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MONOTONE ITERATIVE METHODS FOR INFINITE SYSTEMS OF REACTION-DIFFUSION-CONVECTION EQUATIONS WITH FUNCTIONAL DEPENDENCE

Abstract. We consider the Fourier first initial-boundary value problem for an infinite system of semilinear parabolic differential-functional equations of reaction-diffusion-convection type of the form

$$\mathcal{F}^i[z^i](t,x) = f^i(t,x,z), \quad i \in S,$$

where

$$\mathcal{F}^i := \mathcal{D}_t - \mathcal{L}^i, \quad \mathcal{L}^i := \sum_{j,k=1}^{m} a^i_{jk}(t,x)\mathcal{D}^2_{x_j x_k} + \sum_{j=1}^{m} b^i_j(t,x)\mathcal{D}_{x_j}$$

in a bounded cylindrical domain $(0,T] \times G := D \subset \mathbb{R}^{m+1}$. The right-hand sides of the system are Volterra type functionals of the unknown function z. In the paper, we give methods of the construction of the monotone iterative sequences converging to the unique classical solution of the problem considered in partially ordered Banach spaces with various convergence rates of iterations. We also give remarks on monotone iterative methods in connection with numerical methods, remarks on methods for the construction of lower and upper solutions and remarks concerning the possibility of extending these methods to more general parabolic equations. All monotone iterative methods are based on differential inequalities and, in this paper, we use the theorem on weak partial differential-functional inequalities for infinite systems of parabolic equations, the comparison theorem and the maximum principle. A part of the paper is based on the results of our previous papers. These results generalize the results obtained by several authors in numerous papers for finite systems of semilinear parabolic differential equations to encompass the case of infinite systems of semilinear parabolic differential-functional equations. The monotone iterative schemes can be used for the computation of numerical solutions.

Keywords: infinite systems, reaction-diffusion-convection equations, semilinear parabolic differential-functional equations, Volterra functionals, monotone iterative methods, method of upper and lower solutions.

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CONTENTS

INTRODUCTION

1. Let us consider an infinite weakly coupled[1)] system of semilinear parabolic differential-functional equations of reaction–diffusion-convection type of the form

$$\mathcal{F}^i[z^i](t,x) = f^i(t,x,z), \quad i \in S, \tag{0.1}$$

where:

$$\mathcal{F}^i := \mathcal{D}_t - \mathcal{L}^i, \quad \mathcal{L}^i := \sum_{j,k=1}^{m} a^i_{jk}(t,x)\mathcal{D}^2_{x_j x_k} + \sum_{j=1}^{m} b^i_j(t,x)\mathcal{D}_{x_j}$$

[1)] This means that every equation contains all unknown functions and spatial derivatives of only one unknown function. The strong coupling of a system means that all spatial derivatives up to a given order appear in all equations.

$x = (x_1, \ldots, x_m)$, $(t, x) \in (0, T] \times G := D$, $T < \infty$, $G \subset \mathbb{R}^m$ and G is an open bounded domain with the boundary $\partial G \in C^{2+\alpha}(0 < \alpha < 1)$, $\mathcal{D}_t = \frac{\partial}{\partial t}$, $\mathcal{D}_{x_j} = \frac{\partial}{\partial x_j}$, $\mathcal{D}^2_{x_j x_k} = \frac{\partial^2}{\partial x_j \partial x_k}$ $(j, k = 1, \ldots, m)$, S is an arbitrary set of indices (finite or infinite), z stands for the mapping

$$z \colon S \times \overline{D} \to \mathbb{R}, \quad (i, t, x) \mapsto z(i, t, x) := z^i(t, x),$$

composed of unknown functions z^i, the right-hand sides f^i of the system are functionals of z and we assume that they are the Volterra type.

Let $\mathcal{B}(S)$ be the real Banach space of mappings

$$w \colon S \to \mathbb{R}, \quad i \mapsto w(i) := w^i,$$

with the finite norm

$$\|w\|_{\mathcal{B}(S)} := \sup\left\{\left|w^i\right| : i \in S\right\},$$

where we use the symbol $|\cdot|$ to denote the absolute value of a real number.

Obviously, for a finite S with r elements, there is $\mathcal{B}(S) = \mathbb{R}^r$ and for an infinite countable S we admit $S = \mathbb{N}$ and there is $\mathcal{B}(\mathbb{N}) = l^\infty(\mathbb{N}) := l^\infty$, where $\mathbb{N}$ is the set of natural numbers and l^∞ is the Banach space of all real bounded sequences $w = \{w^i\}_{i \in \mathbb{N}} = (w^1, w^2, \ldots)$ with the finite norm

$$\|w\|_{l^\infty} := \sup\left\{\left|w^i\right| : i \in \mathbb{N}\right\}.$$

Denote by $C_S(\overline{D})$ the real Banach space of mappings

$$w \colon \overline{D} \to \mathcal{B}(S), \quad (t, x) \mapsto w(t, x)$$

and

$$w(t, x) \colon S \to \mathbb{R}, \quad i \mapsto w^i(t, x),$$

where the functions w^i are continuous in $\overline{D}$, i.e., $w^i \in C(\overline{D})$, $i \in S$, with the finite norm

$$\|w\|_0 := \sup\left\{\left|w^i\right|_0 : i \in S\right\},$$

where

$$\left|w^i\right|_0 := \sup\left\{\left|w^i(t, x)\right| : (t, x) \in \overline{D}\right\}$$

is the norm in $C(\overline{D})$.

If S is a finite set with r elements, then we have $C_r(\overline{D})$.

For $w \in C_S(\overline{D})$ and a fixed t, $0 \le t \le T$, we define

$$\|w\|_{0,t} := \sup\left\{\left|w^i(\bar{t}, x)\right| : 0 \le \bar{t} \le t,\ x \in \overline{G},\ i \in S\right\}.$$

Let

$$f^i \colon \overline{D} \times C_S(\overline{D}) \to \mathbb{R}, \quad (t, x, s) \mapsto f^i(t, x, s), \quad i \in S.$$

The notation $f^i(t, x, z)$ means that functions f^i are functionals of the function z. In order to distinguish the function-type dependence from the functional-type dependence of the right-hand sides of the system on the unknown function z, we will write the system (0.1) in the following form

$$\mathcal{F}^i[z^i](t,x) = f^i(t,x,z(t,x),z), \quad i \in S, \tag{0.2}$$

which will be useful in our further considerations.

A mapping $w \in C_S(\overline{D})$ will be called *regular* in $\overline{D}$ if w^i, $i \in S$, have continuous derivatives $\mathcal{D}_t w^i$, $\mathcal{D}_{x_j} w^i$, $\mathcal{D}^2_{x_j x_k} w^i$ in D for $j,k = 1,\dots,m$, i.e., $w \in C_S^{reg}(\overline{D}) := C_S(\overline{D}) \cap C_S^{1,2}(D)$.

For system (0.1), we consider the following Fourier first initial-boundary value problem: Find the *regular solution* (*classical solution*) z of system (0.1) in $\overline{D}$ fulfilling the initial-boundary condition

$$z(t,x) = \phi(t,x) \quad \text{for} \quad (t,x) \in \Gamma, \tag{0.3}$$

where $D_0 := \{(t,x)\colon t = 0,\ x \in \overline{G}\}$, $\sigma := (0,T] \times \partial G$ is the lateral surface, $\Gamma := D_0 \cup \sigma$ is the parabolic boundary of domain D, and $\overline{D} := D \cup \Gamma$.

We will write the initial-boundary condition (0.3) in the another form too, as the initial condition

$$z(0,x) = \phi_0(x) \quad \text{for} \quad x \in G \tag{0.4}$$

and the boundary condition

$$z(t,x) = \psi(t,x) \quad \text{for} \quad (t,x) \in \sigma \tag{0.5}$$

with the compatibility conditions of order[2)] $\left[\frac{\alpha}{2}\right] + 1$, i.e.,

$$\psi(0,x) = \phi_0(x), \quad \mathcal{F}^i[\psi^i](0,x) = f^i(0,x,\psi), \quad i \in S, \quad \text{for} \quad x \in \partial G. \tag{0.6}$$

2. The successive approximation method is among basic and simplest methods for proving existence theorems for certain types of differential equations. Numerous versions of this method are well known, using different constructions of a sequence of successive approximations and different ways of proving the convergence. The variants of the successive approximation method and its modifications have been treated by T. Ważewski [125, 126] and his followers (cp. W. Mlak and C. Olech [74], A. Pelczar [86]).

In this paper, to prove the existence and uniqueness of a solution of this problem, we apply various monotone iterative methods in Banach spaces partially ordered by positive cones. Applying the method of upper and lower solutions requires assuming the monotonicity of the right-hand sides of the system, i.e., the reaction functions $f^i(t,x,y,s)$, $i \in S$, with respect to the argument y and to the functional argument s. We also assume the existence of a pair of a lower and an upper solution of the problem

2) For a definition see O. A. Ladyżenskaja *et al.* ([57], p. 319), G. S. Ladde *et al.* ([56], pp. 138–139). For a positive real number β, by $[\beta]$ we denote the greatest integer not exceeding β.

considered. These assumptions are not typical of existence theorems, but monotone iterative methods are constructive in nature and we obtain the constructive existence theorems. The successive terms of the approximation sequences are defined as the solutions of linear systems of differential equations. These sequences are monotonously convergent to the solution searched for. A basic assumption on $f^i(t,x,y,s)$, $i \in S$, is also the left-hand side Lipschitz condition. The right-hand side Lipschitz condition is only used to ensure the uniqueness of solution. We assume that the reaction functions $f^i(t,x,s)$, $i \in S$, are Volterra functionals with respect to the argument s (satisfy the Volterra condition). This means that the value of these functions depends on the past history of the modelling process.

The plan of the paper is as follows. In Chapter I, we introduce the notations, definitions, assumptions and auxiliary theorems: on the existence and uniqueness of the solutions of linear parabolic equations and on weak parabolic differential-functional inequalities for infinite systems. A part of Chapter II is based on the results contained in our previous papers [16–28, 30] in which we consider the finite and infinite systems of differential and differential-functional equations, which the right-hand sides depends on $z(t,\cdot)$, and its generalizations. These results also generalize the outcomes obtained by several authors for finite systems of parabolic equations to the case of infinite systems of parabolic differential-functional equations. To examine the existence of a solution of the problem considered, six monotone iterative methods have been successivelly used. In the method of direct iteration, Chaplygin method and its modifications, the successive terms of the approximation sequences are defined as solutions of linear equations of the parabolic type. We also present two different variants of a monotone iterative method in which we apply the important idea of a pseudo-linearization of nonlinear problems as introduced by T. Ważewski (the successive terms of approximation sequences are defined as solutions of semilinear differential equations of the parabolic type). We also present the monotone method of direct iterations in unbounded spatial domains, when functions considered satisfy some growth condition. These monotone iterative schemes can be used to the computation of numerical solutions, when differential equations are replaced by suitable finite difference equations. In Chapter III, we give remarks on the monotone iterative methods in connection with the application of numerical methods to solve the problem considered, remarks on the methods of the construction of upper and lower solutions and remarks concerning the possibility of extending these methods to more general equations.

The proofs of theorems are based on Szarski's results concerning the weak partial differential-functional inequalities for infinite systems of parabolic equations, the comparison theorem and the maximum principle ([116,117] and cp. B. Kraśnicka [50]) (for finite systems cp. J. Szarski [111–115]). The maximum principle plays a fundamental role in the construction of monotone approximation sequences. This role is reflected in the so-called positivity lemma, which directly follows from the maximum principle.

The uniqueness of solutions of problems considered is guaranteed by the Lipschitz condition and follows directly from Szarski's uniqueness criterion [116] (cp. B. Kraśnicka [50] and cp. D. Jaruszewska-Walczak [42]).

3. The following examples of Volterra functionals have been considered in papers by H. Bellout [13], K. Nickel [77, 78], R. Redlinger [102, 103], B. Rzepecki [105]:

$$f_1(t,x,z) = \int_0^t m(t-\tau)\ K(z(\tau,x))\ d\tau, \quad f_2(t,x,z) = \int_0^t K(t,\tau,x,z(\tau,x))\ d\tau,$$

$$f_3(t,x,z) = \int_0^t \int_G K(t,\tau,x,\xi,z(\tau,\xi))\ d\tau\ d\xi,$$

$$f_4(t,x,z) = z(\theta t,x) \quad \text{with} \quad 0 \le \theta \le 1, \quad f_5(t,x,z) = z(t-\tau,x) \text{ with } \tau > 0.$$

Equations including such functionals need to be considered in appropriately chosen domains. For instance, the example including $z(t-\tau,x)$ leads to an equation with retarded argument and such equations require a modified domain of initial condition (0.4).

Other examples are (see D. Wrzosek [127, 128])

$$\begin{aligned} f_6^1(t,x,z) &= -z^1(t,x)\sum_{k=1}^{\infty} a_k^1 z^k(t,x) + \sum_{k=1}^{\infty} b_k^1 z^{1+k}(t,x), \\ f_6^i(t,x,z) &= \frac{1}{2}\sum_{k=1}^{i-1} a_k^{i-k} z^{i-k}(t,x) z^k(t,x) - z^i(t,x)\sum_{k=1}^{\infty} a_k^i z^k(t,x) + \\ &\quad + \sum_{k=1}^{\infty} b_k^i z^{i+k}(t,x) - \frac{1}{2} z^i(t,x)\sum_{k=1}^{i-1} b_k^{i-k} \quad \text{for} \quad i = 2,3\ldots, \end{aligned} \tag{0.7}$$

and (see M. Lachowicz and D. Wrzosek [55])

$$\begin{aligned} f_7^1(t,x,z) &= -z^1(t,x)\sum_{k=1}^{\infty}\int_G a_k^1(x,\xi) z^k(t,\xi) d\xi + \sum_{k=1}^{\infty}\int_G B_k^1(x,\xi) z^{1+k}(t,\xi) d\xi, \\ f_7^i(t,x,z) &= \frac{1}{2}\sum_{k=1}^{i-1}\int_{G\times G} A_k^{i-k}(x,\xi,\eta) z^{i-k}(t,\xi) z^k(t,\eta) d\xi d\eta - \\ &- z^i(t,x)\sum_{k=1}^{\infty}\int_G a_k^i(x,\xi) z^k(t,\xi) d\xi + \\ &+ \sum_{k=1}^{\infty}\int_G B_k^i(x,\xi) z^{i+k}(t,\xi) d\xi - \frac{1}{2} z^i(t,x)\sum_{k=1}^{i-1} b_k^{i-k}(x) \text{ for } i = 2,3,\ldots \end{aligned} \tag{0.8}$$

where $\int\limits_G A_k^i(x,\xi,\eta)dx = a_k^i(\xi,\eta)$ and $\int\limits_G B_k^i(x,\xi)dx = b_k^i(\xi)$, are the nonnegative coefficients of coagulation a_k^i and fragmentation b_k^i rates, and f_6^i, f_7^i are Volterra functionals.

The theory presented herein, that is thatof monotone iterative methods, covers some of these examples only, namely the examples of f_1, f_2 and f_3. The theory presented does not cover the examples of f_4 and f_5; the same applies to the examples of f_6^i, f_7^i. Systems with right-hand sides of this type are, however, considered in the literature, and other methods are required to solve them. In the case of examples f_6^i, f_7^i, the truncation method, applied by the authors quoted, may be used.

Lemma 2.1 may be used to verify which of the aforemetioned examples are covered by the theory.

4. Infinite systems of ordinary differential equations, integro-differential equations and differential-functional equations are natural generalizations of finite systems of these equations and we note that these infinite systems play a special role in the mathematical modelling of numerous difficult real-world problems.

Infinite systems of ordinary differential equations can be used to solve some problems for parabolic and hyperbolic equations on the method of lines (see H. Leszczyński [62, 63], Z. Kamont and S. Zacharek [43], Z. Kamont [44]). In this method, space variables only are discretized, leading to an infinite countable system of nonlinear ordinary differential equations.

An infinite system of differential equations was originally introduced by M. Smoluchowski [109] (1917) as a model for coagulation of colloids moving according to a Brownian motion. Some infinite countable systems of reaction-diffusion equations have been considered by D. Wrzosek with Ph. Bénilan, M. Lachowicz and Ph. Laurençot in numerous recently published papers [14, 55, 61, 127–129] as the discrete coagulation-fragmentation models with diffusion. To solve these infinite countable systems the authors apply the truncation method.

Continuous coagulation-fragmentation models may be expressed in terms of infinite uncountable systems of semilinear integro-differential parabolic equations of the reaction-diffusion type (cp. H. Amann [3]).

Infinite countable systems of ordinary differential equations, systems of integral and functional-integral equations have been studied in recently papers by J. Banaś with M. Lecko and K. Sadarangani [6–9] on help of the technique of measures of non-compactness.

CHAPTER 1. PRELIMINARIES

1.1. NOTATIONS, DEFINITIONS AND ASSUMPTIONS

1.1.1. Notations

The notation w denotes that w is regarded as an element of the set of admissible functions, while $w(t,x)$ means the value of this function w at the point (t,x). However, sometimes, to stress the dependence of function w on the variables t and x, we will write $w = w(t,x)$ and hope that this will not raise the reader's doubts.

For any $\eta, y \in \mathcal{B}(S)$ and for every fixed $i \in S$, let $[\eta, y]^i$ denote an element of $\mathcal{B}(S)$ with the description (cp. J. Chandra *et al.* [35])

$$[\eta, y]^i := \begin{cases} y^j & \text{for all } j \neq i,\ j \in S, \\ \eta^i & \text{for } j = i. \end{cases}$$

In the case of an infinite countable set of indices $S = \mathbb{N}$, i.e. for $\eta, y \in l^\infty(\mathbb{N})$, there is

$$[\eta, y]^i := (y^1, y^2, \ldots, y^{i-1}, \eta^i, y^{i+1}, \ldots).$$

For $y, \tilde{y} \in \mathcal{B}(S)$ and for every fixed $i \in S$, we write

$$y \overset{(i)}{\leq} \tilde{y} \iff \begin{cases} y^j \leq \tilde{y}^j & \text{for all } j \neq i,\ j \in S, \\ y^i = \tilde{y}^i & \text{for } j = i \end{cases}$$

and

$$y \underset{(i)}{\leq} \tilde{y} \iff \begin{cases} y^j = \tilde{y}^j & \text{for all } j \neq i,\ j \in S, \\ y^i \leq \tilde{y}^i & \text{for } j = i. \end{cases}$$

For $s, \tilde{s} \in C_S(\overline{D})$ and for every fixed $t, 0 \leq t \leq T$, we write

$$s \overset{(t)}{\leq} \tilde{s} \iff s^i(\bar{t}, x) \leq \tilde{s}^i(\bar{t}, x) \quad \text{for} \quad 0 \leq \bar{t} \leq t,\ x \in \overline{G},\ i \in S$$

and

$$s \overset{(t)}{=} \tilde{s} \iff s^i(\bar{t}, x) = \tilde{s}^i(\bar{t}, x) \quad \text{for} \quad 0 \leq \bar{t} \leq t,\ x \in \overline{G},\ i \in S.$$

If $t \geq T$, then we simply write $s \leq \tilde{s}$ instead of $s \overset{(t)}{\leq} \tilde{s}$.

1.1.2. The Hölder spaces

We give some fundamental information on the Hölder spaces (see A. Friedman [41] pp. 2 and 61–63).

Definition 1.1. *A real function $h = h(x)$ defined on a bounded closed set $A \subset \mathbb{R}^m$ is said to be* Hölder continuous with exponent α $(0 < \alpha < 1)$ *in A if there exists a constant $H > 0$ such that*

$$|h(x) - h(x')| \leq H \|x - x'\|^\alpha \quad \textit{for all} \quad x, x' \in A,$$

where $\|x\| = \left(\sum_{j=1}^{m} x_j^2\right)^{\frac{1}{2}}$. *The smallest* $H = H(h)$ *for which this inequality holds is called the* Hölder coefficient.

If $\alpha = 1$, then we say that $h = h(x)$ is *Lipschitz continuous* in A.

If A is an open set then h is *locally Hölder continuous* with exponent α in A if this inequality holds in every bounded closed subset $B \subset A$, where H may depend on B. If H is independed of B, then we say that h is *uniformly Hölder continuous* in A.

If the function h depends also on a parameter λ, i.e., $h = h(x, \lambda)$, and if the Hölder coefficient H is independent of λ, then we say that h is Hölder continuous in x, uniformly with respect to λ.

Definition 1.2. *The real function* $h = h(t, x)$ *defined on a bounded closed set* $D \subset \mathbb{R}^{m+1}$ *is said to be Hölder continuous with respect to* t *and* x *with exponent* α $(0 < \alpha < 1)$ *in* $\overline{D}$ *if there exists a constant* $H = H(h) > 0$ *such that*

$$|h(t,x) - h(t',x')| \leq H\left(|t - t'|^{\frac{\alpha}{2}} + \|x - x'\|^{\alpha}\right)$$

for all (t, x), $(t', x') \in \overline{D}$.

Definition 1.3. *The* Hölder space $C^{k+\alpha}(\overline{D})$ $(k = 0, 1, 2; 0 < \alpha < 1)$ *is the space of continuous functions* h *in* $\overline{D}$ *whose all derivatives* $\mathcal{D}_t^r \mathcal{D}_x^s h(t, x)$ *such that* $0 \leq 2r+s \leq k$ *exist and are Hölder continuous with exponent* α $(0 < \alpha < 1)$ *in* D, *with the finite norm*

$$|h|_{k+\alpha} := \sup_{\substack{P \in D \\ o \leq 2r+s \leq k}} |\mathcal{D}_t^r \mathcal{D}_x^s h(P)| + \sup_{\substack{P, P' \in D \\ 2r+s=k \\ P \neq P'}} \frac{|\mathcal{D}_t^r \mathcal{D}_x^s h(P) - \mathcal{D}_t^r \mathcal{D}_x^s h(P')|}{[d(P, P')]^{\alpha}},$$

where $d(P, P') = (|t - t'| + \|x - x'\|^2)^{\frac{1}{2}}$ *is the parabolic distance of points* $P = (t, x)$, $P' = (t', x')$ *in* $\mathbb{R}^{m+1}$.

In particular, we have:

$$|h|_{0+\alpha} = |h|_0 + H_{\alpha}^D(h),$$

$$|h|_0 = \sup_{P \in D} |h(P)|,$$

$$H_{\alpha}^D(h) = \sup_{\substack{P, P' \in D \\ P \neq P'}} \frac{|h(P) - h(P')|}{[d(P, P)]^{a}},$$

$$|h|_{1+\alpha} = |h|_{0+\alpha} + \sum_{j=1}^{m} \left|\mathcal{D}_{x_j} h\right|_{0+\alpha},$$

$$|h|_{2+\alpha} = |h|_{0+\alpha} + \sum_{j=1}^{m} \left|\mathcal{D}_{x_j} h\right|_{0+\alpha} + \sum_{j,k=1}^{m} \left|\mathcal{D}_{x_j x_k}^2 h\right|_{0+\alpha} + |\mathcal{D}_t h|_{0+\alpha}.$$

$H^D_\alpha(h) < \infty$ if and only if h is uniformly Hölder continuous (with exponent α) in D, and then $H^D_\alpha(h)$ is the Hölder coefficient of h. It is well known, that the spaces $C^{k+\alpha}(\overline{D})$, $k = 0, 1, 2$ are the Banach spaces.

