of the analogously defined creation and annihilation operators A_a^+, A_a^- in the direction $a \in C^n$. On has replace B_∞ by B_n, l^2 by C^n in (4)–(7). Also (8), (9) are the corresponding characterizations in terms of the coefficients in B_n.

We are using the same symbols A_a^+, A_a^- in both cases ($n < \infty$ and $n = \infty$) to avoid additional subscripts, but their distinction can be made by checking to which of spaces C^n (resp. l^2) does their directional vector „a" belong.

At the end of this section let us note, that using the same method as in the proof of Lemma 6 in [21], it is not difficult to show that

$$LIN\{e_\alpha : \alpha \in \mathbf{T}\} \text{ is a core of } A_a^-. \tag{10}$$

4. GENERALIZED CREATION AND ANNIHILATION OPERATORS IN B_∞ AS INDUCTIVE LIMITS

Let us recall the notion of inductive limit of Hilbert spaces. Suppose we are given a sequence of Hilbert spaces H_k, $k \in \mathbf{N}$. We say that Hilbert space H is an inductive limit of the H_k if there are isometries $\gamma_k^l : H_k \to H_l (k \le l)$ and $\gamma_k : H_k \to H$ such that the following conditions are satisfied:

(i) γ_k^k is the identity on H_k,

(ii) $\gamma_k^m = \gamma_l^m \circ \gamma_k^l$ if $k \le l \le m$,

(iii) $\gamma_k = \gamma_l \circ \gamma_k^l$ if $k \le l$,

(iv) $H = \bigvee\limits_{k=1}^{\infty} \gamma_k H_k$ (the closed linear span of $\bigcup \gamma_k H_k$).

Let us consider a sequence of closable operators L_n defined on dense domain $D_n \subset H_n$, with densely defined adjoints L_n^* with domain D_n^*. Janas in [10] proved the following statement:

Statement J. *Let the following conditions be fulfilled:*

$$\gamma_n^{n+1} D_n \subset D_{n+1} \quad \textit{and} \quad \gamma_n^{n+1} D_n^* \subset D_{n+1}^*,\ n \in \mathbf{N}, \tag{11}$$

for any $\epsilon > 0$ there exists $n_0(\epsilon) \in \mathbf{N}$ such that for every $m > n \ge n_0(\epsilon)$ and any $\phi \in D_n$, $\psi \in D_n^*$ we have the inequalities

$$\|(L_m\gamma_n^m - \gamma_n^m L_n)\phi\| \le \epsilon(\|\phi\| + \|L_m\gamma_n^m\phi\| + \|L_n\phi\|) \tag{12}$$

and

$$\|(L_m^*\gamma_n^m - \gamma_n^m L_n^*)\psi\| \le \epsilon(\|\psi\| + \|L_m^*\gamma_n^m\psi\| + \|L_n\psi\|). \tag{13}$$

Then for any $n \in \mathbf{N}$, $\phi_n \in D_n$, $\psi_n \in D_n^*$ *there exist the following limits*

$$\lim_{m\to\infty} \gamma_m L_m \gamma_n^m \phi_n \quad \textit{and} \quad \lim_{m\to\infty} \gamma_m L_m^* \gamma_n^m \psi_n \tag{14}$$

Consequently, on the domains $D_\infty := \bigcup_{n\in\mathbf{N}} \gamma_n D_n$, $D_\infty^* := \bigcup_{n\in\mathbf{N}} \gamma_n D_n^*$ on can define operators

$$\begin{aligned} L_\infty\phi &:= \lim_{m\to\infty} \gamma_m L_n \gamma_n^m \phi_n \quad \text{for} \quad \phi_n = \gamma_n^{-1}\phi, \phi \in D_\infty \quad \text{and} \\ A_\infty\phi &:= \lim_{m\to\infty} \gamma_m L_m^* \gamma_n^m \psi_n \quad \text{for} \quad \psi_n = \gamma_n^{-1}\psi, \psi \in D_\infty^* \end{aligned} \tag{15}$$

These operators turn out to be closable, densely defined and satisfying $A_\infty \subset L_\infty^*$.