Definition 1.4. *By $C_{k+\alpha,S}(\overline{D}) := C^{k+\alpha}_S(\overline{D})$, we denote the Banach space of mappings $w = \{w^i\}_{i\in S}$ such that $w^i \in C^{k+\alpha}(\overline{D})$ for all $i \in S$, with the finite norm*

$$\|w\|_{k+\alpha} := \sup\left\{\left|w^i\right|_{k+\alpha} : i \in S\right\}.$$

Remark 1.1. *If $\mathcal{B}(S) = \mathbb{R}^r$, i.e., $S = \{1, 2, \ldots, r\}$ then we will denote these spaces by $C_r(\overline{D})$ and $C^{2+\alpha}_r(\overline{D})$, respectively.*

Definition 1.5. *The boundary norm $\|\cdot\|^\Gamma_{k+\alpha}$ of a function $\phi \in C^{k+\alpha}_S(\Gamma)$ is defined as*

$$\|\phi\|^\Gamma_{k+\alpha} := \inf_\Phi \|\Phi\|_{k+\alpha},$$

where the infimum is taken over the set of all possible extensions Φ of ϕ onto $\overline{D}$, i.e., $\Phi(t, x) = \phi(t, x)$ for each $(t, x) \in \Gamma$, such that $\Phi \in C^{k+\alpha}_S(\overline{D})$.

1.1.3. Cones, norms and order

In this paper, we will consider Banach spaces with a partial order induced by a positive cone. Therefore, we recall some definitions and properties regarding cones, norms and order.

Definition 1.6. *Let $\mathcal{X}$ be a real Banach space. A proper, closed, convex subset K of $\mathcal{X}$ is said to be an order cone if $\lambda K \subset K$ for every $\lambda \geq 0$, $K \cup K \subset K$ and $K \cap (-K) = \{0\}$, where 0 denotes the null element of the Banach space $\mathcal{X}$.*

Definition 1.7. *The partial order "$\leq$" in the Banach space $\mathcal{X}$ may be defined by means of the order cone K in the following way:*

$$u \leq v \Longleftrightarrow v - u \in K.$$

Then $\mathcal{X}$ is called an ordered (a partially ordered) Banach space with cone K.

Definition 1.8. *The order cone K induces the order relation "$<$" in the Banach space $\mathcal{X}$ defined by*

$$u < v \Longleftrightarrow v - u \in \overset{\circ}{K},$$

where $\operatorname{int} K := \overset{\circ}{K} := K - \{0\}$ denotes the interior of K and the elements of $\overset{\circ}{K}$ are called positive[3]*.*

[3] More informations on cones and their properties can be found in the works by M. G. Krein and M. A. Rutman [51], M. A. Krasnosel'skiǐ [47, 48].

Remark 1.2. *From this it follows that the inequality $u(t,x) \le v(t,x)$ is to be understood componentwise, i.e., $u^i(t,x) \le v^i(t,x)$ for all $i \in S$.*

Inequality $u \le v$ is to be understood both componentwise and pointwise, i.e., $u^i(t,x) \le v^i(t,x)$ for arbitrary $(t,x) \in \overline{D}$ and all $i \in S$.

Moreover, we accepted that the notation $u < v$ means $u^i(t,x) < v^i(t,x)$ for arbitrary $(t,x) \in \overline{D}$ and all $i \in S$.

The partial order in the space $\mathcal{B}(S)$ is given by the positive cone

$$\mathcal{B}^+(S) := \big\{w\colon w = \{w^i\}_{i\in S} \in \mathcal{B}(S),\ w^i \ge 0 \quad \text{for} \quad i \in S\big\}$$

in the following way

$$u \le v \Longleftrightarrow v - u \in \mathcal{B}^+(S).$$

Analogously, the partial order in the space $C_S(\overline{D})$ is defined by means of the positive cone

$$C_S^+(\overline{D}) := \big\{w\colon w = \{w^i\}_{i\in S} \in C_S(\overline{D}),\ w^i(t,x) \ge 0 \quad \text{for} \quad (t,x) \in \overline{D} \quad \text{and} \quad i \in S\big\}$$

in the following way $n \le v \Leftrightarrow v - u \in C_S^+(\overline{D})$.

Definition 1.9. *The partial ordering in $\mathcal{X}$ induces a corresponding partial ordering also in a subset W of $\mathcal{X}$ and if $u, v \in W$ with $u \le v$, then*

$$\langle u, v\rangle := \{s \in W,\ u \le s \le v\}$$

denotes the sector *(or* order interval, conical segment*) formed by the ordered pair u and v.*

Definition 1.10. *Let $\mathcal{X}$ and $\mathcal{Y}$ be partially ordered sets with the ordering given by cones $K_{\mathcal{X}}$ and $K_{\mathcal{Y}}$, and denoted by "$\le$" in each set. A map $\mathcal{T} : \mathcal{X} \to \mathcal{Y}$ is called* isotone *(monotone increasing) if for each $u, v \in \mathcal{X}$, $u \le v$ implies $\mathcal{T}[u] \le \mathcal{T}[v]$ and* strictly isotone *if $u < v$ implies $\mathcal{T}[u] < \mathcal{T}[v]$.*

Similarly, $\mathcal{T}$ is called antitone *(monotone decreasing) if $u \le v$ implies $\mathcal{T}[u] \ge \mathcal{T}[v]$. Isotone or antitone maps are called* monotone maps.

1.1.4. Fundamental assumptions

We assume that the operators $\mathcal{F}^i$, $i \in S$, are *uniformly parabolic* in $\overline{D}$ (the operators $\mathcal{L}^i$, $i \in S$, are *uniformly elliptic* in $\overline{D}$), i.e., there exists a constant $\mu > 0$ such that

$$\sum_{j,k=1}^{m} a^i_{jk}(t,x)\xi_j\xi_k \ge \mu \sum_{j=1}^{m} \xi_j^2, \quad i \in S, \tag{1.1}$$

hold for all $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m$, $(t,x) \in \overline{D}$.

Assumption $(\mathbf{H}_{\boldsymbol{a}})$**.** *We will assume that all the coefficients* $a^i_{jk} = a^i_{jk}(t,x)$, $a^i_{jk} = a^i_{kj}$ *and* $b^i_j = b^i_j(t,x)$ $(j,k = 1,\dots,m,\ i \in S)$ *of the operators* $\mathcal{L}^i$ *are uniformly Hölder continuous with respect to* t *and* x *in* $\overline{D}$ *with exponent* α $(0<\alpha<1)$ *and they Hölder norms are uniformly bounded, i.e.,*

$$\left|a^i_{jk}\right|_{0+\alpha} \le K_1, \quad \left|b^i_j\right|_{0+\alpha} \le K_1,$$

where K_1 *is a positive constant.*

We will assume that the functions

$$f^i\colon \overline{D}\times\mathcal{B}(S)\times C_S(\overline{D}) \to \mathbb{R}, \quad (t,x,y,s)\mapsto f^i(t,x,y,s), \quad i\in S,$$

are continuous and satisfy the following assumptions:

Assumption $(\mathbf{H}_{\boldsymbol{f}})$**.** *Functions* $f^i(t,x,y,s), i\in S$, *are uniformly Hölder continuous with exponent* α $(0<\alpha<1)$ *with respect to* t *and* x *in* $\overline{D}$, *and Hölder norms* $\left|f^i\right|_{0+\alpha}$ *are uniformly bounded, i.e.,* $f(\cdot,\cdot,s)\in C^{0+\alpha}_S(\overline{D})$.

Assumption (W). *Functions* $f^i(t,x,y,s)$, $i\in S$, *are increasing*[4)] *with respect to the functional argument* s, *i.e., for arbitrary* $s,\tilde{s}\in C_S(\overline{D})$, *there is*

$$s\le\tilde{s} \Longrightarrow f^i(t,x,y,s)\le f^i(t,x,y,\tilde{s}) \quad \textit{for} \quad (t,x)\in\overline{D}, \quad y\in\mathcal{B}(S).$$

Assumption $(\mathbf{W}_+)$**.** *Functions* $f^i(t,x,y,s)$, $i\in S$, *are quasi-increasing with respect to* y *(or satisfy condition* $(W+)$*), that is for arbitrary* $y,\tilde{y}\in\mathcal{B}(S)$ *there is*

$$y \overset{(i)}{\le} \tilde{y} \Longrightarrow f^i(t,x,y,s)\le f^i(t,x,\tilde{y},s) \quad \textit{for} \quad (t,x)\in\overline{D}, \quad s\in C_S(\overline{D}).$$

Assumption (M). *Functions* $f^i(t,x,y,s)$, $i\in S$, *satisfy the following monotonicity condition* (M) *with respect to the functional argument* s*: for every fixed* t, $0\le t\le T$ *and for all functions* $s,\tilde{s}\in C_S(\overline{D})$ *the following implication holds*

$$s \overset{(t)}{\le} \tilde{s} \Longrightarrow f^i(t,x,y,s)\le f^i(t,x,y,\tilde{s}) \quad \textit{for} \quad x\in\overline{G}, \quad y\in\mathcal{B}(S).$$

Assumption (L). *Functions* $f^i(t,x,y,s)$, $i\in S$, *fulfill the Lipschitz condition with respect to* y *and* s, *if for arbitrary* $y,\tilde{y}\in\mathcal{B}(S)$ *and* s, $\tilde{s}\in C_S(\overline{D})$ *the inequality*

$$\left|f^i(t,x,y,s)-f^i(t,x,\tilde{y},\tilde{s})\right| \le L_1\|y-\tilde{y}\|_{\mathcal{B}(S)} + L_2\|s-\tilde{s}\|_0 \quad \textit{for} \quad (t,x)\in\overline{D}$$

holds, where L_1, L_2 *are positive constants.*

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4) Monotonicity of function is understood in the weak sense, i.e., a function $h=h(x)$ is increasing or strictly increasing if $x_1<x_2$ implies $h(x_1)\le h(x_2)$ or $h(x_1)<h(x_2)$, respectively (cp. W. Walter [123]).

Assumption ($\mathbf{L}_l$). *Functions $f^i(t,x,y,s)$, $i \in S$, fulfil the left-hand side Lipschitz condition (resp. right-hand side Lipschitz condition) with respect to y, that is for arbitrary $y, \tilde{y} \in \mathcal{B}(S)$ the inequality*

$$-L_1\|y-\tilde{y}\|_{\mathcal{B}(S)} \leq f^i(t,x,y,s) - f^i(t,x,\tilde{y},\tilde{s}) \quad \textit{for} \quad (t,x) \in \overline{D}, \quad s \in C_S(\overline{D})$$
$$(\text{resp. } f^i(t,x,y,s) - f^i(t,x,\tilde{y},\tilde{s}) \leq L_1\|y-\tilde{y}\|_{\mathcal{B}(S)})$$

hold, where L_1 is a positive constant.

Assumption ($\mathbf{L}_l^i$). *Function $f^i(t,x,y,s)$ for every fixed $i \in S$, satisfies the left-hand side generalized Lipschitz condition with respect to y^i (cp. C.V. Pao [81], pp. 148, 153 and [83], pp. 22, 384, 385) if there exist bounded functions $\underline{l}^i = \underline{l}^i(t,x) \geq 0$ in $\overline{D}$ such that for $y, \tilde{y} \in \mathcal{B}(S)$, $y \leq \tilde{y}$, the following inequality*

$$-\underline{l}^i(t,x)(\tilde{y}^i - y^i) \leq f^i(t,x,\tilde{y},s) - f^i(t,x,y,s) \quad \textit{for} \quad (t,x) \in \overline{D}, \quad s \in C_S(\overline{D})$$

holds.

Assumption ($\mathbf{V}$). *Functions $f^i(t,x,y,s)$, $i \in S$, satisfy the Volterra condition (are Volterra functionals) with respect to the functional argument s, i.e., for arbitrary $(t,x) \in \overline{D}$, $y \in \mathcal{B}(S)$ and for all functions $s, \tilde{s} \in C_S(\overline{D})$ such that $s^j(\bar{t},x) = \tilde{s}^j(\bar{t},x)$ for $0 \leq \bar{t} \leq t$, $j \in S$, there is $f^i(t,x,y,s) = f^i(t,x,y,\tilde{s})$, or shortly*

$$s \overset{(t)}{=} \tilde{s} \Longrightarrow f^i(t,x,y,s) = f^i(t,x,y,\tilde{s}).$$

Assumption ($\mathbf{L}^*$). *Functions $f^i(t,x,y,s)$, $i \in S$, fulfil the following L^*-condition (or the so-called Lipschitz–Volterra condition) with respect to the functional argument s if for arbitrary $s, \tilde{s} \in C_S(\overline{D})$ the inequality*

$$\left|f^i(t,x,y,s) - f^i(t,x,y,\tilde{s})\right| \leq L_3\|s-\tilde{s}\|_{0,t} \quad \textit{for} \quad (t,x) \in \overline{D}, \quad y \in \mathcal{B}(S)$$

holds, where L_3 is a positive constant.

Assumption ($\mathbf{K}$). *We will assume that there exist functions $k^i = k^i(t,x) > 0$, $i \in S$, defined in $\overline{D}$, satisfying the assumption (H_a) with the constant $K_1 > 0$, and such that the function*

$$f_k^i(t,x,y,s) := f^i(t,x,y,s) + k^i(t,x)y^i, \quad \textit{for arbitrary fixed} \quad i, \quad i \in S,$$

is increasing with respect to the variable y^i. Then we will say that $f^i(t,x,y,s)$, $i \in S$ are semi-increasing with respect to y.

Assumption ($\mathbf{K}_{\boldsymbol{\kappa}}$). *We will assume that there exist a constant κ such that each function*

$$f_\kappa^i(t,x,y,s) := f^i(t,x,y,s) + \kappa y^i$$

is increasing with respect to the variable y^i for $i \in S$.

Remark 1.3. *From condition* (L_l^i) *it follows that the function*

$$f_l^i(t,x,y,s) := f^i(t,x,y,s) + \underline{l}^i(t,x)y^i$$

is increasing with respect to the variable y^i, *for* $i \in S$. *If* $\underline{l}^i = \underline{l}^i(t,x)$, $i \in S$, *fullfil the assumption* (H_a), *then this means that the functions* $f^i (i \in S)$ *fulfil the condition* (K) *with* $k^i(t,x) = \underline{l}^i(t,x)$, $i \in S$.

If $f^i(t,x,y,s)$, $i \in S$, *are increasing in* y, *then the condition* (L_l^i) *is satisfied with* $\underline{l}^i \equiv 0$.

If condition (K) *holds and* $\sup\limits_{S\times\overline{D}} \left|k^i(t,x)\right| < \infty$, *then condition* (K_κ) *holds with* $\kappa \geq \sup\limits_{S\times\overline{D}} \left|k^i(t,x)\right|$, *too.*

Moreover, we will assume that:

Assumption $(\mathbf{H}_{\boldsymbol{\phi}})$**.** $\phi \in C_S^{2+\alpha}(\Gamma)$, *where* $0 < \alpha < 1$.

If initial-boundary condition (0.3) is of form (0.4), (0.5), i.e.,

$$\phi(t,x) = \begin{cases} \phi_0(x) & \text{for } t = 0,\ x \in G, \\ \psi(t,x) & \text{for } (t,x) \in \sigma \end{cases}$$

and compatibility conditions (0.6) hold, then we will assume that

Assumption $(\mathbf{H}^*_{\boldsymbol{\phi}})$**.** $\phi_0 \in C_S^{2+\alpha}(G)$, $\psi \in C_S^{2+\alpha}(\sigma)$, *where* $0 < \alpha < 1$.

Remark 1.4. *If* $\phi \in C_S^{2+\alpha}(\Gamma)$ *and the boundary* $\partial G \in C^{2+\alpha}$, *then without loss of generality we can consider the homogeneous initial-boundary condition*

$$z(t,x) = 0 \quad \textit{for} \quad (t,x) \in \Gamma \tag{1.2}$$

for system (0.2).

Indeed, if $\phi(t,x) \not\equiv 0$ *on* Γ, $\phi \in C_S^{2+\alpha}(\Gamma)$ *and* $\partial G \in C^{2+\alpha}$, *then there exists a function* $\Phi \in C_S^{2+\alpha}(\overline{D})$ *such that*

$$\Phi(t,x) = \phi(t,x) \quad \textit{for each} \quad (t,x) \in \Gamma.$$

Let Φ *be a certain given extension of* ϕ *onto* $\overline{D}$ *and* z *be a solution of problem* (0.2), (0.3) *in* $\overline{D}$, $z \in C_S^{2+\alpha}(\overline{D})$. *It is routine to see that the function*

$$\overset{*}{z}(t,x) = z(t,x) - \Phi(t,x)$$

satisfies the following homogeneous problem

$$\begin{cases} \mathcal{F}^i[\overset{*}{z}{}^i](t,x) = \overset{*}{f}{}^i(t,x,\overset{*}{z}(t,x),\overset{*}{z}) & i \in S, \textit{ for } (t,x) \in D, \\ \overset{*}{z}(t,x) = 0 & \textit{for } (t,x) \in \Gamma, \end{cases}$$

where

$$\overset{*}{f}{}^{i}\,(t,x,\overset{*}{z}\,(t,x),\overset{*}{z}) := f^i\left(t,x,\overset{*}{z}\,(t,x)+\Phi(t,x),\overset{*}{z}+\Phi\right)-\mathcal{F}^i[\Phi^i](t,x).$$

Obviously, the functions $\overset{*}{f}{}^{i}$ *have the same property as the functions* f^i. *Accordingly, in what follows, we can confine ourselves to considering homogeneous problem* (0.2), (1.2) *in* $\overline{D}$ *only.*

1.1.5. Interrelations between conditions

Now we prove some relations between monotonicity conditions, Lipschitz conditions and the Volterra condition (cp. S. Brzychczy and J. Janus [29]):

$$\begin{aligned} P_1\colon (M) &\Longrightarrow (W),\\ P_2\colon (M) \text{ and } (L) &\Longrightarrow (L^*),\\ P_3\colon (V) \text{ and } (L) &\Longleftrightarrow (L^*),\\ P_4\colon (V) \text{ and } (W) &\Longleftrightarrow (M). \end{aligned}$$

Proof of P_1. If the functions $f^i(t,x,y,s)$, $i\in S$, satisfy condition (M) then f^i are increasing with respect to s. Indeed, since

$$s \overset{(T)}{\leq} \tilde{s} \Longleftrightarrow s(t,x)\leq \tilde{s}(t,x) \quad \text{for each} \quad t, \quad 0\leq t\leq T, \quad x\in\overline{G},$$

then from condition (M) it follows that

$$f^i(t,x,y,s)\leq f^i(t,x,y,\tilde{s}), \quad i\in S, \quad \text{for each} \quad t, \quad 0\leq t\leq T, \quad x\in\overline{G}.$$

Therefore, f^i are increasing with respect to s.

The reverse implication is not true. Let for example $f^i(t,x,y,s)=s(T,x)$. For every t_0, $0<t_0<T$, there exist functions s, $\tilde{s}\in C_S(\overline{D})$ such that $s \overset{(t_0)}{\leq} \tilde{s}$ and $s(T,x)>\tilde{s}(T,x)$. Hence

$$f^i(t,x,y,s)=s(T,x)>\tilde{s}(T,x)=f^i(t,x,y,\tilde{s}),$$

so f^i does not fulfil condition (M). □

Proof of P_2. When f^i, $i\in S$, satisfy the Lipschitz condition (L) and monotonicity condition (M) then the condition (L^*) holds. Indeed, from condition (M) it follows that $f^i(t,x,y,s)=f^i(t,x,y,\tilde{s})$, $i\in S$, whenever $s(\bar{t},x)=\tilde{s}(\bar{t},x)$ for $0\leq\bar{t}\leq t$, i.e., the condition (V) holds.

For any function $s\in C_S(\overline{D})$ we define a new function $s^{(t)}\in C_S(\overline{D})$ as follows

$$s^{(t)}(\tau,x) := s(\min(\tau,t),x) = \begin{cases} s(\tau,x) & \text{for } 0<\tau\leq t,\ x\in\overline{G},\\ s(t,x) & \text{for } 0<t\leq\tau\leq T,\ x\in\overline{G}. \end{cases}$$

By virtue of (V) and (L), there is

$$\left|f^i(t,x,y,s)-f^i(t,x,y,\tilde{s})\right| = \left|f^i(t,x,y,s^{(t)})-f^i(t,x,y,\tilde{s}^{(t)})\right| \leq$$
$$\leq L\|s^{(t)}-\tilde{s}^{(t)}\|_0 = L\|s-\tilde{s}\|_{0,t}.$$

□

Proof of P_3. The fact that in condition (L^*) there is $\|s-\tilde{s}\|_{0,t}$ means that for functions $s,\tilde{s}\in C_S(\overline{D})$, such that, $s(\bar{t},x)=\tilde{s}(\bar{t},x)$ for $0\leq\bar{t}\leq t$, there is $f^i(t,x,y,s)= = f^i(t,x,y,\tilde{s})$, $i\in S$, i.e., the functions f^i are functionals in s taking same values. Therefore, the functions f^i satisfy Volterra conditions (V), i.e., the functions f^i are functionals in s of the Volterra type. Moreover, if f^i, $i\in S$, satisfy condition (L^*), then Lipschitz condition (L) holds, because $\|s-\tilde{}\|_{0,t}\leq\|s-\tilde{s}\|_0$.

The reverse implication is obvious. □

Proof of P_4. The functions f^i, $i\in S$, satisfy Volterra condition (V) and are increasing with respect to s if and only if f^i, $i\in S$, satisfy condition (M). Indeed, for every fixed t, $0\leq t\leq T$ and for any functions $s,\tilde{s}$ such that $s\overset{(t)}{\leq}\tilde{s}$ we define $s^{(t)}$ and $\tilde{s}^{(t)}$ as in the proof of P_2.

Obviously $s^{(t)}\leq\tilde{s}^{(t)}$ for each t, $0\leq t\leq T$. By virtue of (V) and the fact that f^i, $i\in S$, are increasing we obtain

$$f^i(t,x,y,s)=f^i(t,x,y,s^{(t)})\leq f^i(t,x,y,\tilde{s}^{(t)})=f^i(t,x,y,\tilde{s}),$$

thus condition (M) holds.

The reverse implication follows from the proofs of P_1 and P_2. □

1.1.6. Upper and lower solutions

Definition 1.11. *Functions* $u,v\in C_S^{reg}(\overline{D})$ *satisfying the infinite systems of inequalities*

$$\begin{cases}\mathcal{F}^i[u^i](t,x)\leq f^i(t,x,u(t,x),u),\ i\in S & \text{for } (t,x)\in D,\\ u(t,x)\leq\phi(t,x) & \text{for } (t,x)\in\Gamma,\end{cases}\tag{1.3}$$

$$\begin{cases}\mathcal{F}^i[v^i](t,x)\geq f^i(t,x,v(t,x),v),\ i\in S & \text{for } (t,x)\in D,\\ v(t,x)\geq\phi(t,x) & \text{for } (t,x)\in\Gamma\end{cases}\tag{1.4}$$

are called, respectively, a lower *and an* upper solution *of problem* (0.2), (0.3) *in* $\overline{D}$.

Note that some authors call lower and upper solutions *lower* and *upper functions*, *sub-* and *super-solutions* or *sub-* and *super-functions*, respectively.

In this paper we will always adopt the following fundamental assumption:

Assumption A. *We assume that there exists at least one pair* u_0 *and* v_0 *of a lower and an upper solution of problem* (0.2), (0.3) *in* $\overline{D}$[5], *and* $u_0,v_0\in C_S^{0+\alpha}(\overline{D})$.

[5] More information on the existence of lower and upper solutions can be found in Section 3.2.

Definition 1.12. *A pair of a lower and an upper solution u and v of problem* (0.2)*,* (0.3) *in $\overline{D}$, is called* an ordered pair *(or:* coupled lower and upper solution*), if $u \le v$ in $\overline{D}$.*

We notice that the inequality $u_0 \le v_0$ does not follow directly from inequalities (1.3) and (1.4). Therefore, we will adopt the following assumption:

Assumption $\mathbf{A_0}$. *We assume that there exists at least one ordered pair u_0 and v_0 of a lower and an upper solution of problem* (0.2)*,* (0.3) *in $\overline{D}$, and $u_0, v_0 \in C_S^{0+\alpha}(\overline{D})$.*

Definition 1.13. *For an ordered pair of a lower and an upper solution u_0 and v_0 of problem* (0.2)*,* (0.3) *in $\overline{D}$, we define the* sector *(or* order interval*) $\langle u_0, v_0\rangle$ in the space $C_S(\overline{D})$ as the following* functional interval *formed by u_0 and v_0:*

$$\langle u_0, v_0\rangle := \left\{ w \in C_S(\overline{D})\colon u_0(t,x) \le w(t,x) \le v_0(t,x) \quad \textit{for} \quad (t,x) \in \overline{D} \right\}. \tag{1.5}$$

We define the interval $\langle \underline{m}, \overline{M}\rangle$ in the space $\mathcal{B}(S)$ as follows

$$\langle \underline{m}, \overline{M}\rangle := \left\{ y \in \mathcal{B}(S)\colon \underline{m} \le y \le \overline{M} \right\},$$

where

$$\underline{m}^i = \inf_{\overline{D}} u_0^i(t,x), \quad \underline{m} = \{\underline{m}^i\}_{i\in S}$$
$$\overline{M}^i = \sup_{\overline{D}} v_0^i(t,x), \quad \overline{M} = \{\overline{M}^i\}_{i\in S}.$$

Finally we define the sets

$$\mathcal{K}^* := \left\{ (t,x,s)\colon (t,x) \in \overline{D}, \quad s \in \langle u_0, v_0\rangle \right\}, \tag{1.6}$$
$$\mathcal{K} := \left\{ (t,x,y,s)\colon (t,x) \in \overline{D}, \quad y \in \langle \underline{m}, \overline{M}\rangle, \quad s \in \langle u_0, v_0\rangle \right\}. \tag{1.7}$$

Remark 1.5. *Instead of Assumption A, one may use stronger Assumption A_0. If Assumption A_0 holds, then we define the set $\mathcal{K}$ (or $\mathcal{K}^*$) by* (1.7) *and the other assumptions on the functions f^i may be weakened to hold locally only in the set $\mathcal{K}$ (or $\mathcal{K}^*$). Therefore, all our theorems will be true locally only within the sector $\langle u_0, v_0\rangle$ formed by u_0 and v_0.*

1.2. AUXILIARY THEOREMS AND LEMMAS

1.2.1. Theorem on the existence and uniqueness of solution of linear parabolic initial-boundary value problem in Hölder spaces

Let us consider the following linear parabolic initial-boundary value problem

$$\begin{cases} \mathcal{F}_c^i[u^i](t,x) = g^i(t,x), \; i \in S & \text{for } (t,x) \in D, \\ u(0,x) = \phi_0(x) & \text{for } x \in G, \\ u(t,x) = \psi(t,x) & \text{for } (t,x) \in \sigma, \end{cases} \tag{1.8}$$

with the compatibility conditions (0.6), where

$$\mathcal{F}_c^i := \mathcal{D}_t - \mathcal{L}_c^i,$$
$$\mathcal{L}_c^i := \sum_{j,k=1}^{m} a_{jk}^i(t,x)\mathcal{D}^2_{x_j x_k} + \sum_{j=1}^{m} b_j^i(t,x)\mathcal{D}_{x_j} + c^i(t,x)\mathcal{I}, \quad i \in S,$$

$\mathcal{I}$ is the identity operator and the operators $\mathcal{F}_c^i$, $i \in S$, are uniformly parabolic in $\overline{D}$.

From the theorems on the existence and uniqueness of solutions of the Fourier first initial-boundary value problems for linear parabolic equations in Hölder spaces (see A. Friedman [41], Th. 6 and 7, p. 65 and O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural'ceva [57], pp. 317–321), we directly infer the following theorem.

Theorem 1.1. *Let us consider linear parabolic initial-boundary value problem* (1.8) *and assume that:*

1° *all the coefficients a_{jk}^i, b_j^i and c^i ($j,k = 1,\ldots,n, i \in S$) of the operators $\mathcal{L}_c^i$, $i \in S$, fulfil assumption (H_a);*

2° *the functions g^i, $i \in S$, are uniformly Hölder continuous with respect to t and x in $\overline{D}$, and $g = \{g^i\}_{i\in S} \in C_S^{0+\alpha}(\overline{D})$;*

3° *$\phi_0 \in C_S^{2+\alpha}(G)$ and $\psi \in C_S^{2+\alpha}(\sigma)$;*

4° *the boundary $\partial G \in C^{2+\alpha}$.*

Then problem (1.8) *has the unique solution u and $u \in C_S^{2+\alpha}(\overline{D})$.*

Furthermore, there exists a constant $C > 0$ depending only on the constants μ, K_1, α and on the geometry of the domain D, such that the following a priori estimate of the $(2+\alpha)$*-type holds*

$$\|u\|_{2+\alpha} \le C\left(\|g\|_{0+\alpha} + \|\phi_0\|^G_{2+\alpha} + \|\psi\|^\sigma_{2+\alpha}\right). \tag{1.9}$$

Proof. Observe that system (1.8) has the following property: in the i-th equation, only one unknown function with index i appears. Therefore, system (1.8) is a collection of individual independent equations. Applying the above mentioned theorems, we immediately obtain the estimates

$$\left|u^i\right|_{2+\alpha} \le C_i\left(\left|g^i\right|_{0+\alpha} + \left|\phi_0^i\right|^G_{2+\alpha} + \left|\psi^i\right|^\sigma_{2+\alpha}\right), \quad i \in S, \tag{1.10}$$

in which the constants $C_i > 0$ do not depend on g^i, ϕ_0^i and ψ^i.

From these theorems it follows that constants C_i depend only on the constants μ, K_1, α and on the geometry of the domain D and C_i are uniformly bounded for all $i \in S$. Therefore, there exists a constant $C > 0$ such that $C_i \le C$, for all $i \in S$ (C is independent of both the index i and the functions g^i, ϕ_0^i and ψ^i). Hence, by the definition of the norms in $C_S^{0+\alpha}(\overline{D})$ and $C_S^{2+\alpha}(\overline{D})$, we obtain estimate (1.10). □

1.2.2. Theorem on weak parabolic inequalities

Let us consider an infinite system of semilinear parabolic differential-functional equations of the form

$$\mathcal{F}^i_c[z^i](t,x) = f^i(t,x,z(t,x),z), \quad i \in S, \tag{1.11}$$

where

$$\mathcal{F}^i_c := \mathcal{D}_t - \mathcal{L}^i_c,$$
$$\mathcal{L}^i_c := \sum_{j,k=1}^{m} a^i_{jk}(t,x)\mathcal{D}^2_{x_j x_k} + \sum_{j=1}^{m} b^i_j(t,x)\mathcal{D}_{x_j}$$

and $(t,x) \in (0,T] \times G := D$, $T < \infty$, $G \subset \mathbb{R}^m$, G is an open bounded domain with the boundary $\partial G \in C^{2+\alpha}(0 < \alpha < 1)$, and S is a finite or an infinite set of indices.

Directly from Szarski's theorem on weak parabolic differential-functional inequalities for infinite systems [117], there follows:

Theorem 1.2. Comparison principle. *We assume that:*

1° *the functions $f^i(t,x,y,s)$, $i \in S$, are defined for $(t,x,y,s) \in \overline{D} \times \mathcal{B}(S) \times C_S(\overline{D})$ and the operators $\mathcal{F}^i$, $i \in S$, are uniformly parabolic in $\overline{D}$;*

2° *the functions $f^i(t,x,y,s), i \in S$, satisfy conditions $(W+)$, (L) with respect to y (precisely: satisfy the right-hand side Lipschitz conditions), and conditions (W), (L) and (V) with respect to s;*

3° *the functions $u, v \in C^{reg}_S(\overline{D})$ fulfil the following infinite systems of inequalities:*

$$\mathcal{F}^i[u^i](t,x) \leq f^i(t,x,u(t,x),u),$$
$$\mathcal{F}^i[v^i](t,x) \geq f^i(t,x,v(t,x),v), \;\; i \in S, \quad \textit{for} \quad (t,x) \in D, \textit{and}$$

4° $u(t,x) \leq v(t,x) \quad$ *for* $\quad (t,x) \in \Gamma$.

Under these assumptions, there is

$$u(t,x) \leq v(t,x) \quad \textit{for} \quad (t,x) \in \overline{D}.$$

Corollary 1.1. *If u and v are a lower and an upper solution of problem* (0.2), (0.3) *in $\overline{D}$, respectively, z is a regular solution of this problem and assumptions (W), $(W+)$, (L) and (V) hold, then by the Theorem 1.2 there is*

$$u(t,x) \leq z(t,x) \leq v(t,x) \quad \textit{for} \quad (t,x) \in \overline{D}. \tag{1.12}$$

If the functions u_0 and v_0 are given by Assumption A and assumptions (W), (W_+), (L), (V) hold, then by (1.12) *there holds*

$$u_0(t,x) \leq v_0(t,x) \quad \textit{for} \quad (t,x) \in \overline{D}. \tag{1.13}$$

This means that u_0 and v_0 form an ordered pair. Therefore, Assumption A_0 holds.

As a corollary of the comparison principle for the finite systems (see J. Szarski [113], p. 210) we obtain the maximum principle for finite systems (cp. M. H. Protter and H. F. Weinberger [93], pp. 189–190, R. P. Sperb [110], pp. 20–24).

Theorem 1.3. Maximum principle. *Let we consider the finite system of parabolic differential-functional inequalities of the form*

$$\mathcal{F}^i[z^i](t,x) \leq f^i(t,x,z(t,x),z), \quad i \in S \quad \text{in } D, \tag{1.14}$$

where the operators $\mathcal{F}^i$, $i \in S$, are uniformly parabolic in $\overline{D}$ and the functions $f^i(t,x,y,s)$, $i \in S$, are defined for $(t,x,y,s) \in \overline{D} \times \mathcal{B}(S) \times C_S(\overline{D})$.

Assume that:

1° *the functions $f^i(t,x,y,s)$, $i \in S$, satisfy conditions $(W+)$, (L) with respect to y (precisely: satisfy the right-hand side Lipschitz conditions), and conditions (W), (L) and (V) with respect to s;*

2° *suppose u is a regular solution of (1.14) in $\overline{D}$, satisfying inequalities*

$$u(0,x) \leq C \quad \text{for} \quad x \in G,$$
$$u(t,x) \leq C \quad \text{for} \quad (t,x) \in \sigma,$$

where $C = \{C^i\}_{i \in S} = const.$;

3° *assume finally that*

$$f^i(t,x,C,C) \leq 0, \quad (i \in S), \quad \text{for} \quad (t,x) \in \overline{D}.$$

Under these assumptions, there is

$$u(t,x) \leq C, \quad \text{for} \quad (t,x) \in \overline{D}.$$

In methods of upper and lower solutions for parabolic equations, the theorem on weak inequalities and the maximum principle plays a fundamental role in the construction of monotone approximation sequences. This role is reflected in the so-called *positivity lemma* (cp. C.V. Pao [83], p. 54), which directly follows from the maximum principle.

Lemma 1.1. Positivity lemma. *Let $u \in C_S^{reg}(\overline{D})$ and the following inequalities hold:*

$$\mathcal{F}_c^i[u^i](t,x) \geq 0, \quad i \in S, \quad \text{for} \quad (t,x) \in D,$$
$$u(0,x) \geq 0 \quad \text{for} \quad x \in G,$$
$$u(t,x) \geq 0 \quad \text{for} \quad (t,x) \in \sigma.$$

Then

$$u(t,x) \geq 0 \quad \text{for} \quad (t,x) \in \overline{D}.$$

Remark 1.6. *The comparison principle holds also in unbounded domains $\Omega \subset \mathbb{R}^{m+1}$ for functions w satisfying the estimate $|w(t,x)| \leq Me^{Kx^2}$ in Ω for some constants M, $K > 0$. This theorem does not hold in unbounded domains unless a growth condition holds (see example given by A. Tychonoff [120]).*

CHAPTER 2. FUNDAMENTAL MONOTONE ITERATIVE METHODS

2.1. METHOD OF DIRECT ITERATIONS

Let us consider infinite system of differential-functional equations (0.2) with the homogeneous initial-boundary condition (1.2) in $\overline{D}$, i.e., the problem

$$\begin{cases} \mathcal{F}^i[z^i](t,x) = f^i(t,x,z(t,x),z), \ i \in S & \text{for } (t,x) \in D, \\ z(t,x) = 0 & \text{for } (t,x) \in \Gamma. \end{cases} \tag{2.1}$$

The following theorem holds true.

Theorem 2.1. *Let assumptions A and* (H_a), (H_f), (L), (W), $(W+)$, (V) *hold in the set* $\mathcal{K}$. *If we define the successive terms of the approximation sequences* $\{u_n\}$ *and* $\{v_n\}$ *as regular solutions in* $\overline{D}$ *of the following infinite systems of linear parabolic differential equations*

$$\mathcal{F}^i[u_n^i](t,x) = f^i(t,x,u_{n-1}(t,x),u_{n-1}), \tag{2.2}$$

$$\mathcal{F}^i[v_n^i](t,x) = f^i(t,x,v_{n-1}(t,x),v_{n-1}), \ i \in S, \tag{2.3}$$

for $n = 1,2,\ldots$ *in* D *with homogeneous initial-boundary condition* (1.2), *then:*

1° $\{u_n\}$, $\{u_n\}$ *are well defined and* $u_n, v_n \in C_S^{2+\alpha}(\overline{D})$ *for* $n = 1,2,\ldots$;

2° *the inequalities*

$$u_0(t,x) \le u_n(t,x) \le u_{n+1}(t,x), \quad n = 1,2,\ldots, \tag{2.4}$$

hold for $(t,x) \in \overline{D}$ *and the functions* $u_n (n = 1,2,\ldots)$ *are lower solutions of problem* (2.1) *in* $\overline{D}$, *and analogously*

$$v_{n+1}(t,x) \le v_n(t,x) \le v_0(t,x), \quad n = 1,2,\ldots, \tag{2.5}$$

hold for $(t,x) \in \overline{D}$ *and* v_n $(n = 1,2,\ldots)$ *are upper solutions of problem* (2.1) *in* $\overline{D}$;

3° *the inequalities*

$$u_n(t,x) \le v_n(t,x), \quad n = 1,2,\ldots \tag{2.6}$$

hold for $(t,x) \in \overline{D}$;

4° *the following estimate is true*

$$v_n^i(t,x) - u_n^i(t,x) \le N_0 \frac{[(L_1+L_2)t]^n}{n!}, \quad i \in S, \ n = 1,2,\ldots \tag{2.7}$$

for $(t,x) \in \overline{D}$, *where* $N_0 = \|v_0 - u_0\|_0 = const < \infty$;

5° $\lim\limits_{n\to\infty} \left[v_n^i(t,x) - u_n^i(t,x)\right] = 0$ *uniformly in* $\overline{D}$, $i \in S$;

6° *the function*

$$z = z(t,x) = \lim_{n\to\infty} u_n(t,x)$$

is the unique regular solution of problem (2.1) *within the sector* $\langle u_0, v_0\rangle$, *and* $z \in C_S^{2+\alpha}(\overline{D})$.

Before going into the proof of the theorem, we will introduce the nonlinear Nemytskiĭ operator and prove some lemmas. Since the proofs are straightforward and similar for the lower and upper solutions, we present proof for the upper solution only. We recall that from Assumption A with (W), $(W+)$, (L) and (V) there follows Assumption A_0 (by Corollary 1.1) and the other assumption on the functions f^i may by weakened to hold locally only in the set $\mathcal{K}$ (Remark 1.5).

Let $\beta \in C_S(\overline{D})$ be a sufficiently regular function. Denote by $\mathcal{P}$ the operator

$$\mathcal{P}\colon \beta \mapsto \mathcal{P}[\beta] = \gamma,$$

where γ is the (supposedly unique) solution of the linear initial-boundary value problem

$$\begin{cases} \mathcal{F}^i[\gamma^i](t,x) = f^i(t,x,\beta(t,x),\beta),\ i\in S & \text{for } (t,x)\in D,\\ \gamma(t,x) = 0 & \text{for } (t,x)\in\Gamma. \end{cases} \tag{2.8}$$

The operator $\mathcal{P}$ is the composition of the nonlinear Nemytskiĭ[6)] operator $\mathbf{F} = \{\mathbf{F}^i\}_{i\in S}$ generated by the functions $f^i(t,x,y,s)$, $i\in S$, and defined for any $\beta\in C_S(\overline{D})$ as follows

$$\mathbf{F}\colon \beta \mapsto \mathbf{F}[\beta] = \delta,$$

where

$$\mathbf{F}^i[\beta](t,x) := f^i(t,x,\beta(t,x),\beta) = \delta^i(t,x), \quad i\in S, \tag{2.9}$$

and the operator

$$\mathcal{G}\colon \delta \mapsto \mathcal{G}[\delta] = \gamma,$$

where γ is the (supposedly unique) solution of the linear initial-boundary value problem

$$\begin{cases} \mathcal{F}^i[\gamma^i](t,x) = \delta^i(t,x),\ i\in S & \text{for } (t,x)\in D,\\ \gamma(t,x) = 0 & \text{for } (t,x)\in\Gamma. \end{cases} \tag{2.10}$$

Hence

$$\mathcal{P} = \mathcal{G}\circ\mathbf{F}.$$

Lemma 2.1. *If* $\beta \in C_S^{0+\alpha}(\overline{D})$ *and the function* $f = \{f^i\}_{i\in S}$, *generating the Nemytskiĭ operator* $\mathbf{F}$ *satisfies conditions* (H_f) *and* (L), *then*

$$\mathbf{F}\colon C_S^{0+\alpha}(\overline{D}) \ni \beta \mapsto \mathbf{F}[\beta] = \delta \in C_S^{0+\alpha}(\overline{D}).$$

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6) The nonlinear Nemytskiĭ operator plays an important role in the theory of nonlinear equations. Extensive informations about it can be found in the books by M. A. Krasnosel'skiĭ [47], M. M. Vaĭnberg [121] and the monograph by J. Appell and P. P. Zabreĭko [4].

Proof. Because $\beta \in C_S^{0+\alpha}(\overline{D})$, then

$$\left|\beta^i(t,x) - \beta^i(t',x')\right| \leq H(\beta)\left(|t-t'|^{\frac{\alpha}{2}} + \|x-x'\|^{\alpha}\right)$$

for (t,x), $(t',x') \in \overline{D}$ and all $i \in S$, where $H(\beta) > 0$ is some constant independent of index i.

From (H_f) and (L) it follows that

$$\begin{aligned}
&\left|\delta^i(t,x) - \delta^i(t',x')\right| = \left|\mathbf{F}^i[\beta](t,x) - \mathbf{F}^i[\beta](t',x')\right| = \\
&\quad = \left|f^i(t,x,\beta(t,x),\beta) - f^i(t',x',\beta(t',x'),\beta)\right| \leq \\
&\quad \leq \left|f^i(t,x,\beta(t,x),\beta) - f^i(t',x',\beta(t,x),\beta)\right| + \\
&\qquad + \left|f^i(t',x',\beta(t,x),\beta) - f^i(t',x',\beta(t',x'),\beta)\right| \leq \\
&\quad \leq H(f)\left(|t-t'|^{\frac{\alpha}{2}} + \|x-x'\|^{\alpha}\right) + L_1\|\beta(t,x) - \beta(t',x')\|_{\mathcal{B}(S)} \leq \\
&\quad \leq H(f)\left(|t-t'|^{\frac{\alpha}{2}} + \|x-x'\|^{\alpha}\right) + L_1 H(\beta)\left(|t-t'|^{\frac{\alpha}{2}} + \|x-x'\|^{\alpha}\right) \leq \\
&\quad \leq H^*\left(|t-t'|^{\frac{\alpha}{2}} + |x-x'|^{\alpha}\right)
\end{aligned}$$

where $H^* = H(f) + L_1 H(\beta)$ for all $(t,x), (t',x') \in \overline{D}$, $i \in S$. Therefore, $\mathbf{F}[\beta] \in$ $\in C_S^{0+\alpha}(\overline{D})$. □

From Theorem 1.1 and Lemma 2.1, the next lemma follows directly.

Lemma 2.2. *If* $\delta \in C_S^{0+\alpha}(\overline{D})$ *and all the coefficients of the operators* $\mathcal{L}^i$, $i \in S$, *satisfy condition* (H_a), *then problem* (2.10) *has the unique regular solution* γ *and* $\gamma \in C_S^{2+\alpha}(\overline{D})$.

Corollary 2.1. *From Lemmas 2.1 and 2.2 it follows that*

$$\mathcal{P} = \mathcal{G} \circ \mathbf{F} \colon C_S^{0+\alpha}(\overline{D}) \ni \beta \mapsto \mathcal{P}[\beta] = \gamma \in C_S^{2+\alpha}(\overline{D}).$$

Lemma 2.3. *Let all the assumptions of Lemmas 2.1 and 2.2 hold,* β *be an upper solution and* α *be a lower solution of problem* (2.1) *in* $\overline{D}$, α, $\beta \in \langle u_0, v_0\rangle$ *and conditions* (W), $(W+)$ *hold. Then*

$$\alpha(t,x) \leq \mathcal{P}[\beta](t,x) \leq \beta(t,x) \quad \text{in} \quad \overline{D} \tag{2.11}$$

and $\gamma = \mathcal{P}[\beta]$ *is an upper solution of problem* (2.1) *in* $\overline{D}$, *and analogously*

$$\alpha(t,x) \leq \mathcal{P}[\alpha](t,x)) \leq \beta(t,x) \quad \text{in } \overline{D} \tag{2.12}$$

and $\eta = \mathcal{P}[\alpha]$ *is a lower solution of problem* (2.1) *in* $\overline{D}$.