It is not difficult to show that $B_n = B_k \otimes B_l$, where $n = k + l$, and that B_∞ is an inductive limit of B_k with $\gamma_m^n\phi := \phi \otimes \mathbf{1}_{n-m}$, $\gamma_n\psi := \psi \otimes \mathbf{1}$, where $\mathbf{1}_{n-m} \in B_{n-m}$, $\mathbf{1} \in B$ and $\mathbf{1}_{n-m}(z) = 1$ for all $z \in \mathbf{C}^{n-m}$ and $\mathbf{1}(z) = 1$ for all $z \in l^2$.

Let next $P_k \colon l^2 \to \mathbf{C}^k$ denote the projections onto the first k coordinates: $P_k(z_1, z_2, \ldots) := (z_1, \ldots, z_k)$.

Now we show that the operators A_a^+ and A_a^- are inductive limits of the families of operators $\{A_{(P_k a)}^+ : k \in \mathbf{N}\}$ and $\{A_{(P_k a)}^- : k \in \mathbf{N}\}$ respectively, where $a \in l^2$. Let in the sequel: $a \in l^2$, $L_k := A_{(P_k a)}^+$, $L_k^* := A_{(P_k a)}^-$, $D_k = D(A_{(P_k a)}^+)$, $D_k^* = D(A_{(P_k a)}^-)$, $H_k := B_k$, $H := B_\infty$ and γ_m^n, γ_n are defined as above.

Theorem 4.1. *The following conditions are fulfilled:*

$$\gamma_n^{n+1} D_n \subset D_{n+1}, \quad n \in \mathbf{N}, \tag{16}$$

$$\gamma_n^{n+1} D_n^* \subset D_{n+1}^*, \quad n \in \mathbf{N}, \tag{17}$$

$$\overline{L_\infty} = A_a^+, \tag{18}$$

$$\overline{A_\infty} = A_a^- = L_\infty^*, \tag{19}$$

where A_∞, L_∞ *are defined as in Statement J and* $\overline{A_\infty}$, $\overline{L_\infty}$ *denote the closures of* A_∞ *and* L_∞ *resp.*

Proof. To verify (16), let us take $\phi \in D_n$.

Then $\phi(z) = \sum_{\beta\in\mathbf{Z}_+^n} \phi_\beta \frac{z^\beta}{\sqrt{\beta!}}$ and

$$\gamma_n^m\phi(z) = \phi \oplus \mathbf{1}(z) = \sum_{\beta\in\mathbf{Z}_+^n} \phi_\beta \frac{z^\beta}{\sqrt{\beta!}}.$$

From the other side we have:

$$\gamma_n^m\phi(z) = \sum_{\alpha\in\mathbf{Z}_+^m} (\gamma_n^m\phi)_\alpha \frac{z^\alpha}{\sqrt{\alpha!}} \quad \text{because of} \quad \gamma_n^m\phi \in H_m.$$

So it must be

$$\left[\begin{array}{l}(\gamma_n^m\phi)_{(\alpha_1,\ldots,\alpha_n,0,\ldots,0)} = \phi_{(\alpha_1,\ldots,\alpha_n)}\\ \text{and}\\ (\gamma_n^m\phi)_\alpha = 0 \quad \text{if} \quad \alpha_{n+1}^2+\ldots+\alpha_m^2>0.\end{array}\right. \tag{20}$$

Now with help of the above property we obtain:

$$\sum_{\alpha\in\mathbf{Z}_+^{n+1}}\left|\sum_{j=1}^{n+1}(\gamma_n^{n+1}\phi)_{\alpha-\delta_j}\overline{a_j}\sqrt{\alpha_j}\right|^2 =$$

$$= \sum_{\substack{\alpha\in\mathbf{Z}_+^{n+1}\\ \alpha_{n+1}=0}}\left|\sum_{j=1}^{n}(\gamma_n^{n+1}\phi)_{\alpha-\delta_j}\right|\overline{a}_j\sqrt{\alpha_j}^2 + \sum_{\substack{\alpha\in\mathbf{Z}_+^{n+1}\\ \alpha_{n+1}=1}}\left|(\gamma_n^{n+1}\phi)_{\alpha-\delta_j}\overline{a}_{n+1}\right|^2 =$$

$$= \sum_{\beta\in\mathbf{Z}_+^n}\left|\sum_{j=1}^{n}\phi_{\beta-\delta_j}\overline{a_j}\sqrt{\beta_j}\right|^2 + \sum_{\beta\in\mathbf{Z}_+^n}|\phi_\beta\overline{a_{n+1}}|^2 \le$$

$$\le \sum_{\beta\in\mathbf{Z}_+^n}\left|\sum_{j=1}^{n}\phi_{\beta-\delta_j}\overline{a_j}\sqrt{\beta_j}\right|^2 + \left(\sum_{\beta\in\mathbf{Z}_+^n}|\phi_\beta^2|\right)\cdot|a_{n+1}|^2 =$$

$$= \sum_{\beta\in\mathbf{Z}_+^n}\left|\sum_{j=1}^{n}\phi_{\beta-\delta_j}\overline{a_j}\sqrt{\beta_j}\right|^2 + |\phi|^2\,|a_{n+1}|\,.$$

From this and the condition (8) we have: $j_n^{n+1}(D_n)\subset D_{n+1}$. This finishes the proof of (16). Similarly, we have for $\psi\in D_n^*$, $n\in\mathbf{N}$ the equalities:

$$\sum_{\alpha\in\mathbf{Z}_+^{n+1}}\left|\sum_{i=1}^{n+1}\left(\gamma_n^{n+1}\psi\right)_{\alpha+\delta_i}a_i\sqrt{\alpha_i+1}\right|^2 = \sum_{\substack{\alpha\in\mathbf{Z}_+^{n+1}\\ \alpha_{n+1}=0}}\left|\sum_{i=1}^{n}\left(\gamma_n^{n+1}\psi\right)_{\alpha+\delta_i}a_i\sqrt{\alpha_i+1}\right|^2 =$$

$$= \sum_{\beta\in\mathbf{Z}_+^n}\left|\sum_{i=1}^{n}\psi_{\beta+\delta_i}a_i\sqrt{\beta_i+1}\right|^2.$$

So from the above and the condition (9) it follows that the statement (17) is true, too.

Our proof of (18) and (19) will be based on Statement J, whose assumption (11) has just been verified. Now we show that the assumption (12) is also true. Let $\phi\in D_n$, n, $m\in\mathbf{N}$, $n\le m$. Then from the conditions (20) and (8) we obtain:

$$\|L_m\gamma_n^m\phi-\gamma_n^m L_n\phi\|^2 =$$

$$= \left\|\sum_{\alpha\in\mathbf{Z}_+^m}\left[\sum_{i=1}^{m}\left(\gamma_n^m\psi\right)_{\alpha-\delta_i}\overline{a}_i\sqrt{\alpha_i}\right]e_\alpha - \left(\sum_{\beta\in\mathbf{Z}_+^n}\left[\sum_{i=1}^{n}\psi_{\beta-\delta_i}\overline{a}_i\sqrt{\beta_i}\right]e_\beta\right)\otimes\mathbf{1}_{m-n}\right\|^2 =$$

$$= \Bigg\| \underbrace{\sum_{\substack{\alpha\in\mathbf{Z}_+^m\\ \alpha_{n+1}=\ldots=\alpha_m=0}} \left[\sum_{i=1}^{n} (\gamma_n^m\psi)_{\alpha-\delta_i}\, \overline{a_i}\sqrt{\alpha_i}\right] e_\alpha}_{(1)} +$$

$$+ \underbrace{\sum_{j=n+1}^{m} \sum_{\substack{\alpha\in\mathbf{Z}_+^m\\ \alpha_{n+1}^2+\ldots+\alpha_m^2=1\\ \alpha_j=1}} \left[(\gamma_n^m\psi)_{\alpha-\delta_j}\, \overline{a_j}\right] e_\alpha}_{(2)} -$$