Proof. If β is an upper solution, then by virtue of (1.4) there is

$$\mathcal{F}^i[\beta^i](t,x) \geq f^i(t,x,\beta(t,x),\beta), \; i \in S, \text{ in } D.$$

From the definition (2.8) of the operator $\mathcal{P}$ it follows that

$$\mathcal{F}^i[\gamma^i](t,x) = f^i(t,x,\beta(t,x),\beta), \quad i \in S \quad \text{in } D.$$

Therefore

$$\mathcal{F}^i[\beta^i - \gamma^i](t,x) \geq 0, \quad i \in S, \quad \text{in} \quad D$$

and

$$\beta(t,x) - \gamma(t,x) = 0 \quad \text{on} \quad \Gamma.$$

Hence, by Lemma 1.1 there is

$$\beta(t,x) - \gamma(t,x) \leq 0 \quad \text{in } \overline{D}$$

so

$$\gamma(t,x) = \mathcal{P}[\beta](t,x) \leq \beta(t,x) \quad \text{in} \quad \overline{D}.$$

From (2.3), (2.8) and conditions (W), $(W+)$, we obtain

$$\mathcal{F}^i[\gamma^i](t,x) - f^i(t,x,\gamma(t,x),\gamma) = f^i(t,x,\beta(t,x),\beta) - f^i(t,x,\gamma(t,x),\gamma) \geq 0$$

in D, $i \in S$, and

$$\gamma(t,x) = 0 \quad \text{on} \quad \Gamma.$$

From Corollary 2.1 we infer that γ is a regular function and $\gamma \in C_S^{2+\alpha}(\overline{D})$, so it is an upper solution of problem (2.1) in $\overline{D}$ and from Corollary 1.1 there follows

$$\alpha(t,x) \leq \mathcal{P}[\beta](t,x) \leq \beta(t,x) \quad \text{in } \overline{D}.$$

□

Lemma 2.4. *If all the assumptions of Lemma 2.3 hold, $\alpha, \beta \in \langle u_0, v_0 \rangle$ and $\alpha(t,x) \leq \beta(t,x)$ in $\overline{D}$, then*

$$\mathcal{P}[\alpha](t,x) \leq \mathcal{P}[\beta](t,x) \quad \text{in } \overline{D} \tag{2.13}$$

i.e., the operator $\mathcal{P}$ is isotone in the sector $\langle u_0, v_0 \rangle$.

Proof. Let $\eta = \mathcal{P}[\alpha]$ and $\gamma = \mathcal{P}[\beta]$, then by (2.8)

$$\mathcal{F}^i[\eta^i](t,x) = f^i(t,x,\alpha(t,x),\alpha), \quad i \in S, \text{ in } D,$$
$$\mathcal{F}^i[\gamma^i](t,x) = f^i(t,x,\beta(t,x),\beta), \quad i \in S, \text{ in } D,$$

and

$$\eta(t,x) = \gamma(t,x) = 0 \quad \text{on} \quad \Gamma.$$

By (W), $(W+)$, there is

$$\mathcal{F}^i[\eta^i - \gamma^i](t,x) = f^i(t,x,\beta(t,x),\beta) - f^i(t,x,\alpha(t,x),\alpha) \geq 0 \quad i \in S \text{ in } D,$$

and

$$\gamma(t,x) - \eta(t,x) = 0 \quad \text{on} \quad \Gamma.$$

By Lemma 1.1, there is

$$\gamma(t,x) \geq \eta(t,x), \quad \text{in } \overline{D}, \quad \text{i.e.,} \quad \mathcal{P}[\alpha](t,x) \leq \mathcal{P}[\beta](t,x) \quad \text{in } \overline{D}.$$

This means that the operator $\mathcal{P}$ is isotone in the sector $\langle u_0, v_0 \rangle$. □

From Corollary 2.1, Lemmas 2.3 and 2.4, the next corollary follows

Corollary 2.2. *If all the assumptions of Lemma 2.3 hold and if α and β are a lower and an upper solution of problem* (2.1) *in $\overline{D}$, respectively, α, $\beta \in \langle u_0, v_0 \rangle$, and $\alpha(t,x) \leq \beta(t,x)$ then*

$$\alpha(t,x) \leq \mathcal{P}[\alpha](t,x) \leq \mathcal{P}[\beta](t,x) \leq \beta(t,x) \quad \text{in } \overline{D}. \tag{2.14}$$

This means that $\mathcal{P}[\langle \alpha, \beta \rangle] \subset \langle \alpha, \beta \rangle$.

Proof. Starting from the lower solution u_0 and the upper solution v_0, we define by induction two sequences of functions $\{u_n\}$ and $\{v_n\}$ as regular solutions of systems (2.2) and (2.4). Therefore we have

$$\begin{aligned} u_1 &= \mathcal{P}[u_0], \qquad u_n = \mathcal{P}[u_{n-1}], \\ v_1 &= \mathcal{P}[v_0], \qquad v_n = \mathcal{P}[v_{n-1}] \quad \text{for} \quad n = 1, 2, \ldots . \end{aligned}$$

From Lemmas 2.1, 2.2 and 2.4 it follows that u_n and v_n, for $n = 1, 2, \ldots$, are well defined and are the lower and the upper solutions of problem (2.1) in $\overline{D}$, respectively.

Using mathematical induction, from Lemma 2.3, we obtain

$$u_n(t,x) \leq \mathcal{P}[u_n](t,x) = u_{n+1}(t,x), \quad n = 1, 2, \ldots$$

and

$$v_{n+1}(t,x) = \mathcal{P}[v_n](t,x) \leq v_n(t,x), \quad n = 1, 2, \ldots, \quad \text{for} \quad (t,x) \in \overline{D}.$$

Therefore, the inequalities (2.4) and (2.5) hold.

Analogously, using mathematical induction, from Lemma 2.4 and Assumption A_0 we obtain inequalities (2.6), too.

Using mathematical induction, we will prove the inequalities

$$0 \leq v_n^i(t,x) - u_n^i(t,x) := w_n^i(t,x) \leq N_0 \frac{[(L_1 + L_2)t]^n}{n!}, \quad i \in S, \tag{2.15}$$

$n = 1, 2, \ldots$ for $(t,x) \in \overline{D}$.

It is obvious that inequality (2.15) holds for w_0. Let inequality (2.15) hold for w_n. The functions $f^i(t,x,y,s)$, $i \in S$, fulfil Lipschitz condition (L) with respect to y

and s, and condition (V). Therefore (see property $\mathcal{P}_3$, p. 43), f^i fulfil L^*-condition. By (2.2), (2.3) and (2.15), there is

$$\mathcal{F}^i[w^i_{n+1}](t,x) = f^i(t,x,v_n(t,x),v_n) - f^i(t,x,u_n(t,x),u_n) \leq$$
$$\leq L_1\|w_n(t,x)\|_{\mathcal{B}(S)} + L_2\|w_n\|_{0,t}.$$

By the definition of the norm $\|\cdot\|_{0,t}$ in the space $C_S(\overline{D})$ and by inequality (2.15),

$$\|w_n\|_{0,t} \leq \frac{[(L_1+L_2)t]^n}{n!}$$

so we finally obtain

$$\mathcal{F}^i[w^i_{n+1}] \leq N_0\frac{(L_1+L_2)^{n+1}t^n}{n!}, \quad i \in S, \quad \text{for} \quad (t,x) \in D, \tag{2.16}$$

and

$$w_{n+1}(t,x) = 0 \quad \text{for} \quad (t,x) \in \Gamma. \tag{2.17}$$

Let us consider the comparison system of equations

$$\mathcal{F}^i[M^i_{n+1}] = N_0\frac{(L_1+L_2)^{n+1}t^n}{n!}, \quad i \in S, \quad \text{for} \quad (t,x) \in D, \tag{2.18}$$

with the initial-boundary condition

$$M_{n+1}(t,x) \geq 0 \quad \text{for} \quad (t,x) \in \Gamma. \tag{2.19}$$

It is obvious that the functions

$$M^i_{n+1}(t,x) = N_0\frac{[(L_1+L_2)t]^{n+1}}{(n+1)!}, \quad i \in S, \quad \text{for} \quad (t,x) \in \overline{D}$$

are the regular solutions of the comparison problem (2.18), (2.19) in $\overline{D}$.

Applying the theorem on weak differential inequalities of parabolic type (J. Szarski [111], Th. 64.1, pp. 195) to systems (2.16), (2.17) and (2.18), (2.19), we obtain

$$w^i_{n+1}(t,x) \leq M^i_{n+1}(t,x) = N_0\frac{[(L_1+L_2)t]^{n+1}}{(n+1)!}, \quad i \in S, \quad \text{for} \quad (t,x) \in \overline{D},$$

so the induction step is proved and so is inequality (2.15).

As a direct conclusion from formula (2.15) we obtain

$$\lim_{n\to\infty}[v^i_n(t,x) - u^i_n(t,x)] = 0 \quad \text{uniformly in} \quad \overline{D},\ i \in S. \tag{2.20}$$

The sequences of functions $\{u_n(t,x)\}$ and $\{v_n(t,x)\}$ are monotonous and bounded, and (2.20) holds, so there exists a continuous function $U = U(t,x)$ in $\overline{D}$ such that

$$\lim_{n\to\infty} u^i_n(t,x) = U^i(t,x) \text{ and } \lim_{n\to\infty} v^i_n(t,x) = U^i(t,x) \text{ uniformly in } \overline{D},\ i \in S. \tag{2.21}$$

Since functions f^i, $i \in S$, are monotonous (conditions (W), $(W+)$), from (2.4) it follows that the functions $f^i(t,x,u_{n-1}(t,x),u_{n-1})$, $i \in S$, are uniformly bounded in D with respect to n. Hence we conclude by Theorem 1.1 that all the functions $u_n \in \in C_S^{2+\alpha}(\overline{D})$ for $n = 1, 2, \ldots$ satisfy the Hölder condition with a constant independent of n. Hence $U \in C_S^{0+\alpha}(\overline{D})$.

If we now consider the system of equations

$$\mathcal{F}^i[z^i](t,x) = f^i(t,x,U(t,x),U) := \mathbf{F}^i[U](t,x), \quad i \in S, \quad \text{for} \quad (t,x) \in D, \qquad (2.22)$$

with initial-boundary condition (1.2), then by Lemma 2.1 there is $\mathbf{F}^i[U] \in C_S^{0+\alpha}(\overline{D})$. Therefore, by virtue of Lemma 2.2 this problem has the unique regular solution z, and $z \in C_S^{2+\alpha}(\overline{D})$.

Let us now consider systems (2.2) and (2.22) together. Let us apply, to these systems, J. Szarski's theorem ([111], Th. 51.1, pp. 147) on the continuous dependence of the solution of the first problem on the initial-boundary values and on the right-hand sides of systems. Since the functions f^i, $i \in S$, satisfy the Lipschitz condition (L), by (2.21), there is

$$\lim_{n\to\infty} f^i(t,x,u_n(t,x),u_n) = f^i(t,x,U(t,x),U) \quad \text{uniformly in} \quad \overline{D},\ i \in S.$$

Hence

$$\lim_{n\to\infty} u_n^i(t,x) = z^i(t,x), \quad i \in S. \qquad (2.23)$$

By virtue of (2.21) and (2.23),

$$z = z(t,x) = U(t,x) \quad \text{for} \quad (t,x) \in \overline{D}$$

is the regular solution of problem (2.1) in $\overline{D}$ and $z \in C_S^{2+\alpha}(\overline{D})$.

The uniqueness of the solution follows directly from J. Szarski's uniqueness criterion [116] (cp. B. Kraśnicka [50] and D. Jaruszewska-Walczak [42]). It also follows directly from inequality (2.15). Because the assumptions of our theorem hold only in the set $\mathcal{K}$, then the uniqueness of a solution is ensured only with respect to the given upper and lower solutions, and it does not rule out the existence of other solutions outside the sector $\langle u_0, v_0 \rangle$.

Thus the theorem is proved. □

Remark 2.1. *In the case of estimate* (2.15) *we say that the sequences of successive approximations* $\{u_n\}$ *and* $\{v_n\}$ *defined by* (2.2) *and* (2.3) *converge to the searched solution* z *with the power speed.*

Remark 2.2. *The convergence of the method of direct iterations may also be derived from the comparison theorem proved in J. Szarski's monograph ([111], Th. 49.1, p. 139). The comparison theorem allows us to prove that the sequences of direct iterations* $\{u_n\}$ *and* $\{v_n\}$ *are Cauchy sequences. Thus the monotonicity assumption would not be required. However, this way we would not obtain estimate* (2.7)*, which we are going to use later one.*

From Theorem 1.2, as a particular case there follows the theorem on the existence and uniqueness of a solution of the Fourier first problem for infinite system of semilinear differential equations of parabolic type. This theorem is a generalization of the well-known T. Kusano theorem ([54], p. 113, Th. 2.1) to the case of infinite system.

Theorem 2.2. *Let us consider the first initial-boundary value problem for the infinite system of semilinear parabolic differential equations of the form*

$$\begin{cases} \mathcal{F}^i[z^i](t,x) = g^i(t,x,z(t,x)), \ i \in S, & \text{for } (t,x) \in D, \\ z(t,x) = 0 & \text{for } (t,x) \in \Gamma. \end{cases} \tag{2.24}$$

Let the following assumption be satisfied:

1° *the operators $\mathcal{F}^i$, $i \in S$, are uniformly parabolic in $\overline{D}$;*

2° *assumption A holds;*

3° *all the coefficients of the operators $\mathcal{F}^i$, $i \in S$, fulfil condition (H_a);*

4° *the functions $g^i(t,x,y)$, $i \in S$, are defined for $(t,x,y) \in \overline{D} \times \mathcal{B}(S)$, satisfy assumptions (H_f), Lipschitz condition (L) and condition (W) with respect to y;*

5° *the above assumptions hold locally in the set $\mathcal{K}$ formed by u_0 and v_0.*

Under these assumptions, problem (2.24) *possesses the unique regular solution z within the sector $\langle u_0, v_0 \rangle$, and $z \in C_S^{2+\alpha}(\overline{D})$.*

2.2. CHAPLYGIN METHOD

We consider problem (2.1) in the domain $\overline{D}$. To solve this problem, we now apply another monotone iterative method, namely the Chaplygin method, in which we use the linearization with respect to the nonfunctional argument y only. The main difference between the Chaplygin method and the previous method of direct iteration lies in the definitions of the approximation sequences. This method will require stronger assumptions on the functions f^i, i.e., the convexity assumption, but on the other hand, the Chaplygin method yields sequences of successive approximations converging to the solution searched for more quickly than the iterative sequences $\{u_n\}$ and $\{v_n\}$ constructed with the previous method, under the obvious assumption that both iterative methods start from the same pair of a lower u_0 and an upper v_0 solution, whose existence we assume. This convergence rate is quadratic.

We assume that assumption A holds and the functions $f^i(t,x,y,s)$, $i \in S$, satisfy conditions (H_f), (L), (W) and (V) with respect to y and s in the set $\mathcal{K}$.

We additionally assume that each function $f^i(t,x,y,s)$, $i \in S$, has the continuous derivatives $\mathcal{D}_{y^i} f^i := \frac{\partial f^i}{\partial y^i} := f^i_{y^i}(t,x,y,s)$, $i \in S$ which satisfy the following assumptions:

$(\mathbf{H_p})$ fulfil condition (H_f);

$(\mathbf{L_p})$ satisfy the Lipschitz condition with respect to y and s;

$(\mathbf{W_p})$ are increasing with respect to y and s.

Theorem 2.3. *Let assumptions A and* (H_a), (H_f), (H_p), (W), (W_p), (L), (L_p), (V) *hold in the set* $\mathcal{K}$. *Let us assume that the successive terms of approximation sequences* $\{\hat{u}_n\}$ *and* $\{\hat{v}_n\}$ *are defined as regular solutions in* $\overline{D}$ *of the following infinite systems of linear parabolic differential equations*

$$\begin{aligned}\mathcal{F}^i[\hat{u}^i_n](t,x) &= f^i(t,x,\hat{u}_{n-1}(t,x),\hat{u}_{n-1})+\\ &\quad + f^i_{y^i}(t,x,\hat{u}_{n-1}(t,x),\hat{u}_{n-1})\cdot\left[\hat{u}^i_n(t,x)-\hat{u}^i_{n-1}(t,x)\right],\end{aligned} \tag{2.25}$$

$$\begin{aligned}\mathcal{F}^i[\hat{v}^i_n](t,x) &= f^i(t,x,\hat{v}_{n-1}(t,x),\hat{v}_{n-1})+\\ &\quad + f^i_{y^i}(t,x,\hat{v}_{n-1}(t,x),\hat{v}_{n-1})\cdot[\hat{v}^i_n(t,x)-\hat{v}^i_{n-1}(t,x)],\quad i\in S,\end{aligned} \tag{2.26}$$

for $n = 1,2,\ldots$ *in* $\overline{D}$ *with homogeneous initial-boundary condition* (1.2) *and let* $\hat{u}_0 = u_0$, $\hat{v}_0 = v_0$.

Then:

1° $\{\hat{u}_n\}$, $\{\hat{v}_n\}$ *are well defined and* $\hat{u}_n, \hat{v}_n \in C^{2+\alpha}_S(\overline{D})$ *for* $n = 1,2,\ldots$;

2° *the inequalities*

$$\begin{aligned}&u_0(t.x)\le\hat{u}_n(t,x)\le\hat{u}_{n+1}(t,x)\le\hat{v}_{n+1}(t,x)\le\hat{v}_n(t,x)\le v_0(t,x),\\ &n=1,2,\ldots\end{aligned} \tag{2.27}$$

hold for $(t,x)\in\overline{D}$, *and the functions* $\hat{u}_n$ *and* $\hat{v}_n$ *for* $n=1,2,\ldots$, *are lower and upper solutions of problem* (2.1) *in* $\overline{D}$, *respectively;*

3° *the following inequalities*

$$u_n(t,x)\le\hat{u}_n(t,x)\le\hat{v}_n(t,x)\le v_n(t,x),\quad n=1,2,\ldots, \tag{2.28}$$

hold for $(t,x)\in\overline{D}$, *where the sequences* $\{u_n\}$ *and* $\{u_n\}$ *are defined by* (2.2), (2.3);

4° *the following estimate*

$$\hat{v}^i_n(t,x)-\hat{u}^i_n(t,x)\le N_0\frac{[(L_1+L_2)t]^n}{n!},\quad i\in S,\ n=1,2,\ldots \tag{2.29}$$

holds for $(t,x)\in\overline{D}$, *where* $N_0=\|v_0-u_0\|_0 = const<\infty$;

5°

$$\lim_{n\to\infty}\left[\hat{v}^i_n(t,x)-\hat{u}^i_n(t,x)\right]=0\quad \textit{uniformly in}\quad \overline{D},\ i\in S;$$

6° *the function*

$$z=z(t,x)=\lim_{n\to\infty}\hat{u}_n(t,x)$$

is the unique regular solution of problem (2.1) *within the sector* $\langle u_0,v_0\rangle$, *and* $z\in C^{2+\alpha}_S(\overline{D})$.

Before we begin proving Theorem, we introduce the Nemytskiĭ nonlinear operators and prove some lemmas.

Let $\beta \in C_S(\overline{D})$ be a sufficiently regular function. Let $\hat{\mathcal{P}}$ denote the operator

$$\hat{\mathcal{P}}\colon \beta \mapsto \hat{\mathcal{P}}[\beta] = \gamma,$$

where γ is the (supposedly unique) solution of the following problem

$$\begin{cases} \mathcal{F}^i[\gamma^i](t,x) + f^i_{y^i}(t,x,\beta(t,x),\beta) \cdot \gamma^i(t,x) = \\ = f^i(t,x,\beta(t,x),\beta) + f^i_{y^i}(t,x,\beta(t,x),\beta) \cdot \beta^i(t,x), \;\; i \in S, \text{ for } (t,x) \in D, \\ \gamma(t,x) = 0 \text{ for } (t,x) \in \Gamma. \end{cases} \tag{2.30}$$

It is convenient to define two Nemytskiĭ operators related to the functions $f^i(t,x,y,s)$ and $f^i_{y^i}(t,x,y,s)$, $i \in S$, to examine them separately. They are: the operator $\mathbf{C} = \{\mathbf{C}^i\}_{i \in S}$

$$\mathbf{C}\colon \beta \mapsto \mathbf{C}[\beta] = \eta,$$

where

$$\mathbf{C}^i[\beta](t,x) := f^i_{y^i}(t,x,\beta(t,x),\beta) = \eta^i(t,x), \quad i \in S \tag{2.31}$$

and the operator $\hat{\mathbf{F}} = \{\hat{\mathbf{F}}^i\}_{i \in S}$

$$\hat{\mathbf{F}}\colon \beta \mapsto \hat{\mathbf{F}}[\beta] = \delta,$$

where

$$\begin{aligned} \hat{\mathbf{F}}^i[\beta](t,x) &:= f^i(t,x,\beta(t,x),\beta) + f^i_{y^i}(t,x,\beta(t,x),\beta) \cdot \beta^i(t,x) = \\ &= f^i(t,x,\beta(t,x),\beta) + \mathbf{C}^i[\beta](t,x) \cdot \beta^i(t,x) := \delta^i(t,x), \quad i \in S. \end{aligned} \tag{2.32}$$

One may use the notation just introduced to write problem (2.30) simpler way.

The operator $\hat{P}$ is the composition of the Nemytskiĭ operators $\hat{\mathbf{F}}$ and $\mathbf{C}$ with the operator

$$\hat{\mathcal{G}}\colon \delta \mapsto \mathcal{G}[\delta] = \gamma,$$

where γ is the (supposedly unique) solution of the linear problem

$$\begin{cases} \mathcal{F}^i[\gamma^i](t,x) + \mathbf{C}^i[\beta](t,x)\gamma^i(t,x) = \hat{\mathbf{F}}^i[\beta](t,x), \;\; i \in S, & \text{in } D, \\ \gamma(t,x) = 0 & \text{on } \Gamma. \end{cases} \tag{2.33}$$

Hence

$$\hat{\mathcal{P}} = \hat{\mathcal{G}} \circ \hat{\mathbf{F}} \circ \mathbf{C}.$$

The following lemmas hold, too.

Lemma 2.5. (i) *If* $\beta \in C_S^{0+\alpha}(\overline{D})$ *and the functions* $f_{y^i} = \{f^i_{y^i}\}_{i\in S}$ *generating the Nemytskiĭ operator* $\mathbf{C}$ *satisfy conditions* (H_p) *and* (L_p)*, then*

$$\mathbf{C}[\beta] = \eta \in C_S^{0+\alpha}(\overline{D}).$$

(ii) *If* $\beta \in C_S^{0+\alpha}(\overline{D})$ *and the functions* $f = \{f^i\}_{i\in S}$ *and* $f_{y^i} = \{f^i_{y^i}\}_{i\in S}$ *generating the Nemytskiĭ operator* $\hat{\mathbf{F}}$ *satisfy conditions* (H_f)*,* (L)*,* (H_p) *and* (L_p)*, then*

$$\hat{\mathbf{F}}[\beta] = \delta \in C_S^{0+\alpha}(\overline{D}).$$

Proof. It runs analogously to the proof of Lemma 2.1. □

From Theorem 1.1 and Lemma 2.5, one may derive the following statements.

Lemma 2.6. *If* $\beta \in C_S^{0+\alpha}(\overline{D})$*, all the coefficients of the operators* $\mathcal{L}^i$*,* $i \in S$*, satisfy assumption* (H_a)*, then problem* (2.27) *has exactly one regular solution* $\gamma \in C_S^{2+\alpha}(\overline{D})$*.*

Corollary 2.3. *There follows from Lemmas 2.5 and 2.6 that*

$$\hat{\mathcal{P}}\colon C_S^{0+\alpha}(\overline{D}) \ni \beta \mapsto \hat{\mathcal{P}}[\beta] = \gamma \in C_S^{2+\alpha}(\overline{D}).$$

Lemma 2.7. *Let all the assumptions of Lemmas 2.5 and 2.6 hold,* β *be an upper solution and* α *be a lower solution of problem* (2.1) *in* $\overline{D}$*, and conditions* (W)*,* (W_p) *hold. Then*

$$\hat{\mathcal{P}}[\beta](t,x) \le \beta(t,x) \quad \text{in } \overline{D} \tag{2.34}$$

and $\gamma = \hat{\mathcal{P}}[\beta]$ *is an upper solution of problem* (2.1) *in* $\overline{D}$*, and analogously*

$$\hat{\mathcal{P}}[\alpha](t,x) \ge \alpha(t,x) \quad \text{in } \overline{D} \tag{2.35}$$

and $\eta = \hat{\mathcal{P}}[\alpha]$ *is a lower solution of problem* (2.1) *in* $\overline{D}$*.*

Proof. If β is an upper solution, then due to (1.4) and the notation introduced, there is

$$\mathcal{F}^i[\beta^i](t,x) \ge f^i(t,x,\beta(t,x),\beta), \quad i \in S, \quad \text{in } D,$$

thus β satisfies the following system of inequalities

$$\mathcal{F}^i[\tilde{z}^i](t,x) \ge f^i(t,x,\beta(t,x),\beta) + f^i_{y^i}(t,x,\beta(t,x),\beta)\cdot\left[\tilde{z}^i(t,x) - \beta^i(t,x)\right], \tag{2.36}$$

$i \in S$, in D, and initial-boundary condition (1.2).

From definition (2.30) of the operator $\hat{\mathcal{P}}$ it follows that the function γ is a solution of the system of equations

$$\mathcal{F}^i[\tilde{\tilde{z}}^i](t,x) = f^i(t,x,\beta(t,x),\beta) + f^i_{y^i}(t,x,\beta(t,x),\beta)\cdot\left[\tilde{\tilde{z}}^i(t,x) - \beta^i(t,x)\right] \tag{2.37}$$

$i \in S$, in D, with condition (1.2).

Applying Theorem 1.2 to systems (2.36) and (2.37) we obtain

$$\tilde{\tilde{z}}^i(t,x) \le \tilde{z}^i(t,x), \quad i \in S, \quad \text{in } \overline{D},$$

so,

$$\gamma(t,x) = \hat{\mathcal{P}}[\beta](t,x) \le \beta(t,x) \quad \text{in } \overline{D}.$$

Since the functions $f^i_{y^i}(t,x,y,s)$ are increasing in variable y^i (condition (W_p)), then the functions $f^i(t,x,y,s)$ are convex in y^i. Now (2.30), (2.34) and condition (W) give

$$\mathcal{F}^i\left[\gamma^i\right](t,x) = f^i(t,x,\beta(t,x),\beta) + f^i_{y^i}(t,x,\beta(t,x),\beta)\cdot\left[\gamma^i(t,x) - \beta^i(t,x)\right] \ge \\ \ge f^i(t,x,\gamma(t,x),\beta) \ge f^i(t,x,\gamma(t,x),\gamma), \quad i \in S, \quad \text{in } D,$$

and

$$\gamma(t,x) = 0 \quad \text{on} \quad \Gamma.$$

Thus from (1.4) and Corollary 2.2 it follows that the function γ is an upper solution of problem (2.1) in $\overline{D}$. □

Proof of Theorem. One may prove statements 1°–3° of Theorem using induction argument. Using Assumption A it is easy to see that those statements are immediate consequences of Lemmas 2.5–2.7 and Corollary 2.2.

Indeed, due to the definition of $\hat{\mathcal{P}}$, (2.30), (2.25) and (2.26), we obtain

$$\hat{u}_1 = \hat{\mathcal{P}}[\hat{u}_0], \qquad \hat{v}_1 = \hat{\mathcal{P}}[\hat{v}_0],$$
$$\hat{u}_n = \hat{\mathcal{P}}[\hat{u}_{n-1}], \quad \hat{v}_n = \hat{\mathcal{P}}[\hat{v}_{n-1}], \quad n = 1,2,\ldots$$

which means that the sequences $\{\hat{u}_n\}$ and $\{\hat{v}_n\}$ are well defined.

Statement 4° follows immediately from the comparison of system (2.25) and (2.26) with system (2.2) and (2.3) defining the sequences $\{u_n\}$ and $\{v_n\}$.

Indeed, due to (2.30) and condition (W_p), there is

$$\mathcal{F}^i[\hat{u}^i_1](t,x) - \mathcal{F}^i[u^i_1](t,x) = f^i(t,x,u_0(t,x),u_0) + \\ + f^i_{y^i}(t,x,u_0(t,x),u_0)\cdot\left[\hat{u}^i_1(t,x) - u^i_0(t,x)\right] - f^i(t,x,u_0(t,x),u_0) = \\ = f^i_{y^i}(t,x,u_0(t,x),u_0)\cdot\left[\hat{u}^i_1(t,x) - u^i_0(t,x)\right] \ge 0, \quad i \in S, \quad \text{in } D,$$

and

$$\hat{u}_1(t,x) - u_1(t,x) = 0 \quad \text{on} \quad \Gamma.$$

Consequently, by Lemma 1.1

$$\hat{u}_1(t,x) \ge u_1(t,x) \quad \text{in } \overline{D}.$$

Using (1.13) and (1.14), by mathematical induction, we obtain

$$u_n(t,x) \le \hat{u}_n(t,x) \le \hat{v}_n(t,x) \le v_n(t,x),$$

$n = 1,2,\ldots$ for $(t,x) \in \overline{D}$.

The other statements, 5° and 6°, follow immediately from inequality (2.29) and Theorem 1.1.

Thus the theorem is proved. □

Remark 2.3. *This method has been introduced by S. A. Chaplygin in [36] for ordinary differential equations and developed by N. Lusin [69]. Next, P. K. Zeragia [131–133], W. Mlak [73] and S. Brzychczy [16, 17] have applied this method to parabolic differential equations and J. P. Mysovskikh [75] to elliptic equations. The Chaplygin method was generalized and extended in several directions and to a larger class of equations and problems; has several versions in literature and appears also as Zeragia method (see R. Rabczuk [96]). Turning of Chaplygin's idea R. Bellman and R. Kalaba [12] develope this method as method of quasilinearization and after the publication of interesting several articles by V. Lakshmikantham et al. (see e.g. [60]) as the generalized quasilinearization method. In this method the convergence rate is quadratic.*

2.3. CERTAIN VARIANTS OF CHAPLYGIN METHOD

Now, we present the next two monotone iterative methods being certain variants of the Chaplygin method, applicable to problem (2.1), whose right-hand sides are semi-increasing functions (more precisely, they meet condition (K)). The last of these methods consists in adding an appropriate linear term including the unknown functions to the both sides of equations (2.1), in order to render the new sides monotonous. Thus we obtain the system of equations

$$\mathcal{F}^i_k[z^i](t,x) + k^i(t,x)z^i(t,x) = f^i(t,x,z(t,x),z) + k^i(t,x)z^i(t,x), \quad i \in S,$$

whose right-hand sides are increasing with respect to the variable y^i for each i, $i \in S$. Setting

$$\mathcal{F}^i_k := \mathcal{D}_t - \mathcal{L}^i + k^i\mathcal{I},$$

we obtain the system of the following form

$$\mathcal{F}^i_k[z^i](t,x) = f^i(t,x,z(t,x),z) + k^i(t,x)z^i(t,x), \quad i \in S \tag{2.38}$$

to which monotone iterative methods, including in particular the simple iteration method, are applicable.

Theorem 2.4. *Let assumptions A and (H_a), (H_f), (W), (L), (K), (V) hold in the set $\mathcal{K}$ and let the successive terms of approximation sequences $\{\overset{*}{u}_n\}$ and $\{\overset{*}{v}_n\}$ be defined as regular solutions in $\overline{D}$ of the following infinite systems of linear parabolic differential equations*

$$\mathcal{F}^i_k[\overset{*}{u}{}^i_n](t,x) = f^i(t,x,\overset{*}{u}_{n-1}(t,x),\overset{*}{u}_{n-1}) + k^i(t,x)\,\overset{*}{u}{}^i_{n-1}(t,x), \tag{2.39}$$

$$\mathcal{F}^i_k[\overset{*}{v}{}^i_n](t,x) = f^i(t,x,\overset{*}{v}_{n-1}(t,x),\overset{*}{v}_{n-1}) + k^i(t,x)\,\overset{*}{v}{}^i_{n-1}(t,x), \quad i \in S, \tag{2.40}$$

for $n = 1, 2, \ldots$ in D with homogeneous initial-boundary condition (1.2) *and let $\overset{*}{u}_0 = u_0$, $\overset{*}{v}_0 = v_0$.*

Then the statements of Theorem 2.3 hold, thus there exists the unique regular solution z of problem (2.1) *within the sector* $\langle u_0, v_0\rangle$*, and* $z \in C_S^{2+\alpha}(\overline{D})$.

Proof of Theorem. After some minute changes in technical details, the proof of this theorem is identical to those of previous theorems, so we may omit here. □

If condition (K_κ) holds, then we will define the approximation sequences as follows.

Theorem 2.5. *Let assumptions A and* (H_a), (H_f), (K_κ), (L), (V) *hold in the set* $\mathcal{K}$ *and let the successive terms of the approximation sequences* $\{\overset{\circ}{u}_n\}$ *and* $\{\overset{\circ}{v}_n\}$ *be defined as regular solutions in* $\overline{D}$ *of the following infinite systems of linear parabolic equations*

$$\mathcal{F}^i_\kappa[\overset{\circ}{u}{}^i_n](t,x) = f^i(t,x,\overset{\circ}{u}_{n-1}(t,x),\overset{\circ}{u}_{n-1}) + \kappa\,\overset{\circ}{u}{}^i_{n-1}(t,x), \tag{2.41}$$

$$\mathcal{F}^i_\kappa[\overset{\circ}{v}{}^i_n](t,x) = f^i(t,x,\overset{\circ}{v}_{n-1}(t,x),\overset{\circ}{v}_{n-1}) + \kappa\,\overset{\circ}{v}{}^i_{n-1}(t,x), \quad i \in S, \tag{2.42}$$

for $n = 1, 2, \ldots$ *in* D *with the condition* (1.2)*, where*

$$\mathcal{F}^i_\kappa := \mathcal{D}_t - \mathcal{L}^i + \kappa\mathcal{I}.$$

Then there exists the unique regular solution z *of problem* (2.1)

$$z = z(t,x) = \lim_{n\to infty} \overset{\circ}{u}_n(t,x) = \lim_{n\to infty} \overset{\circ}{v}_n(t,x)$$

within the sector $\langle u_0, v_0\rangle$*, and* $z \in C_S^{2+\alpha}(\overline{D})$.

Remark 2.4. *This variant of Chaplygin method has frequently been applied by several authors (see e.g. H. Amann [2], O. Dickman and N. M. Temme [39], C. V. Pao [81], D. H. Sattinger [106], J. Smoller [108]) to prove the existence of solution of nonlinear parabolic and elliptic equations.*

2.4. CERTAIN VARIANT OF MONOTONE ITERATIVE METHOD (WAŻEWSKI METHOD)

Previously, the four monotone iterative methods have been used to examine the existence of a solution of problem (0.2), (1.2). Here we shall apply two other monotone iterative methods. These methods make it again possible to build sequences of successive approximations that converge to a solution sought for at a rate higher than in the case of the iterative sequences of successive approximations defined by (2.2), (2.3). In general, it consists in what follows: if we consider some nonlinear system of equations whose right-hand sides are functions of the form $f = f(t,x,s,s)$, then the successive approximation sequence $\{\tilde{u}_n\}$ arising from the iteration $f(t,x,\tilde{u}_n,\tilde{u}_{n-1})$ is considered. Thus it is a pseudo-linearization of the nonlinear problem which has been proposed by T. Ważewski [125,126]. Applying the above iterative method to the parabolic problem here considered has been suggested by A. Pelczar [86]. Under appropriate assumptions on the functions f, the sequence $\{\tilde{u}_n\}$ tends to the searched-for exact solution at a rate not lower then that of the successive approximation sequence $\{u_n\}$ given by the iteration $f(t, x, u_n, u_n)$.

We will define the successive terms of approximation sequences $\{\tilde{u}_n\}$ and $\{\tilde{v}_n\}$ as regular solutions of the following infinite systems of semilinear parabolic differential equations

$$\mathcal{F}^i[\tilde{u}_n^i](t,x) = f^i(t, x, \tilde{u}_n(t,x), \tilde{u}_{n-1}), \tag{2.43}$$

$$\mathcal{F}[\tilde{v}_n^i](t,x) = f^i(t, x, \tilde{v}_n(t,x), \tilde{v}_{n-1}), \quad i \in S, \tag{2.44}$$

for $n = 1, 2, \ldots$ in D, satisfying the homogeneous initial-boundary condition (1.2).

Theorem 2.6. *Under assumptions A and* (H_a), (H_f), (W), (W_+), (L), (V), *if we start approximation process from the same pair of a lower* u_0 *and an upper* v_0 *solution of problem* (2.1) *in* $\overline{D}$, *then the sequences* $\{\tilde{u}_n\}$ *and* $\{\tilde{v}_n\}$ *given by* (2.43), (2.44) *and* (1.2) *are well defined in* $C_S^{2+\alpha}(\overline{D})$, *the functions* $\tilde{u}_n$ *and* $\tilde{v}_n$, $(n = 1, 2, \ldots)$ *are lower and upper solutions of problem* (2.1) *in* $\overline{D}$, *respectively, and these sequences converge monotonously and uniformly to a solution* z *of problem* (2.1) *in* $\overline{D}$ *at a rate not lower then that of the iterative sequences* $\{u_n\}$ *and* $\{v_n\}$ *defined by* (2.2), (2.3) *and* (1.2) *in* $\overline{D}$, *namely the inequalities*

$$u_0(t,x) \le \tilde{u}_n(t,x) \le \tilde{u}_{n+1}(t,x) \le \tilde{v}_{n+1}(t,x) \le \tilde{v}_n(t,x) \le v_0(t,x) \tag{2.45}$$

and

$$u_n(t,x) \le \tilde{u}_n(t,x) \le \tilde{v}_n(t,x) \le v_n(t,x), \tag{2.46}$$

hold for $n = 1, 2, \ldots$ *and* $(t,x) \in \overline{D}$, *and the function*

$$z = z(t,x) = \lim_{n\to\infty} \tilde{u}_n(t,x) \quad \textit{uniformly in} \quad \overline{D}$$

is the unique regular solution of problem (2.1) *within the sector* $\langle u_0, v_0 \rangle$, *and* $z \in$ $\in C_S^{2+\alpha}(\overline{D})$.

Proof. From Theorem 2.2 it follows that the succesive terms of approximation sequences $\tilde{u}_n$ and $\tilde{v}_n$ are well defined and $\tilde{u}_n, \tilde{v}_n \in C_S^{2+\alpha}(\overline{D})$ for $n = 1, 2, \ldots$

To proof of theorems we only show inequality (2.46), because the other statements of theorem are obvious. To this we use the mathematical induction. Indeed, for $n = 1$, by (2.45), (2.43), (2.2) and (W) we obtain

$$\mathcal{F}^i[\tilde{u}_1^i](t,x) - \mathcal{F}^i[u_1^i](t,x) = f^i(t, x, \tilde{u}_1(t,x), u_0) - f^i(t, x, u_0(t,x), u_0) \ge 0,$$

$i \in S$, in D, with initial-boundary condition (1.2). Hence, by virtue of Lemma 1.1, we obtain

$$\tilde{u}_1(t,x) \ge u_1(t,x) \quad \text{in } \overline{D}.$$

If now

$$\tilde{u}_{n-1}(t,x) \ge u_{n-1}(t,x) \quad \text{in } \overline{D},$$

then by (2.43), (2.45), (2.3) and (W) we come to the inequalities

$$\mathcal{F}^i[\tilde{u}_n^i](t,x) - \mathcal{F}^i[u_n^i](t,x) = f^i(t,x,\tilde{u}_n(t,x),\tilde{u}_{n-1}) - f^i(t,x,u_{n-1}(t,x),u_{n-1}) \geq$$
$$\geq f^i(t,x,\tilde{u}_{n-1}(t,x),\tilde{u}_{n-1}) - f^i(t,x,u_{n-1}(t,x),u_{n-1}) \geq 0, \quad i \in S, \quad \text{in } D$$

with condition (1.2). Hence by Lemma 1.1 we obtain

$$\tilde{u}_n(t,x) \geq u_n(t,x) \quad \text{in } \overline{D}.$$

Analogously

$$v_n(t,x) \geq \tilde{v}_n(t,x) \quad \text{in } \overline{D}.$$

Therefore, by induction, inequality (2.46) is proved. □

2.5. ANOTHER VARIANT OF MONOTONE ITERATIVE METHOD (MLAK–OLECH METHOD)

Let us consider the infinite countable system of equations of the form (0.1), i.e.,

$$\mathcal{F}^i[z^i](t,x) = f^i(t,x,z), \quad i \in \mathbb{N}, \quad \text{in } D \tag{2.47}$$

with homogeneous initial-boundary condition (1.2).

Applying also the idea given by T. Ważewski [125, 126], we will define the successive terms of the approximation sequences of problem (2.47), (1.2) as solutions of following infinite countable systems of semilinear equations

$$\mathcal{F}^i\left[\overline{u}_n^i\right](t,x) = f^i(t,x,[\overline{u}_n,\overline{u}_{n-1}]^i]), \tag{2.48}$$
$$\mathcal{F}^i\left[\overline{v}_n^i\right](t,x) = f^i(t,x,[\overline{v}_n,\overline{v}_{n-1}]^i), \quad i \in \mathbb{N}, \tag{2.49}$$

for $n = 1,2,\ldots$ in D, with initial-boundary condition (1.2).

We note, that this method of constructions of approximate sequences has been applied to ordinary differential equations by W. Mlak and C. Olech [74].

Theorem 2.7. *Let assumptions A and (H_a), (H_f), (W), (L), (V) hold in the set K^*. If we define the successive terms of approximation sequences $\{\overline{u}_n\}$ and $\{\overline{v}_n\}$ as regular solutions in $\overline{D}$ of systems (2.48), (2.49) with homogeneous initial-boundary condition (1.2) and if $\overline{u}_0 = u_0$, $\overline{v}_0 = v_0$, then:*

1° *$\{\overline{u}_n\}$, $\{\overline{v}_n\}$ are well defined and $\overline{u}_n$, $\overline{v}_n \in C^{2+\alpha}_{\mathbb{N}}(\overline{D})$ for $n = 1,2,\ldots$;*

2° *the inequalities*

$$u_0(t,x) \leq \overline{u}_n(t,x) \leq \overline{u}_{n+1}(t,x) \leq \overline{v}_{n+1}(t,x) \leq \overline{v}_n(t,x) \leq v_0(t,x), \tag{2.50}$$

$n = 1,2,\ldots$ hold for $(t,x) \in \overline{D}$;

3° *the functions $\overline{u}_n$ and $\overline{v}_n$ for $n = 1,2,\ldots$ are lower and upper solutions of problem (2.46), (1.2) in $\overline{D}$, respectively;*

4° *if we start two approximation processes from the same pair of a lower solution* u_0 *and upper solution* v_0 *of problem* (2.47), (1.2) *in* $\overline{D}$, *then the sequences* $\{\overline{u}_n\}$ *and* $\{\overline{v}_n\}$ *converge monotonously and uniformly to a solution* z *of problem* (2.47), (1.2) *in* $\overline{D}$ *at a rate not lower than that of the iterative sequences* $\{u_n\}$ *and* $\{v_n\}$ *defined by* (2.2), (2.3) *and* (1.2) *in* $\overline{D}$, *i.e. the inequalities*

$$u_n(t,x) \leq \overline{u}_n(t,x) \leq \overline{v}_n(t,x) \leq v_n(t,x), \tag{2.51}$$

$n = 1, 2, \ldots$ *hold for* $(t,x) \in \overline{D}$;

5° *the function* $z = z(t,x) = \lim_{n\to\infty} \overline{u}_n(t,x)$ *uniformly in* $\overline{D}$ *is the unique regular solution of problem* (2.47), (1.2) *within the sector* $\langle u_0, v_0\rangle$ *and* $z \in C^{2+\alpha}_{\mathbb{N}}(\overline{D})$.

Proof. Using the auxiliary lemmas we prove the theorem by induction. The proof is simple and similar for the lower and upper solutions, so we present the proof for lower solutions in the both cases only.

Since $u_0 \in C^{0+\alpha}_{\mathbb{N}}(\overline{D})$, then from Lemma 2.1 and the theorem on the existence and uniqueness of the solution of the Fourier first initial-boundary value problem for finite system of parabolic differential–functional equations (see S. Brzychczy [20]; [25], Th. 3.1, pp. 26–27), it follows that there exists the regular unique solution $\overline{u}_1$ of problem (2.48), (1.2) in $\overline{D}$ and $\overline{u}_1 \in C^{2+\alpha}_{\mathbb{N}}(\overline{D})$. Analogously, if $\overline{u}_{n-1} \in C^{2+\alpha}_{\mathbb{N}}(\overline{D})$, then there exists the regular unique solution $\overline{u}_n$ of problem (2.49), (1.2) in $\overline{D}$, $\overline{u}_n \in C^{2+\alpha}_{\mathbb{N}}(\overline{D})$ and 1° is proved by induction.

Since u_0 is a lower solution, it satisfies the inequalities

$$\mathcal{F}^i[u_0^i](t,x) \leq f^i(t,x,u_0) = f^i(t,x,[u_0,u_0]^i), \quad i \in \mathbb{N}, \quad \text{in } D,$$

with condition (1.2). The function $\overline{u}_1$ is a solution of the equation

$$\mathcal{F}^i[\overline{u}_1^i](t,x) = f^i(t,x,[\overline{u}_1,u_0]^i), \quad i \in \mathbb{N}, \quad \text{in } D,$$

with condition (1.2). Therefore, by Lemma 1.1 we obtain

$$u_0(t,x) \leq \overline{u}_1(t,x) \quad \text{for} \quad (t,x) \in \overline{D}. \tag{2.52}$$

Moreover, by (2.52) and (W) there is

$$\mathcal{F}^i[\overline{u}_1^i](t,x) = f^i(t,x,[\overline{u}_1,u_0]^i) \leq f^i(t,x,[\overline{u}_1,\overline{u}_1]^i), \quad i \in \mathbb{N}, \quad \text{in } D,$$

so the function $\overline{u}_1$ is a lower solution of problem (2.47), (1.2) in $\overline{D}$.

Analogously, if $\overline{u}_{n-1}$ is a lower solution, then by (1.3) we obtain

$$\mathcal{F}^i[\overline{u}_{n-1}^i](t,x) \leq f^i(t,x,\overline{u}_{n-1}), \quad i \in \mathbb{N}, \quad \text{in } D,$$

and $\overline{u}_n$ is a solution of system (2.48) with condition (1.2). Therefore, by Lemma 1.1 we obtain

$$\overline{u}_{n-1}(t,x) \leq \overline{u}_n(t,x) \quad \text{for} \quad (t,x) \in \overline{D}. \tag{2.53}$$

Moreover, by (2.53) and (W) there is

$$\mathcal{F}^i[\overline{u}_n^i](t,x) = f^i(t,x,[\overline{u}_n,\overline{u}_{n-1}]^i) \leq f^i(t,x,[\overline{u}_n,\overline{u}_n]^i) = \\ = f^i\left(t,x,\overline{u}_n\right), \quad i \in \mathbb{N}, \quad \text{in } D,$$

so the function $\overline{u}_n$ is a lower solution of problem (2.47), (1.2) in $\overline{D}$.