$$- \underbrace{\sum_{\beta\in\mathbf{Z}_+^n} \left[\sum_{i=1}^{n} \psi_{\beta-\delta_i}\overline{a_i}\sqrt{\beta_i}\right] e_\beta}_{(3)} \Bigg\|^2 =$$

$$= \left\| \sum_{j=n+1}^{m} \sum_{\beta\in\mathbf{Z}_+^n} (\phi_\beta\overline{a_j})\, e_{i_n(\beta)+\delta_j} \right\|^2 = \sum_{j=n+1}^{m} \sum_{\beta\in\mathbf{Z}_+^n} |\phi_\beta|^2\,|a_j|^2 = \|\phi\|^2\cdot \sum_{j=n+1}^{m} |a_j|^2,$$

where the expressions (1) and (3) cancel and $i_n(\beta) := (\beta_1,\ldots,\beta_n,0,\ldots,0)\in\mathbf{Z}_+^m$. But $a\in l^2$, so for any $\epsilon>0$ there exists $n_0(\epsilon)$ such that for every $m>n\geq n_0(\epsilon)$ $\sum\limits_{j=n+1}^{m} |a_j|^2 < \epsilon^2$ and as consequence we obtain:

$$\|(L_m\gamma_n^m - \gamma_n^m L_n)\phi\| < \epsilon\cdot\|\phi\| \quad \phi\in D_n.$$

Let now $\psi\in D_n^*$. Then from the conditions (20) and (9) we obtain:

$$\|(L_m^*\gamma_n^m - \gamma_n^m L_n^*)\psi\|^2 =$$

$$= \left\| \sum_{\alpha\in\mathbf{Z}_+^m} \left[\sum_{i=1}^{m} (\gamma_n^m\psi)_{\alpha+\delta_i}\, a_i\sqrt{\alpha_{i+1}}\right] e_\alpha - \left(\sum_{\beta\in\mathbf{Z}_+^m} \left[\sum_{i=1}^{n} \psi_{\beta+\delta_i} a_i\sqrt{\beta_{i+1}}\right] e_\beta\right)\otimes\mathbf{1}_{m-n} \right\| =$$

$$= \left\| \sum_{\substack{\alpha\in\mathbf{Z}_+^m\\ \alpha_{n+1}=\ldots=\alpha_m=0}} \left[\sum_{i=1}^{n} (\gamma_n^m\psi)_{\alpha+\delta_i}\, a_i\sqrt{\alpha_i+1}\right] e_\alpha - \sum_{\beta\in\mathbf{Z}_+^n} \left[\sum_{i=1}^{n} \psi_{\beta+\delta_i} a_i\sqrt{\beta_i+1}\right] e_\beta \right\|^2 =$$

$$= 0.$$

So really, the assumption (12) is fulfilled. Now we can use Statement J and we obtain that the conditions (14) and (15) are true in our situation described at the beginning of Section 4. It is plain that $\gamma_n D_n \subset D(A_a^+)$ and $\gamma_n D_n^* \subset D(A_a^-)$ for every $n \in \mathbf{N}$. So we have

$$D_\infty \subset D(A_a^+) \quad \text{and} \quad D_\infty^* \subset D(A_a^-).$$

Next we note:

$$\begin{aligned} L_\infty \phi_n(z) &= \lim_{m\to\infty} \gamma_m L_m \gamma_n^m \phi_n(z) = \\ &= \lim_{n\to\infty} \left(\gamma_m \left[\langle \cdot, P_m a\rangle_m \gamma_n^m \phi_n(\cdot)\right]\right) z = \\ &= \lim_{m\to\infty} \left(\langle \cdot, P_m a\rangle_m \gamma_n^m \phi_n(\cdot) \otimes \mathbf{1}\right) z = \\ &= \lim_{m\to\infty} \langle P_m z, P_m a\rangle_m \gamma_n^m \phi_n(P_m z) = \\ &= \langle z, a\rangle \gamma_n \phi_n(z), \quad \phi_n \in D_n, \quad n \in \mathbf{N}, \end{aligned}$$

because of the facts, that B_n, B_∞ are the Hilbert spaces with reproducing kernels (so, the evaluation functions on these spaces are continuous) and the observations:

$$\langle P_m z, P_m a\rangle_m \longrightarrow \langle z, a\rangle \quad \text{and} \quad \gamma_n^m \phi_n(P_m z) \longrightarrow \gamma_n \phi_n(z) \quad \text{if} \quad m \to \infty.$$

The above allows us to write: $L_\infty \subset A_a^+$.

Let $\mathcal{M} := LIN\{e_\alpha \colon \alpha \in T\}$. Then we have $\mathcal{M} \subset D_\infty \cap D_\infty^*$ and $A_a^*|_{\mathcal{M}} \subset L_\infty \subset \subset \overline{L_\infty} \subset A_a^+$, where $A_a^+|_{\mathcal{M}}$ denotes a restriction of the operator A_a^+ to the linear space $\mathcal{M}$ and $\overline{L_\infty}$ denotes a closure of the operator L_∞ (from the Statement J it follows that L_∞ is closable). But $\mathcal{M}$ is a core of A_a^+ [21]. So we have:

$$A_a^+ = \overline{A_a^+|_{\mathcal{M}}} \subset \overline{L_\infty} \subset A_a^+$$

and at last $\overline{L_\infty} = A_a^+$. This finishes the proof of the condition (18).

Similarly as above we obtain:

$$\begin{aligned} A_\infty \psi_n(z) &= \lim_{m\to\infty} \gamma_m L_m^* \psi_n(z) = \\ &= \lim_{m\to\infty} \frac{d}{d\lambda} \gamma_n^m \psi_n(P_m z + \lambda P_m a)|_{\lambda=0} = \\ &= \lim_{m\to\infty} \gamma_n^m \frac{d}{d\lambda} \psi_n(P_n z + \lambda P_n a)|_{\lambda=0} = \gamma_n \frac{d}{d\lambda} \psi_n(P_n z + \lambda P_n a)|_{\lambda=0} = \\ &= \frac{d}{d\lambda} \gamma_n \psi_n(z + \lambda a)|_{\lambda=0} \quad \text{for} \quad \psi_n \in D_n^* \quad \text{and} \quad n \in \mathbf{N}, \end{aligned}$$

because of $\frac{d}{d\lambda}\left[f(\lambda,\cdot) \otimes g(\cdot)\right]|_{\lambda=\lambda_0} = \left(\frac{d}{d\lambda} f(\lambda,\cdot)|_{\lambda=\lambda_0}\right) \otimes g(\cdot)$ and the other same reasons as by the calculation at the preceding page.

From this it immediately follows that

$$A_a^-|_{\mathcal{M}} \subset A_\infty \subset \overline{A_\infty} \subset A_a^-.$$

But $\mathcal{M}$ is core of A_a^- (see the condition (10)). Thus

$$A_a^- = \overline{A_a^-|_{\mathcal{M}}} \subset \overline{A_\infty} \subset A_a^-$$

and finally $\overline{A_\infty} = A_a^-$, which finishes the proof of (19) and of our all Theorem. □

Janas in [10] showed that an inductive limit of normal operators is a normal operator under assumption describing the behavior of bounded vectors of operators from the inductive sequence. He obtained also hyponormality (cohyponormality) for the decreasing sequence of appropriate tensor products of hyponormal (cohyponormal) operators under some normalization assumption. In [21] it is shown that the generalized creation operators A_a^+ are subnormal. Our Theorem shows us an example of a sequence of comparatively simple subnormal operators, which has also subnormal inductive limit, having minimal normal extensions on $L^2(u_n)$ such, that their inductive limit, although subnormal, has no straightforward normal extension on L^2-space ([21], see also [17]).

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