By inequalities (2.52), (2.53) for lower solutions and the analogous inequalities for upper solutions, using mathematical induction, we obtain inequality (2.50).

We now prove inequality (2.51) by induction. From (2.2), (2.53) for $n = 1$, (2.52) and (W) we obtain

$$\mathcal{F}^i[\overline{u}_1^i](t,x) - \mathcal{F}^i[u_1^i](t,x) = f^i(t,x,[\overline{u}_1,u_0]^i) - f^i(t,x,[u_0,u_0]^i) \geq 0, \quad i \in \mathbb{N}, \text{ in } D,$$

with condition (1.2). By virtue of Lemma 1.1 we obtain

$$\overline{u}_1(t,x) \geq u_1(t,x) \quad \text{in } \overline{D}. \tag{2.54}$$

Let now

$$\overline{u}_{n-1}(t,x) \geq u_{n-1}(t,x) \quad \text{in } \overline{D}, \tag{2.55}$$

then by (W), because u_{n-1} is a lower solution of problem (2.47), (1.2) in $\overline{D}$, we derive

$$\mathcal{F}^i[u_{n-1}^i](t,x) \leq f^i(t,x,[u_{n-1},u_{n-1}]^i) \leq f^i(t,x,[u_{n-1},\overline{u}_{n-1}]^i), \quad i \in \mathbb{N}, \quad \text{in } D,$$

with the condition (1.2). Therefore, by (2.48) and Lemma 1.1 we have

$$\overline{u}_n(t,x) \geq u_{n-1}(t,x) \quad \text{in } \overline{D}. \tag{2.56}$$

From (2.2), (2.48), (2.55), (2.56) we get

$$\mathcal{F}^i\left[\overline{u}_n^i\right](t,x) - \mathcal{F}^i\left[u_n^i\right](t,x) = f^i(t,x,[\overline{u}_n,\overline{u}_{n-1}]^i) - f^i(t,x,[u_n,u_{n-1}]^i) \geq \\ \geq f^i(t,x,[\overline{u}_n,u_{n-1}]^i) - f^i(t,x,[u_n,u_{n-1}]^i) \geq 0, \quad i \in \mathbb{N}, \quad \text{in } D,$$

with condition (1.2). By virtue of Lemma 1.1 we obtain

$$\overline{u}_n(t,x) \geq u_n(t,x) \quad \text{in } \overline{D}. \tag{2.57}$$

From (2.50), (2.54), (2.57) we finally obtain inequality (2.51).

By (2.51) and (2.15) there is

$$\overline{v}_n^i(t,x) - \overline{u}_n^i(t,x) \leq N_0\frac{(Lt)^n}{n!}, \quad i \in \mathbb{N}, \quad n = 1,2,\ldots \tag{2.58}$$

for $(t,x) \in \overline{D}$. As a direct consequence, we obtain

$$\lim_{n\to\infty}\left[\overline{v}_n^i(t,x) - \overline{u}_n^i(t,x)\right] = 0 \quad \text{uniformly in} \quad \overline{D}, \quad i \in \mathbb{N}.$$

By arguments similar to that used in section 2.1 we show that

$$z = z(t,x) := \lim_{n\to\infty} \overline{u}_n(t,x) \quad \text{uniformly in} \quad \overline{D}, \tag{2.59}$$

is the unique regular solution of problem (2.47), (1.2) within the sector $\langle u_0, v_0\rangle$. Obviously, $z \in C^{2+\alpha}_{\mathbb{N}}(\overline{D})$. □

Remark 2.5. *It is easy to see that the method used in the proof does not need the assumption that the considered system is countable. Therefore, in Theorem 2.7 the countable system may be replated by an arbitrary infinite system (which was observed by W. Mlak and C. Olech [74], p. 110).*

2.6. MONOTONE METHOD OF DIRECT ITERATIONS IN UNBOUNDED DOMAINS

Now we consider infinite system of equations (0.1) with initial-boundary condition (0.3), i.e., the problem

$$\begin{cases} \mathcal{F}^i[z^i](t,x) = f^i(t,x,z) & i \in S, \text{ for } (t,x) \in \Omega, \\ z(t,x) = \phi(t,x) & \text{for } (t,x) \in \Gamma_\Omega, \end{cases} \tag{2.60}$$

where Ω is an arbitrary open domain in the time-space $\mathbb{R}^{m+1}$, unbounded with respect to x, Γ_Ω is the parabolic boundary of Ω, S is an arbitrary set of indices (finite or infinite) and z stands for the mapping

$$z\colon S \times \overline{\Omega} \in \mathbb{R}, \quad (i,t,x) \mapsto z^i(t,x),$$

composed of unknown functions z^i.

Analogously as in section 1.1 we define the Banach space $\mathcal{B}(S)$ and denote by $C_S(\overline{\Omega})$ the Banach space of mappings

$$w\colon \overline{\Omega} \to \mathcal{B}(S), \quad (t,x) \mapsto w(t,x),$$

and

$$w(t,x)\colon S \to \mathbb{R}, \quad i \mapsto w^i(t,x),$$

where the functions w^i are continuous in $\overline{\Omega}$, with the finite norm

$$\|w\|_0 := \sup\left\{ \left|w^i(t,x)\right| : (t,x) \in \overline{\Omega}, i \in S \right\}.$$

We will assume that the unbounded domain Ω has the following *property* $(\mathcal{P})$ (cp. J. Szarski [116, 117]):

1° the projection of the interior of Ω on the t-axis is the interval $(0,T)$, where $0 < T < \infty$;

2° for every $(\tilde{t},\tilde{x}) \in \Omega$ there is a positive number r such that the lower half neighbourhood is contained in Ω, i.e.,

$$\left\{(t,x)\colon (t-\tilde{t})^2 + \|x-\tilde{x}\|^2 < r^2, \quad t \le \tilde{t}\right\} \subset \Omega.$$

We will assume that the boundary $\partial\Omega$ of domain Ω consists of m-dimensional (bounded or unbounded) domains Ω_0 and Ω_T lying on the hyperplanes $t = 0$ and $t = T$, respectively, and a certain manifold σ (not necessarily connected) of class $C_*^{2+\alpha}$ (i.e., it consists of a finite number of manifolds of class $C^{2+\alpha}$, not overlapping but having common boundary points) lying in the zone $0 < t < T$ which is not tangent to any hyperplane $t = \text{const}$. We will assume that the number T is sufficiently small. Precisely, $T \leq h_0 = \text{const}$, where h_0 is some positive constant depending on problem (2.60) and defined by M. Krzyżański in [52] (cp. also P. Besala [15]).

We will denote $\Gamma_\Omega := \Omega_0 \cup \sigma$, $\overline{\Omega} := \Omega \cup \Gamma_\Omega$. Moreover, we will denote by Ω_R the part of the domain Ω contained inside the cylindric surface Σ_R described by the equation $\sum_{j=1}^n x_j^2 = R^2$ and $\Gamma_R := \partial\Omega_R \setminus \Omega_T$.

For a fixed τ, $0 < \tau \leq T$, we define $\Omega^\tau := \Omega \cap \{(t, x) \colon 0 < t \leq \tau, x \in \mathbb{R}^m\}$, $\sigma^\tau := \sigma \cap \{(t, x) \colon 0 < t \leq \tau, x \in \mathbb{R}^m\}$, $\Gamma^\tau := \Omega_0 \cup \sigma^\tau$, $\overline{\Omega}^\tau := \Omega^\tau \cup \Gamma^\tau$. Obviously, $\Omega^T = \Omega$.

In the theory of parabolic equations it is well-known that initial-value problems in unbounded spatial domains are considered with the growth condition $|w(t, x)| \leq$ $\leq M \exp(K\|x\|^2)$, otherwise these problems are ill posed (see A. Tychonoff [120] and cp. E. DiBenedetto [38], pp. 237–238).

By $E_2(M, K; \Omega)$ or shortly E_2 we denote the class of functions $w = w(t, x)$ for which there exist positive constants M and K such that the following growth condition is fulfilled

$$|w(t, x)| \leq M \exp(K\|x\|^2) \quad \text{for} \quad (t, x) \in \Omega.$$

Denote by $C_{S,E_2}(\overline{\Omega})$ the space of mappings $w \in C_S(\overline{\Omega})$ belonging to the class $E_2(M, K; \Omega)$ with the finite weighted norm

$$\|w\|_0^{E_2} := \sup\left\{\left|w^i(t, x)\right| \exp(-K\|x\|^2) \colon (t, x) \in \overline{\Omega},\ i \in S\right\}. \tag{2.61}$$

For $w \in C_{S,E_2}(\overline{\Omega})$ and for a fixed t, $0 \leq t \leq T$, we define

$$\|w\|_{0,t}^{E_2} := \sup\left\{\left|w^i\left(\tilde{t}, \tilde{x}\right)\right| \exp(-K\|\tilde{x}\|^2) \colon \left(\tilde{t}, \tilde{x}\right) \in \overline{\Omega}^t, i \in S\right\}. \tag{2.62}$$

Analogously as in section 1.1 we define the Hölder space $C^{k+\alpha}(\overline{\Omega})$, $k = 0, 1, 2$. By $C_{S,E_2}^{k+\alpha}(\overline{\Omega})$ we denote the spaces of mappings $w \in C_S^{k+\alpha}(\overline{\Omega})$ belonging to the class $E_2(M, K; \Omega)$ with the finite weighted norm

$$\|w\|_{k+\alpha}^{E_2} := \sup\left\{\left|w^i\right|_{k+\alpha} \exp(-K\|x\|^2) \colon (t, x) \in \overline{\Omega}, i \in S\right\},$$

where $K > 0$ is a constant.

In the Banach space $C_{S,E_2}(\overline{\Omega})$ the partial order is defined by means of the positive cone $C_{S,E_2}^+(\overline{\Omega})$.

We will assume that the operators $\mathcal{F}^i$, $i \in S$, are *uniformly parabolic* in $\overline{\Omega}$.

Assumption ($\tilde{\mathbf{H}}_{\boldsymbol{a}}$). *We will assume that the coefficients $a^i_{jk} = a^i_{jk}(t,x)$, $a^i_{jk} = a^i_{kj}$, $b^i_j = b^i_j(t,x)$ $(j,k = 1,\dots,m,\ i \in S)$ of the operators $\mathcal{L}^i$, $i \in S$, are continuous with respect to t and x in $\overline{\Omega}$, bounded and locally Hölder continuous with exponent α $(0 < \alpha < 1)$ with respect to t and x in $\overline{\Omega}$ and they Hölder norms are uniformly bounded.*

We will assume that the functions

$$f^i : \overline{\Omega} \times C_S(\overline{\Omega}) \to \mathbb{R}, \quad (t,x,s) \mapsto f^i(t,x,s), \quad i \in S$$

are continuous and satisfy the following assumptions:

Assumption ($\tilde{\mathbf{H}}_{\boldsymbol{f}}$). *The functions f^i, $i \in S$, are locally Hölder continuous with exponent α $(0 < \alpha < 1)$ with respect to t and x in $\overline{\Omega}$, and they Hölder norms are uniformly bounded.*

Assumption ($\mathbf{E}_{\boldsymbol{f}}$). *$f^i(t,x,0) \in E_2(M_f, K_f; \Omega)$, $i \in S$.*

Assumption ($\tilde{\mathbf{L}}_{\mathbf{E_2}}$). *The functions $f^i(t,x,s)$, $i \in S$, fulfil the Lipschitz condition with respect to s: for arbitrary s, $\tilde{s} \in C_{S,E_2}(\overline{\Omega})$ there is*

$$\left|f^i(t,x,s) - f^i(t,x,\tilde{s})\right| \le L_1 \|s - \tilde{s}\|_0^{E_2} \quad \textit{for} \quad (t,x) \in \Omega,$$

where $L_1 > 0$ is a constant.

Assumption ($\tilde{\mathbf{L}}^*_{\mathbf{E_2}}$). *We say that the functions $f^i(t,x,s)$, $i \in S$, fulfil the Lipschitz–Volterra condition with respect to s if for arbitrary $s, \tilde{s} \in C_{S,E_2}(\overline{\Omega})$ there is*

$$\left|f^i(t,x,s) - f^i(t,x,\tilde{s})\right| \le L_2 \|s - \tilde{s}\|_{0,t}^{E_2} \quad \textit{for} \quad (t,x) \in \Omega,$$

where $L_2 > 0$ is a constant.

Moreover, we will assume that

($\tilde{\mathbf{H}}_{\boldsymbol{\phi}}$) *$\phi \in C_S^{2+\alpha}(\Gamma_\Omega)$, where $0 < \alpha < 1$;*

($\tilde{\mathbf{E}}_{\boldsymbol{\phi}}$) *$\phi \in E_2(M_\phi, K_\phi; \Gamma_\Omega)$.*

Functions u, $v \in C^{reg}_{S,E_2}(\overline{\Omega})$ satisfying the infinite systems of inequalities

$$\begin{cases} \mathcal{F}^i[u^i](t,x) \le f^i(t,x,u) & i \in S, \text{ for } (t,x) \in \Omega, \\ u(t,x) \le \phi(t,x) & \text{for } (t,x) \in \Gamma_\Omega, \end{cases} \tag{2.63}$$

$$\begin{cases} \mathcal{F}^i[v^i](t,x) \ge f^i(t,x,v) & i \in S, \text{ for } (t,x) \in \Omega, \\ v(t,x) \ge \phi(t,x) & \text{for } (t,x) \in \Gamma_\Omega \end{cases} \tag{2.64}$$

are called, respectively, a *lower* and an *upper solution* of problem (2.60) in $\overline{\Omega}$.

Assumption $\tilde{\boldsymbol{A}}$. *We assume that there exists at least one pair u_0 and v_0 of a lower and an upper solution of problem* (2.60) *in $\overline{\Omega}$ and $u_0, v_0 \in C_S^{0+\alpha}(\overline{\Omega}) \cap E_2(M_0, K_0; \Omega)$.*

Remark 2.6. *If u and v are a lower and an upper solution of problem (2.60) in $\overline{\Omega}$, respectively, and z is a regular solution of this problem and assumptions (W), $(\tilde{L}_{E_2})$ and (V) holds, then by the Szarski theorem on differential-functional inequations for infinite systems of parabolic type in an arbitrary domain (see [117] Theorem 1.2 and Remark 1.6) there is*

$$u(t,x) \leq z(t,x) \leq v(t,x) \quad \text{for} \quad (t,x) \in \overline{\Omega} \tag{2.65}$$

and in particular there is

$$u_0(t,x) \leq z(t,x) \leq v_0(t,x) \quad \text{for} \quad (t,x) \in \overline{\Omega}. \tag{2.66}$$

Assumption $\tilde{A}_0$. *Therefore, we will assume that there exists at least one ordered pair of these functions u_0, $v_0 \in C_S^{0+\alpha}(\overline{\Omega}) \cap E_2(M_0, K_0, \Omega)$.*

Let $\beta \in C_S(\overline{\Omega})$ be a sufficiently regular function. Denote by $\tilde{\mathcal{P}}$ the operator

$$\tilde{\mathcal{P}} \colon \beta \mapsto \tilde{\mathcal{P}}[\beta] = \gamma,$$

where γ is the (supposedly unique) solution of the initial-boundary value problem

$$\begin{cases} \mathcal{F}^i[\gamma^i](t,x) = f^i(t,x,\beta) & i \in S, \text{ for } (t,x) \in \Omega, \\ \gamma(t,x) = \phi(t,x) & \text{for } (t,x) \in \Gamma_\Omega. \end{cases} \tag{2.67}$$

The operator $\tilde{\mathcal{P}}$ is the composition of the nonlinear Nemytskiĭ operator $\tilde{\mathbf{F}} = \{\tilde{\mathbf{F}}^i\}_{i\in S}$ generated by the functions $f^i(t,x,s)$, $i \in S$, and defined for any $\beta \in C_S(\overline{\Omega})$ as follows

$$\tilde{\mathbf{F}}^i \colon \beta \mapsto \tilde{\mathbf{F}}^i[\beta] = \delta^i,$$

where

$$\tilde{\mathbf{F}}^i[\beta](t,x) := f^i(t,x,\beta) = \delta^i(t,x), \quad i \in S, \tag{2.68}$$

and the operator

$$\mathcal{G} \colon \delta \mapsto \mathcal{G}[\delta] = \gamma,$$

where γ is the (supposedly unique) solution of the linear problem

$$\begin{cases} \mathcal{F}^i[\gamma^i](t,x) = \delta^i(t,x) & i \in S, \text{ for } (t,x) \in \Omega, \\ \gamma(t,x) = \phi(t,x) & \text{for } (t,x) \in \Gamma_\Omega. \end{cases} \tag{2.69}$$

Hence

$$\tilde{\mathcal{P}} = \mathcal{G} \circ \tilde{\mathbf{F}}.$$

Lemma 2.8. *If $\beta \in C_{S,E_2}^{0+\alpha}(\overline{\Omega})$ and the function $f = \{f^i\}_{i\in S}$ generating the Nemytskiĭ operator, satisfies conditions $(\tilde{H}_f)$, (E_f) and $(\tilde{L}_{E_2})$, then*

$$\tilde{\mathbf{F}} \colon C_{S,E_2}^{0+\alpha}(\overline{\Omega}) \ni \beta \mapsto \tilde{\mathbf{F}}[\beta] = \delta \in C_{S,E_2}^{0+\alpha}(\overline{\Omega}).$$

Proof. Using the same argument as in the proof of Lemma 2.3 we obtain that $\delta \in C^{0+\alpha}_S(\overline{\Omega})$ and by (E_f) and $(\tilde{L}_{E_2})$ there is $\delta(t,x) \in E_2(M_\delta, K_\delta; \Omega)$. □

Lemma 2.9. *If assumptions* $(\tilde{H}_a)$, $(\tilde{H}_f)$, (E_f), $(\tilde{H}_\phi)$, (E_ϕ) *and* $(\tilde{L}_{E_2})$ *hold, then the operator* $\tilde{\mathcal{P}}$ *is well defined for* $\beta \in C^{0+\alpha}_{S,E_2}(\overline{\Omega})$, *where* $T \le h_0$ *and*

$$\tilde{\mathcal{P}}\colon C^{0+\alpha}_{S,E_2}(\overline{D}) \ni \beta \mapsto \gamma = \tilde{\mathcal{P}}[\beta] \in C^{2+\alpha}_{S,E_2}(\overline{\Omega}).$$

Proof. Observe that system (2.67) has the following property: the i-th equation depends on the i-th unknown function only. This fact, Lemma 2.8 and assumptions on the domain Ω imply that the M. Krzyżański [52,53] theorem on the existence and uniqueness of solution for a linear parabolic problem in an unbounded domain holds. Therefore, problem (2.67) has exactly one regular solution $\gamma \in C^{2+\alpha}_{S,E_2}(\overline{\Omega})$ provided $T \le h_0$ (where h_0 is formerly defined). □

Using the same arguments as in section 1.1 we will prove the following lemma.

Lemma 2.10. *Let all the assumptions of Lemmas 2.8 and 2.9 hold,* β *be an upper solution and* α *be a lower solution of problem* (2.60) *in* $\overline{\Omega}$, $\alpha, \beta \in \langle u_0, v_0\rangle$ *and condition* (W) *hold. Then*

$$\alpha(t,x) \le \tilde{\mathcal{P}}[\beta](t,x) \le \beta(t,x) \quad in \quad \overline{\Omega} \tag{2.70}$$

and $\gamma = \tilde{\mathcal{P}}[\beta]$ *is an upper solution of problem* (2.60) *in* $\overline{\Omega}$, *and analogously*

$$\alpha(t,x) \le \tilde{\mathcal{P}}[\alpha](t,x)) \le \beta(t,x) \quad in \quad \overline{\Omega} \tag{2.71}$$

and $\eta = \tilde{\mathcal{P}}[\alpha]$ *is a lower solution of problem* (2.60) *in* $\overline{\Omega}$.

Lemma 2.11. *If the assumptions of Lemma 2.9 and condition* (W) *hold, then the operator* $\tilde{\mathcal{P}}$ *is monotone increasing (isotone).*

Proof. Of course, $\tilde{\mathcal{P}}$ is monotone increasing, because $\mathbf{F}$ is monotone increasing by assumption (W) and $\mathcal{G}$ is monotone increasing by the maximum principle. □

Theorem 2.8. *Let assumptions* $\tilde{A}$ *and* $(\tilde{H}_a)$, $(\tilde{H}_f)$, (E_f), $(\tilde{H}_\phi)$, (E_ϕ), $(\tilde{L}_{E_2})$, (W), $(\tilde{L}^*_{E_2})$ *hold in the set* $\tilde{\mathcal{K}}$

$$\tilde{\mathcal{K}} := \big\{(t,x,s)\colon (t,x) \in \overline{\Omega}, \quad s \in \langle u_0, v_0\rangle\big\}$$

and now

$$\langle u_0, v_0\rangle := \big\{w \in C_S(\overline{\Omega})\colon u_0(t,x) \le w(t,x) \le v_0(t,x) \quad for \quad (t,x) \in \overline{\Omega}\big\}.$$

Consider the following infinite systems of linear equations:

$$\mathcal{F}^i[u^i_n](t,x) = f^i(t,x,u_{n-1}), \tag{2.72}$$

$$\mathcal{F}^i[v^i_n](t,x) = f^i(t,x,v_{n-1}) \quad i \in S, \ for \ (t,x) \in \Omega, \tag{2.73}$$

for $n = 1, 2, \dots$ *with initial-boundary condition* (0.3) *in* $\overline{\Omega}$ *and let* $\tilde{N}_0 = \|v_0 - u_0\|^{E_2}_0 <$ $< \infty$.

Then:

1° *there exist regular unique solutions* u_n *and* v_n $(n = 1, 2, \dots)$ *of systems* (2.72) *and* (2.73) *with boundary condition* (0.3) *in* $\overline{\Omega}$ *and* $u_n,\ v_n \in C^{2+\alpha}_{S,E_2}(\overline{\Omega})$*;*

2° *the inequalities*

$$\begin{aligned} &u_0(t,x) \le u_n(t,x) \le u_{n+1}(t,x) \le v_{n+1}(t,x) \le v_n(t,x) \le v_0(t,x), \\ &n = 1, 2, \dots \end{aligned} \tag{2.74}$$

hold for $(t,x) \in \overline{\Omega}$*;*

3° *the functions* u_n *and* v_n $(n = 1, 2, \dots)$ *are lower and upper solution of problem* (2.60) *in* $\overline{\Omega}$*, respectively;*

4° *the following estimate*

$$w_n^i(t,x) \le \tilde{N}_0 \frac{(Lt)^n}{n!}, \quad i \in S,\ n = 1, 2, \dots, \quad \textit{for} \quad (t,x) \in \overline{\Omega}, \tag{2.75}$$

holds, where

$$w_n^i(t,x) := v_n^i(t,x) - u_n^i(t,x) \ge 0 \quad i \in S, \quad \textit{in} \quad \overline{\Omega};$$

5° $\lim_{n\to\infty} [v_n^i(t,x) - u_n^i(t,x)] = 0$ *almost uniformly in* Ω*,* $i \in S$*;*

6° *the function*

$$z = z(t,x) = \lim_{n\to\infty} u_n(t,x)$$

is the unique regular solution of problem (2.60) *within the sector* $\langle u_0, v_0 \rangle$*, and* $z \in C^{2+\alpha}_{S,E_2}(\overline{\Omega})$.

Proof. Starting from the lower solution u_0 and the upper solution v_0 of problem (2.60), we define by induction the successive terms of the iteration sequences $\{u_n\}$, $\{v_n\}$ as solutions of systems of linear equations (2.72), (2.73) with boundary conditions (0.3) in $\overline{\Omega}$, or shortly

$$\begin{aligned} &u_1 = \tilde{\mathcal{P}}[u_0], \quad u_n = \tilde{\mathcal{P}}[u_{n-1}], \\ &v_1 = \tilde{\mathcal{P}}[v_0], \quad v_n = \tilde{\mathcal{P}}[v_{n-1}] \quad \text{for} \quad n = 1, 2, \dots \end{aligned}$$

From Lemmas 2.8, 2.9 and 2.10 it follows that u_n and v_n, for $n = 1, 2, \dots$ exist, $u_n, v_n \in C^{2+\alpha}_{S,E_2}(\overline{\Omega})$ and are the lower and the upper solution of problem (2.60) in $\overline{\Omega}$, respectively.

By induction, from Lemma 2.10, we derive

$$\begin{aligned} &u_{n-1}(t,x) \le \tilde{\mathcal{P}}[u_{n-1}](t,x) = u_n(t,x), \\ &\quad v_n(t,x) = \tilde{\mathcal{P}}[v_{n-1}](t,x) \le v_{n-1}(t,x), \quad n = 1, 2, \dots, \quad \text{for} \quad (t,x) \in \Omega. \end{aligned}$$

Therefore, inequalities (2.74) hold.

From (2.74) and Assumption $\tilde{A}$ it follows that $u_n, v_n \in E_2(M_0, N_0; \Omega)$ for $n = = 1, 2, \ldots$

We now show by induction that (2.75) hold. It is obvious that (2.75) holds for w_0. Suppose it holds for w_n. Since the functions f^i $(i \in S)$ satisfy the Lipschitz–Volterra condition $(\tilde{L}^*_{E_2})$, by (2.72)–(2.75) we obtain

$$\mathcal{F}^i[w^i_{n+1}](t,x) = f^i(t,x,v_n) - f^i(t,x,u_n) \le L\|w_n\|^{E_2}_{0,t}.$$

By the definition of the norm $\|\cdot\|^{E_2}_{0,t}$ and by (2.75) we obtain

$$\|w_n\|^{E_2}_{0,t} \le \tilde{N}_0 \frac{(Lt)^n}{n!},$$

so we finally obtain

$$\mathcal{F}^i[w^i_{n+1}](t,x) \le \tilde{N}_0 \frac{L^{n+1}t^n}{n!}, \quad i \in S, \quad \text{for} \quad (t,x) \in \Omega \tag{2.76}$$

and

$$w_{n+1}(t,x) = 0 \quad \text{for} \quad (t,x) \in \Gamma. \tag{2.77}$$

Consider the comparison system

$$\mathcal{F}^i[M^i_{n+1}](t,x) = \tilde{N}_0 \frac{L^{n+1}t^n}{n!}, \quad i \in S, \quad \text{for} \quad (t,x) \in \Omega, \tag{2.78}$$

with the boundary condition

$$M^i_{n+1}(t,x) \ge 0 \quad \text{for} \quad (t,x) \in \Gamma. \tag{2.79}$$

It is obvious that the functions

$$M^i_{n+1}(t,x) = \tilde{N}_0 \frac{(Lt)^{n+1}}{(n+1)!}, \quad i \in S,$$

are regular solutions of (2.78), (2.79) in $\overline{\Omega}$.

Applying the theorem on weak differential inequalities of parabolic type in an unbounded domain (see P. Bessala [15]) to systems (2.76) and (2.78) we get

$$w^i_{n+1}(t,x) \le M^i_{n+1}(t,x) = \tilde{N}_0 \frac{(Lt)^{n+1}}{(n+1)!}, \quad i \in S, \quad \text{for} \quad (t,x) \in \overline{\Omega}, \tag{2.80}$$

so the induction step is proved by inequality (2.75).

As a direct consequence of (2.75) we obtain

$$\lim_{n\to\infty} [v^i_n(t,x) - u^i_n(t,x)] = 0 \quad \text{almost uniformly in} \quad \Omega, \ i \in S. \tag{2.81}$$

The iteration sequences $\{u_n\}$ and $\{v_n\}$ are monotone and bounded, and (2.81) holds, so there is a continuous function $U = U(t,x)$ in $\overline{\Omega}$ such that

$$\lim_{n\to\infty} u^i_n(t,x) = U^i(t,x), \quad \lim_{n\to\infty} v^i_n(t,x) = U^i(t,x) \tag{2.82}$$

almost uniformly in Ω, $i \in S$, and this function satisfies boundary condition (0.3).

To prove that the function $U(t,x)$ defined by (2.82) is the regular solution of system (0.1) in Ω it is enough to show that it fulfils (0.1) in any compact set contained in Ω.

Consequently, because of the definition of Ω_R, we only need to prove that it is a regular solution in Ω_R for any $R > 0$ (cp. M. Nowotarska [79]).

Since the functions f^i, $i \in S$, are monotone (condition (W)) and from (2.74) it follows that the functions $f^i(t,x,u_{n-1})$, $i \in S$, are uniformly bounded in Ω with respect to n.

On the basis of Pogorzelski's results (see [91]; [92], pp. 140–160) concerning the properties of weak singular integrals, by means of which the solution of the linear system of equations is expressed

$$\mathcal{F}^i[u_n^i](t,x) = f^i(t,x,u_{n-1}), \quad i \in S, \quad \text{for} \quad (t,x) \in \Omega_R, \tag{2.83}$$

we conclude that the function $u_n(t,x)$ satisfies locally the Lipschitz condition with respect to x, with a constant independent on n. Hence by (2.82), we conclude that the boundary function $U(t,x)$ satisfies locally the Lipschitz condition with respect to variable x.

If we now take the system of equations

$$\mathcal{F}^i[z^i](t,x) = f^i(t,x,U), \quad i \in S, \quad \text{for} \quad (t,x) \in \Omega_R, \tag{2.84}$$

with the initial-boundary condition

$$z(t,x) = U(t,x) \quad \text{for} \quad (t,x) \in \Gamma_R, \tag{2.85}$$

then the last property of $U(t,x)$ together with conditions (H_f) and (L), implies that the right-hand sides of system (2.84) are continuous with respect to t, x in Ω_R and locally Hölder continuous with respect to x.

Hence, by Lemma 2.9, there exists the unique regular solution z of problem (2.84), (2.85) in $\overline{\Omega}_R$ and $z \in C_S^{2+\alpha}(\overline{\Omega}_R)$.

On the other hand, using (2.82), we conclude that the right-hand sides of (2.83) converge uniformly in Ω_R to the right-hand sides of (2.84),

$$\lim_{n\to\infty} f^i(t,x,u_n) = f^i(t,x,U) \quad \text{uniformly in} \quad \overline{\Omega}_R, \ i \in S. \tag{2.86}$$

Moreover, the boundary values of $u_n(t,x)$ converge uniformly on Γ_R to the respective values of $U(t,x)$. Hence, using the theorem on the continuous dependence of the solution on the right-hand sides of the system and on the initial-boundary conditions (see J. Szarski [111], Th. 51.1, p. 147) to systems (2.83) and (2.84), we obtain

$$\lim_{n\to\infty} u_n^i(t,x) = z^i(t,x) \quad \text{uniformly in} \quad \overline{\Omega}_R, \ i \in S. \tag{2.87}$$

By (2.82) and (2.87), there is

$$z = z(t,x) = U(t,x) \quad \text{in} \quad \overline{\Omega}_R$$

for an arbitrary R, which means that

$$z = z(t,x) = U(t,x) \quad \text{for} \quad (t,x) \in \overline{\Omega},$$

i.e., z is the regular solution of problem (2.60) within the sector $\langle u_0, v_0 \rangle$, and $z \in$ $\in C^{2+\alpha}_{S,E_2}(\overline{\Omega})$.

Moreover, by (2.74) and Lemma 2.9, there is $z \in E_2(M, K; \overline{\Omega})$, where $M =$ $= M(M_0, M_f, M_\phi)$ and $K = K(K_0, K_f, K_\phi)$.

The uniqueness of the solution of this problem follows directly from the uniqueness criterion of Szarski [116] (cp. [50]), which ends the proof. □

CHAPTER 3. REMARKS ON MONOTONE ITERATIVE METHODS

3.1. SOME REMARKS IN CONNECTION WITH APPLICATIONS OF NUMERICAL METHODS

For certain numerical methods (cp. M. Malec [71]) it is essential that the derivatives $\mathcal{D}_t w^i (i \in S)$ of the functions $w = \{w^i\}_{i \in S}$ searched for exist and are continuous not only for $t \in (0, T)$, but also for $t \in [0, T)$. It is so, as this fact is used to construct appropriate difference schemes and to prove the consistency and convergence of the numerical method used. Thus, the assumption that a solution of problem (0.1), (0.2) is a $C_S^{2+\alpha}(\overline{D})$ — function is not sufficient. One has to consider functions continuous in $\overline{D}$, with continuous derivatives $\mathcal{D}_t w^i$, $\mathcal{D}_{x_j} w^i$ and $\mathcal{D}^2_{x_j x_k} w^i$ $(j, k = 1, \ldots, m;\ i \in S)$ in $\overline{D}$. This means one has to consider the Hölder spaces $H^{k+\alpha, \frac{k+\alpha}{2}}(\overline{D})$ in Ladyženskaja's sense (see O. A. Ladyženskaja *et al.* [57], pp. 7–8).

Definition 3.1. *The Hölder space* $H^{l, \frac{l}{2}}(\overline{D}) := H^{k+\alpha, \frac{k+\alpha}{2}}(\overline{D})$ $(k = 0, 1, 2,\ 0 < < \alpha < 1,\ l = k + \alpha)$ *is the space of continuous functions* h *in* $\overline{D}$ *whose all derivatives* $\mathcal{D}_t^r \mathcal{D}_x^s h(t, x)$ $(0 \le 2r + s \le k)$ *exist and are Hölder continuous with exponent* α $(0 < \alpha < 1)$ *in* $\overline{D}$*, with the finite norm*

$$|h|^{k+\alpha} := \langle h \rangle^{k+\alpha} + \sum_{l=1}^{k} \langle h \rangle^{l},$$

where the components $\langle h \rangle^{k+\alpha}$ *and* $\langle h \rangle^{l}$ *are defined as in [ibid.].*

Definition 3.2. *By* $H_S^{l, \frac{l}{2}}(\overline{D}) := H_S^{k+\alpha, \frac{k+\alpha}{2}}(\overline{D})$ *we denote the Banach space of mappings* w *such that* $w^i \in H^{l, \frac{l}{2}}(\overline{D})$ *for all* $i \in S$*, with the finite norm*

$$\|w\|^{k+\alpha} := \sup\{ \left|w^i\right|^{k+\alpha} : i \in S \}.$$

Remark 3.1. *We have*

$$H^{\alpha, \frac{\alpha}{2}}(\overline{D}) = C^{0+\alpha}(\overline{D}),$$
$$H^{2+\alpha, 1+\frac{\alpha}{2}}(\overline{D}) \subset C^{2+\alpha}(\overline{D}),$$

where $C^{k+\alpha}(\overline{D})$ *are the Hölder spaces in Friedman's sense (see A. Friedman [41, pp. 61–63] and Definition 1.3).*

Definition 3.3. *A mapping* $w = \{w^i\}_{i \in S} \in C_S(\overline{D})$ *will be called *-regular in* $\overline{D}$ *if the functions* w^i*,* $i \in S$*, have continuous derivatives* $\mathcal{D}_t w^i$*,* $\mathcal{D}_{x_j} w^i$ *and* $\mathcal{D}^2_{x_j x_k} w^i$ *in* $\overline{D}$ *for* $j, k = 1, \ldots, m$*.*

We will consider system (0.1) with initial-boundary condition (0.3), i.e., the problem

$$\begin{cases} \mathcal{F}^i[z^i](t, x) = f^i(t, x, z) & \text{for } (t, x) \in D,\ i \in S \\ z(t, x) = \phi(t, x) & \text{for } (t, x) \in \Gamma. \end{cases} \tag{3.1}$$

We assume that $\partial G \in H^{2+\alpha}$ and

Assumption ($\overline{\mathbf{H}}_{\boldsymbol{a}}$). *All the coefficients* $a^i_{jk} = a^i_{jk}(t,x)$, $a^i_{jk} = a^i_{kj}$ *and* $b^i_j = b^i_j(t,x)$ $(j,k = 1,\dots,m,\ i \in S)$ *of the operators* $\mathcal{L}^i$, $i \in S$, *are locally Hölder continuous with exponent* α $(0 < \alpha < 1)$ *with respect to* t *and* x *in* $\overline{D}$ *and they Hölder norms are uniformly bounded.*

Assumption ($\overline{\mathbf{H}}_{\boldsymbol{f}}$). $f(\cdot,\cdot,s) \in H_S^{\alpha,\frac{\alpha}{2}}(\overline{D})$, *where* $0 < \alpha < 1$.

Assumption ($\overline{\mathbf{H}}_{\boldsymbol{\phi}}$). $\phi \in H_S^{2+\alpha,1+\frac{\alpha}{2}}(\Gamma)$, *where* $0 < \alpha < 1$.

Assumption $\overline{A}$. *We will assume that there exists at least one pair* u_0 *and* v_0 *of a lower and an upper solution of problem* (3.1) *in* $\overline{D}$, *respectively, and* u_0, $v_0 \in H_S^{2+\alpha,1+\frac{\alpha}{2}}(\overline{D})$.

Using a similar argument as in section 2.1 we will prove the following auxiliary lemmas and main theorem.

Lemma 3.1. *If* $\beta \in H_S^{\alpha,\frac{\alpha}{2}}(\overline{D})$ *and the function* $f = \{f^i\}_{i\in S}$ *generating the Nemytskiǐ operator* $\overline{\mathbf{F}} = \{\overline{\mathbf{F}}^i\}_{i\in S}$

$$\overline{\mathbf{F}}\colon \beta \mapsto \overline{\mathbf{F}}[\beta] = \overline{\delta},$$

where

$$\overline{\mathbf{F}}^i[\beta](t,x) := f^i(t,x,\beta) = \overline{\delta}^i(t,x), \quad i \in S, \tag{3.2}$$

satisfies assumptions $(\overline{H}_f)$ *and* (L), *then*

$$\overline{\mathbf{F}}[\beta] = \overline{\delta} \in H_S^{\alpha,\frac{\alpha}{2}}(\overline{D}).$$

Lemma 3.2. *If assumptions* $(\overline{H}_a)$, $(\overline{H}_g)$, $(\overline{H}_\phi)$ *hold and* $\partial G \in H^{2+\alpha}$, *then the problem*

$$\begin{cases} \mathcal{F}^i[z^i](t,x) = g^i(t,x) & i \in S, \text{ for } (t,x) \in D, \\ z(t,x) = \phi(t,x) & \text{for } (t,x) \in \Gamma \end{cases} \tag{3.3}$$

*has the unique *-regular solution* γ, *and* $\gamma \in H_S^{2+\alpha,1+\frac{\alpha}{2}}(\overline{D})$.

(For a proof, see [57], Th. 5.2, p. 320).

Corollary 3.1. *If the assumptions of Lemmas 3.1 and 3.2 hold, then the operator* $\overline{\mathcal{P}}$ *defined analogously as in section 2.1 has the following property:*

$$\overline{\mathcal{P}}\colon H_S^{\alpha,\frac{\alpha}{2}}(\overline{D}) \ni \beta \mapsto \overline{\mathcal{P}}[\beta] = \gamma \in H_S^{2+\alpha,1+\frac{\alpha}{2}}(\overline{D}). \tag{3.4}$$

The following theorem is true.

Theorem 3.1. *Let assumptions* $\overline{A}$ *and* $(\overline{H}_a)$, $(\overline{H}_f)$, $(\overline{H}_\phi)$, (W), (L), (V) *hold in the set* $\mathcal{K}^*$. *If the successive terms of the approximation sequences* $\{u_n\}$ *and* $\{v_n\}$

*are defined as *-regular solutions of the following infinite systems of linear parabolic differential equations*

$$\begin{cases} \mathcal{F}^i[u_n^i](t,x) = f^i(t,x,u_{n-1}), & i \in S, \text{ for } (t,x) \in D, \\ u_n(t,x) = \phi(t,x) & \text{for } (t,x) \in \Gamma \end{cases} \tag{3.5}$$

and

$$\begin{cases} \mathcal{F}^i[v_n^i](t,x) = f^i(t,x,v_{n-1}), & i \in S, \text{ for } (t,x) \in D, \\ v_n(t,x) = \phi(t,x) & \text{for } (t,x) \in \Gamma, \end{cases} \tag{3.6}$$

*i.e., if we use the method of direct iterations, then there exists the unique *-regular solution z of problem* (3.1) *within the sector $\langle u_0, v_0 \rangle$, and $z \in H_S^{2+\alpha,1+\frac{\alpha}{2}}(\overline{D})$.*

3.2. ON CONSTRUCTIONS OF UPPER AND LOWER SOLUTIONS; POSITIVE SOLUTIONS

3.2.1. Fundamental example

A basic difficulty in applying monotone iterative methods lies in the construction of a pair of a lower and an upper solution of the given problem. The otherwise ample literature on monotone methods describes no general way to build such functions. However, the right-hand sides of the equations discussed, i.e., the functions $f^i(t,x,s)$, should be bounded in the domain considered. A lower and an upper solution can be easily built by means of the Green function for the equation and domain considered.

Let functions $f^i(t,x,s)$, $i \in S$, and operators $\mathcal{F}^i$ fulfil the suitable assumptions in the set $\overline{D} \times C_S(\overline{D})$. Using the fundamental solutions $\Gamma^i(t,x;\tau,\xi)$ for the equations $\mathcal{F}^i[z^i](t,x) = 0$, $i \in S$, we define the Green functions $\mathcal{G}^i(t,x;\tau,\xi)$ for the domain D (see O. A. Ladyżenskaja *et al.* [57], pp. 412, 413 and S. Brzychczy [25], pp. 32–33) and consider the functions

$$U_0^i(t,x) = m_i(2\sqrt{\pi})^{-m} \int_0^t \int_G \mathcal{G}^i(t,x;\tau,\xi)d\tau d\xi, \tag{3.7}$$

$$V_0^i(t,x) = M_i(2\sqrt{\pi})^{-m} \int_0^t \int_G \mathcal{G}^i(t,x;\tau,\xi)d\tau d\xi, \quad i \in S, \tag{3.8}$$

where

$$m_i = \inf_{\overline{D} \times C_S(\overline{D})} f^i(t,x,s), \quad M_i = \sup_{\overline{D} \times C_S(\overline{D})} f^i(t,x,s).$$

These functions fulfil homogeneous initial-boundary condition (1.2) and the equations

$$\mathcal{F}^i[U_0^i](t,x) = m_i,$$
$$\mathcal{F}^i[V_0^i](t,x) = M_i, \quad i \in S, \quad \text{for} \quad (t,x) \in D.$$

Therefore, the following differential inequalities hold

$$\mathcal{F}^i[U_0^i](t,x) - f^i(t,x,U_0) = m_i - f^i(t,x,U_0) \le 0,$$
$$\mathcal{F}^i[V_0^i](t,x) - f^i(t,x,V_0) = M_i - f^i(t,x,V_0) \ge 0, \quad i \in S,$$

for $(t,x) \in D$. Consequently, the functions U_0 and V_0 are the lower and the upper solution of problem (0.1), (1.2) in $\overline{D}$, respectively, and one may use them as zero-approximations.

3.2.2. Other examples

In papers K. Y. K. Ng [118] and K. K. Tam [119] have given the construction of a lower and an upper solution of a flow past a non-uniformly heated plate and for a problem in combustion theory. The construction procedure itself is interesting and instructive.

These authors consider the problem of forced heat convection over a heated flat plate and apply the monotone iterative method to solve it. A lower and an upper solution are constructed by applying the so-called modified Oseen linearization or by use of a comparison theorem. In both these papers, while constructing a lower and an upper solution, the knowledge of the processes described and interpretation of coefficients appearing in the equations play a crucial role.

In the second paper basing on results of D.H. Sattinger [107], the author describe a procedure which makes use of the comparison principle (Theorem 1.2) in the construction of a lower and an upper solution of the following semilinear equation which governs the combustion of a material

$$\begin{aligned} &\frac{\partial \Theta}{\partial t} = \Delta\Theta + \delta \exp\left(\frac{\alpha\Theta}{\alpha+\Theta}\right) \quad \text{in } D, \\ &\Theta(0,x) = h(x) \quad \text{on } G, \\ &\Theta(t,x) = 0 \quad \text{on } \sigma, \end{aligned} \tag{3.9}$$

where $\Theta = \Theta(t,x)$ is the temperature, δ and α are positive parameters and $D = (0,T] \times G$, where G is a long cylinder or a sphere.

In the case of $\Delta = \frac{\partial^2}{\partial x^2}$ and $0 < x < 1$, the author seeks steady-state lower and upper solutions in the form

$$u_0(x) = C \sin^2 \pi x,$$
$$v_0(x) = 4kx(1-x),$$

where C, $k > 0$ are constants to be determined.

The solution Θ of the problem considered is such that

$$C \sin^2 \pi x \le \lim_{t \to +\infty} \Theta(t,x) \le 4kx(1-x)$$

for some constants C and k.

3.2.3. Positive solutions

For the purposes of applications, an important task is to find positive (nonnegative) solutions of the problem considered. Namely, in many physical problems only positive solutions are of interested. In general, this is a difficult question. The papers of H. Amann [1] and C. V. Pao [81,83], V. Lakshmikantham and Z. Drici [59] play a crucial role in this field.

The following theorem may be directly proved.

Theorem 3.2. *Let we consider problem* (0.1), (0.4), (0.5) *and* (0.6) *in* $\overline{D}$, *i.e.:*

$$\begin{cases} \mathcal{F}^i[z^i](t,x) = f^i(t,x,z), & i \in S, \textit{ for } (t,x) \in D, \\ z(0,x) = \phi_0(x) & \textit{for } x \in G, \\ z(t,x) = \psi(t,x) & \textit{for } (t,x) \in \sigma \end{cases} \tag{3.10}$$

and

1° *there exists a positive upper solution* v_0 *of this problem in* $\overline{D}$*;*

2° *the following inequalities:*

$$f^i(t,x,0) \geq 0, \;\; i \in S \textit{ in } D, \quad \phi_0(x) \geq 0 \textit{ on } G, \quad \psi(t,x) \geq 0 \textit{ on } \sigma \tag{3.11}$$

hold and not all the three functions are identically zero;

3° *the functions* $f^i(t,x,s)$, $i \in S$, *satisfy the condition* (W), (V) *and the left-hand side Lipschitz condition* (L) *with respect to* s *for* $s \in \langle 0, v_0 \rangle$.

Then problem (3.10) *has at last one positive solution* z *within the sector* $\langle 0, v_0 \rangle$. *Moreover, if* $f^i(t,x,0) = \phi_0(x) = \psi(t,x) = 0$, *then* $z = z(t,x) = 0$ *in* $\overline{D}$.

3.3. ESTIMATION OF CONVERGENCE RATES FOR DIFFERENT ITERATIVE METHODS

Let $\{u_n\}$ be a sequence of successive approximations which converges to a solution z of the problem considered (0.2), (0.3):

$$z = z(t,x) = \lim_{n\to\infty} u_n(t,x) \quad \text{for} \quad (t,x) \in \overline{D}.$$

We shall say that the sequence $\{u_n\}$ converge to the solution z with a *geometrical rate*, if the following inequalities hold

$$\|u_n - z\|_0 \leq cq^n, \quad n = 1,2,\ldots, \quad \text{in} \quad \overline{D}, \tag{3.12}$$

where $c = \text{const} > 0$ and $0 < q < 1$.

If

$$\|u_n - z\|_0 \leq c\frac{K^n}{n!}, \quad n = 1,2,\ldots, \quad \text{in} \quad \overline{D}, \tag{3.13}$$

where c and K are nonnegative constants, then we say that $\{u_n\}$ converges to z with a *power rate*.

The convergence in this sense is essentially faster than the convegence with a geometrical rate.

We shall say that $\{u_n\}$ converges to z with a *Newton rate*, if

$$\|u_n - z\|_0 \le c\delta^{2^n}, \quad n = 1, 2, \dots, \quad \text{in} \quad \overline{D}, \tag{3.14}$$

where $c = \text{const} > 0$ and $0 < \delta < 1$. This convergence is essentially faster than the previous one.

If the successive terms of sequences $\{u_n\}$ and $\{v_n\}$ are lower and upper solutions of problem (0.2), (0.3), defined respectively as solutions of systems (2.2) and (2.3), then we have estimates

$$0 \le v_n^i(t,x) - u_n^i(t,x) \le N_0 \frac{[(L_1 + L_2)t]^n}{n!}, \quad n = 0, 1, 2, \dots, i \in S,$$

for $(t,x) \in \overline{D}$,where $N_0 = \|v_0 - u_0\|_0 < \infty$.

Therefore, by (2.7) these sequence converge to the exact solution with a power rate.

Analogously, by (2.46), the sequences $\{\overline{u}_n\}$ and $\{\overline{v}_n\}$ defined as solutions of systems (2.43) and (2.44), converge to z with at least power rate.

Almost the same we have for Chaplygin's sequences $\{\hat{u}_n\}$ and $\{\hat{v}_n\}$ defined as solutions of equations (2.25) and (2.26). Estimate (2.28) proves that these sequences converge with at least power rate.

W. Mlak [73] (see also P. K. Zeragia [133]) has applied Chaplygin's method for the Fourier first initial-boundary problem for the nonlinear parabolic differential equation of the form

$$\frac{\partial z}{\partial t} - \frac{\partial^2 z}{\partial x^2} = f(t,x,z(t,x)) \quad \text{for} \quad (t,x) \in D := [0,T] \times [a,b],$$
$$z(t,x) = \psi(t,x) \quad \text{for} \quad (t,x) \in \Gamma.$$

He assumed that there existed a lower u_0 and an upper v_0 solutions and defined a sequence of lower function $\{u_n\}$ as regular solutions of the following linear equations

$$\frac{\partial u_n}{\partial t} - \frac{\partial^2 u_n}{\partial x^2} = f(t,x,u_{n-1}(t,x)) + \\ + f_y(t,x,u_{n-1}(t,x)) \cdot [u_n(t,x) - u_{n-1}(t,x)], \quad n = 1, 2, \dots.$$

Next he studied the convergence of the sequence $\{u_n\}$ to the exact solution of this problem, assuming additionaly that the function $f(t,x,y)$ is sufficiently regular and the derivative $f_y(t,x,y)$ satisfies the Lipschitz condition with respect to y. In particular, assuming that

$$\sup_{(t,x,y)\in K^*} |f_y(t,x,y)| = c_0 < +\infty,$$
$$\sup_{(t,x,y)\in K^*} |f_{yy}(t,x,y)| = H < +\infty$$

and $f_{yy} > 0$ he proved the estimates

$$|u_n(t,x) - z(t,x)| \leq \frac{2C}{2^{2^n}}, \quad n = 1, 2, \ldots, \quad \text{in } \overline{D},$$

where

$$C = \frac{1}{2HTe^{c_0 T}},$$

which is analogous to the one obtained by N. N. Lusin [69] for ordinary differential equations.

This means that the convergence rate is quadratic and the sequence of successive approximations converges uniformly to the unique solution with the *Newton speed*.

We cannot directly repeat the above results on the rate of convergence of successive approximations in the case of a differential-functional equation of parabolic type of the form

$$\frac{\partial z}{\partial t} - \frac{\partial^2 z}{\partial x^2} = f(t, x, z(t,x), z). \tag{3.15}$$

It is so because even in the simple case considered by us the algorithms have not guaranteed convergence with this rate. To prove results similar to the above ones we need stronger assumptions about the regularity of function $f(t,x,y,s)$, or have to use the full quasilineralization of equation (3.15) with respect to the both arguments y and s, simultaneously. This means that we need to define the sequence of successive approximations $\{u_n^*\}$ as solutions of the following linear differential-functional equations

$$\begin{aligned} \frac{\partial u_n^*}{\partial t} - \frac{\partial^2 u_n^*(t,x)}{\partial x^2} &= f\left(t, x, u_{n-1}^*(t,x), u_{n-1}^*\right) + \\ &+ f_y\left(t, x, u_{n-1}^*, u_{n-1}^*\right) \cdot \left[u_n^*(t,x) - u_{n-1}^*(t,x)\right] + \\ &+ f_s\left(t, x, u_{n-1}^*(t,x), u_{n-1}^*\right) \cdot \left[u_n^* - u_{n-1}^*\right], \quad n = 1, 2, \ldots, \end{aligned} \tag{3.16}$$

where f_s is the Fréchet derivative of a function $f = f(t,x,y,s)$ with respect to the functional argument s.

3.4. EXTENSIONS OF MONOTONE ITERATIVE METHODS TO MORE GENERAL EQUATIONS

3.4.1. Weakly coupled systems

We have confined ourselves to studying weakly coupled systems in view of the results of A. Pliś [88, 89] and the usefulness of such systems in numerous applications. In the case of strongly coupled systems, the situation gets intrinsically complex.

There is the well-known fundamental example, given by A. Pliś, of the nonuniqueness of the Cauchy problem for the strongly coupled system of linear first order partial differential equations of the form

$$\frac{\partial u_j}{\partial x} = \sum_{k=1}^{2} a_{jk}(x,y) \frac{\partial u_k}{\partial y} \quad \text{for} \quad (x,y) \in \mathbb{R}^2 \quad (j = 1, 2), \tag{3.17}$$

with the initial condition

$$u_j(0,y) = 0 \quad \text{for} \quad y \in \mathbb{R} \quad (j = 1,2), \tag{3.18}$$

where the coefficients a_{jk} $(j,k = 1,2)$ are defined in the whole plane $\mathbb{R}^2$ and $a_{jk} \in$ $\in C^\infty(\mathbb{R}^2)$. Problem (3.17), (3.18) does not have a unique solution of the class $C^\infty(\mathbb{R}^2)$. Precisely, this problem has a solution of the class $C^\infty(\mathbb{R}^2)$, vanishing together with all its derivatives for $x = 0$, and not vanishing identically in any neighbourbood of the point $(0,0)$.

3.4.2. Examples of finite and infinite important systems

1. Let us consider the differential equations of heat conduction in homogeneous isotropic solids (see H. S. Carslaw and J. C. Jaeger [35], pp. 1–49)

$$c\rho \frac{\partial u}{\partial t} - \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\left(\kappa \frac{\partial u}{\partial x_j}\right) = f(t,x,u), \tag{3.19}$$

where $x = (x_1, x_2, x_3)$, $u = u(t,x)$ is the temperature, ρ and c are the average density and the specific heat of the material, κ is the thermal conductivity coefficient and f is the reaction function. If the thermal properties of solid depend on the temperature (as in the solidification of castings, cp. S. Brzychczy *et al.* [19]), then the situation is more complicated, since the equation becomes nonlinear. In particular, if $\kappa = \kappa(u)$, then we obtain the following equation

$$c\rho \frac{\partial u}{\partial t} - \kappa(u)\Delta u - \frac{d\kappa}{du} \sum_{j=1}^{3} \left(\frac{\partial u}{\partial x_j}\right)^2 = f(t,x,u). \tag{3.20}$$

In the case of anisotropic solids (crystals, laminated materials such as transformer cores), if the thermal conductivity coefficients depend on the temperature, i.e., $\kappa_{jk} =$ $= \kappa_{jk}(u)$, then we obtain the quasilinear equation (with cubic anisotropy)

$$c\rho \frac{\partial u}{\partial t} - \sum_{j,k=1}^{3} \kappa_{jk}(u) \frac{\partial^2 u}{\partial x_j \partial x_k} - \sum_{j,k=1}^{3} \frac{d\kappa_{jk}}{du} \left(\frac{\partial u}{\partial x_j}\right)\left(\frac{\partial u}{\partial x_k}\right) = f(t,x,u). \tag{3.21}$$

2. As a particular case of system (0.2), when S is a finite set of indices with r elements, we may consider the classic system of reaction-diffusion-convection equations with multicomponent diffusion, which can be writen in the following conventional form

$$\mathcal{D}_t z^i(t,x) - \nabla \circ \left(a^i(t,x)\nabla z^i(t,x)\right) + \overrightarrow{v}^i(t,x) \circ \nabla z^i(t,x) = f^i\left(t,x,z(t,x)\right), \tag{3.22}$$

where $i = 1,\ldots,r$, $z = (z^1,\ldots,z^r)$, $\nabla := \mathcal{D}_x := (\mathcal{D}_{x_1},\ldots,\mathcal{D}_{x_m})$ is the gradient operator, $\overrightarrow{v}^i := \overrightarrow{v}^i(t,x) = \left(v_1^i(t,x),\ldots,v_m^i(t,x)\right)$ is the drift vector, $a^i = (a^i_{jk}(t,x))$, $j,k = 1,\ldots,m$, are coefficients of multicomponent diffusion and

$$\overrightarrow{v}^i(t,x) \circ \nabla z^i(t,x) := v_1^i(t,x)\mathcal{D}_{x_1} z^i(t,x) + \cdots + v_m^i(t,x)\mathcal{D}_{x_m} z^i(t,x).$$

3. In numerous papers [14, 61, 127–129], D. Wrzosek with Ph. Bénilan and Ph. Laurençot studied the phenomenon of coagulation and fragmentation of clusters. The discrete coagulation-fragmentation models with diffusion are expressed in terms of the infinite countable systems of reaction-diffusion equations of the form

$$\mathcal{D}_t z^i(t,x) - d_i \Delta z^i(t,x) = f_6^i(t,x,z), \quad i \in \mathbb{N}, \tag{3.23}$$

for $(t,x) \in (0,T) \times G = D$, $G \subset \mathbb{R}^m$ is a bounded domain with sufficiently smooth boundary ∂G, with the initial condition

$$z(0,x) = U_0(x) \geq 0 \quad \text{on } G, \tag{3.24}$$

and the boundary condition of the Neumann type

$$\frac{\partial z^i}{\partial \nu} = 0, \quad i \in \mathbb{N}, \quad \text{on } \sigma, \tag{3.25}$$

where the functions f_6^i are given by (0.7), the coagulation a_k^i and fragmentation b_k^i rates are nonnegative constants and the diffusion coefficients d_i are positive for $i, k = 1, 2, \ldots$.

4. M. Lachowicz and D. Wrzosek in interesting article [55] proposes a new nonlocal model of cluster coagulation and fragmentation. This model is expressed in terms of the infinite countable system of semilinear integro-differential parabolic equations of the form

$$\mathcal{D}_t z^i(t,x) - \mathcal{A}^i[z^i](t,x) = f_7^i(t,x,z), \quad i \in \mathbb{N}, \tag{3.26}$$

in $(0,\infty) \times G = D$, with initial and boundary conditions (3.24), (3.25) where the diffusion operators

$$\mathcal{A}^i[z^i](t,x) := \sum_{k,l=1}^{m} \mathcal{D}_{x_k} \left(d_{kl}^i(x) \mathcal{D}_{x_l} z^i(t,x) \right), \quad i \in \mathbb{N}$$

are uniformly elliptic in $\overline{D}$, the functions f_7^i are given by (0.8) (these are nonlocal coagulation-fragmentation operators), and the diffusion coefficients d_{kl}^i are positive.

To solve the problems considered for systems (3.23) and (3.26) the authors apply the truncation method.

5. The more general reaction-diffusion operators and suitable infinite uncountable systems of equations of the form

$$\begin{aligned} \mathcal{D}_t z(t,x) - \nabla \circ \Big(a(t,x) \nabla z(t,x) + \overrightarrow{b}(t,x) z(t,x) \Big) + \overrightarrow{c}(t,x) \circ \nabla z(t,x) + \\ + d(t,x) z(t,x) = [\mathcal{D}_t z]_{coag} + [\mathcal{D}_t z]_{frag} + h, \end{aligned} \tag{3.27}$$

where the diffusion matrix a, the drift vectors $\overrightarrow{b}$ and $\overrightarrow{c}$, and the absorption rate d are sufficiently smooth functions, has been studied by H. Amann in [3] and the existence and uniqueness has been proved in some class of volume preserving solutions.

3.4.3. Extensions of monotone iterative methods to more general equations

The theorems and methods given in this paper can be extended to more general infinite systems of quasilinear parabolic differential–functional equations with a full functional dependence of the right-hand sides of the systems on the unknown function z and on its spatial derivatives $\mathcal{D}_x z$, of the form

$$\begin{aligned}\mathcal{D}_t z^i(t,x) - \sum_{j,k=1}^{m} a^i_{jk}\left(t,x,z^i(t,x),\mathcal{D}_x z^i(t,x)\right)\mathcal{D}^2_{x_j x_k} z^i(t,x) = \\ = f^i\left(t,x,z(t,x),\mathcal{D}_x z^i(t,x), z, \mathcal{D}_x z^i\right), \quad i \in S,\end{aligned} \tag{3.28}$$

with classic or nonlocal initial-boundary conditions. The remarks and algorithms given by W. Mlak [73], T. Kusano [54], H. Leszczyński [63,64] and A. Bychowska [32] for finite systems of similar equations may by used to construct iterative sequences for infinite systems.

The generalized quasilinearization method for infinite systems of semilinear differential-functional equations of parabolic type of the form (0.2) we will study in a forthcoming paper.

3.5. SOME REMARKS ON THE TRUNCATION METHOD[7)]

Let us consider the infinite countable system of semilinear parabolic equations of the reaction-diffusion-convection type of the form (0.1), i.e.,

$$\mathcal{F}^i[z^i](t,x) = f^i(t,x,z) := f^i(t,x,z^1,z^2,\ldots), \quad i \in \mathbb{N}, \quad \text{for} \quad (t,x) \in D, \tag{3.29}$$

where the operators $\mathcal{F}^i$, $i \in \mathbb{N}$, are uniformly parabolic in $\overline{D}$ and the functions $f^i(t,x,s)$, $i \in \mathbb{N}$, where

$$f^i \colon \overline{D} \times C_{\mathbb{N}}(\overline{D}) \to \mathbb{R}, \quad (t,x,s) \mapsto f^i(t,x,s), \quad i \in \mathbb{N},$$

are functionals of the unknown function $z = \{z^i\}_{i\in\mathbb{N}} = (z^1, z^2, \ldots)$.

For system (3.29) we will consider the initial-boundary condition (0.3), i.e.,

$$z(t,x) = \phi(t,x) \quad \text{for} \quad (t,x) \in \Gamma. \tag{3.30}$$

We will study the solvability of problem (3.29), (3.30) in some real Banach space $\mathfrak{B}$. This means that we will be interested in the existence of solutions $z = z(t,x)$, which are defined for $(t,x) \in \overline{D}$ and such that $z(t,x) \in \mathfrak{B}$, for each $(t,x) \in \overline{D}$.

A solution z of the infinite countable system (3.29) is defined as a limit in a Banach space $\mathfrak{B}$ of sequence of successive approximations $\{z_N\}_{N=1,2,\ldots}$, where

7) In this paper we give only some general remarks on the truncation method, however this method we will study in a separate paper.

$z_N = (z_N^1, z_N^2, \cdots, z_N^N)$ is defined as a solution of a suitable finite system of N equations (N is an arbitrary fixed natural number) of the following form

$$\mathcal{F}^j[z_N^j](t,x) = F_N^j(t,x,z_N) := F_N^j(t,x,z_N^1,z_N^2,\ldots,z_N^N), \quad j=1,2,\ldots,N, \tag{3.31}$$

for $(t,x) \in D$, with the initial-boundary condition

$$z_N^j(t,x) = \phi^j(t,x), \quad j=1,2,\ldots,N, \quad \text{for} \quad (t,x) \in \Gamma. \tag{3.32}$$

where $\psi(t,x) = \phi(t,x)$ for $(t,x) \in \Gamma$ and the functions F_N^j are defined in a special way. Moreover, the remaining functions $z_N^{N+1}, z_N^{N+2}, \ldots$ will be defined as follows

$$z_N^j(t,x) := \psi^j(t,x), \quad j = N+1, N+2, \ldots, \quad \text{for} \quad (t,x) \in \overline{D}, \tag{3.33}$$

where ψ^i are functions defined also in a special way.

We adhere to the convention that every finite sequence $z_N = \left(z_N^1, z_N^2, \cdots, z_N^N\right)$ is treated as an infinite one

$$z_{N,\psi} = \left(z_{N,\psi}^1, z_{N,\psi}^2, \cdots, z_{N,\psi}^N, \psi^{N+1}, \psi^{N+2}, \cdots\right).$$

Now, we will study the methods of constructed an auxiliary finite system of N equations. First, we present the method of substitution given by W. Mlak and C. Olech [74]. This method has been applied as a method of integration of infinite countable systems of ordinary differential equations.

If α is a lower solution of problem (3.29), (3.30) in $\overline{D}$ and

$$\alpha(t,x) = \phi(t,x) \quad \text{for} \quad (t,x) \in \Gamma,$$

then we construct the finite system of N equations by substituting

$$\begin{aligned} \mathcal{F}^j[z_{N,\alpha}^j](t,x) &= f^j\left(t,x,z_{N,\alpha}^1, z_{N,\alpha}^2, \ldots, z_{N,\alpha}^N, \alpha^{N+1}, \alpha^{N+2}, \ldots\right) := \\ &:= F_{N,\alpha}^j\left(t,x,z_{N,\alpha}^1, z_{N,\alpha}^2, \ldots, z_{N,\alpha}^j\right), \quad j=1,2,\ldots,N, \quad \text{for} \quad (t,x) \in D, \end{aligned} \tag{3.34}$$

with the initial-boundary condition (3.32) and we define the remaining functions

$$z_{N,\alpha}^j(t,x) := \alpha^j(t,x), \quad j = N+1, N+2, \ldots, \quad \text{for} \quad (t,x) \in \overline{D}. \tag{3.35}$$

The sequence $\{z_{N,\alpha}\}$ is defined as follows

$$z_{N,\alpha} = \left(z_{N,\alpha}^1, z_{N,\alpha}^2, \ldots, z_{N,\alpha}^N, \alpha^{N+1}, \alpha^{N+2}, \ldots\right).$$

Analogously, if β is an upper solution of problem (3.29), (3.30) in $\overline{D}$ and

$$\beta(t,x) = \phi(t,x) \quad \text{for} \quad (t,x) \in \Gamma,$$

then we construct the finite system of N equations by substituting

$$\begin{aligned}\mathcal{F}^j[z^j_{N,\beta}](t,x) = f^j\left(t,x,z^1_{N,\beta},z^2_{N,\beta},\dots,z^N_{N,\beta},\beta^{N+1},\beta^{N+2},\dots\right) :=\\ := F^j_{N,\beta}\left(t,x,z^1_{N,\beta},z^2_{N,\beta},\dots,z^j_{N,\beta}\right), \quad j=1,2,\dots,N, \quad \text{for} \quad (t,x)\in D,\end{aligned} \tag{3.36}$$

with initial-boundary condition (3.32), and we define

$$z^j_{N,\beta}(t,x) := \beta^j(t,x), \quad j = N+1, N+2, \dots, \quad \text{for} \quad (t,x)\in \overline{D}. \tag{3.37}$$

The suitable sequence $\{z_{N,\beta}\}$ is defined as follows

$$z_{N,\beta} = \left(z^1_{N,\beta}, z^2_{N,\beta}, \dots, z^N_{N,\beta}, \beta^{N+1}, \beta^{N+2}, \dots\right).$$

If $u_0 = u_0(t,x) \equiv 0$ in $\overline{D}$ is the lower solution of homogeneous problem (3.29), (1.2) in $\overline{D}$, then we construct the finite system of N equations in the following form

$$\begin{aligned}\mathcal{F}^j[z^j_{N,0}](t,x) = f^j\left(t,x,z^1_{N,0},z^2_{N,0},\dots,z^N_{N,0},0,0,\dots\right) :=\\ := F^j_{N,0}\left(t,x,z^1_{N,0},z^2_{N,0},\dots,z^j_{N,0}\right), \quad j=1,2,\dots,N, \quad \text{for} \quad (t,x)\in D,\end{aligned} \tag{3.38}$$

with the homogeneous initial-boundary condition

$$z^j_{N,0}(t,x) = 0, \quad j = 1,2,\dots,N, \quad \text{for} \quad (t,x)\in \Gamma \tag{3.39}$$

and we define

$$z^j_{N,0}(t,x) = 0, \quad j = N+1, N+2, \dots, \quad \text{for} \quad (t,x)\in \overline{D}. \tag{3.40}$$

The sequence $\{z_{N,0}\}$ is defined as follows

$$z_{N,0} = \left(z^1_{N,0}, z^2_{N,0}, \dots, z^N_{N,0}, 0, 0, \dots\right).$$

On the other hand several authors to solve infinite countable systems of ordinary and partial differential equations (cp. J. M. Ball and J. Carr [5], M. Lachowicz and D. Wrzosek [55], B. Rzepecki [105], D. Wrzosek [127–129]) apply the truncation method: first one studies finite systems obtained by truncating to the first N equations and next they pass to the limit as N tends to inifinity. Proceeding this way, they obtain a system of N equations which is identical to system (3.38).

Example 3.1. *As an example of the use of truncation method, we will consider this method for the infinite countable system of differential-functional equations of the form (see M. Lachowicz and D. Wrzosek [55])*

$$\mathcal{D}_t z^i(t,x) - \mathcal{A}^i[z^i](t,x) = f^i_7(t,x,z), \quad i\in\mathbb{N}, \quad (t,x)\in D \tag{3.41}$$

where the operators

$$\mathcal{A}^i[z^i](t,x) := \sum_{k,l=1}^{m} \mathcal{D}_{x_k}\left(d^i_{kl}(x)\mathcal{D}_{x_l} z^i(t,x)\right), \quad i \in \mathbb{N}$$

are uniformly elliptic in $\overline{D}$*, the coefficients of multicomponent diffusion* d^i_{kl} *are positive and* f^i_7 *are nonlocal operators given by formula* (0.8)*, i.e.,*

$$\begin{aligned}
f^1_7(t,x,z) &= -z^1(t,x)\sum_{k=1}^{\infty}\int_G a^1_k(x,\xi)z^k(t,\xi)d\xi + \sum_{k=1}^{\infty}\int_G B^1_k(x,\xi)z^{1+k}(t,\xi)d\xi,\\
f^i_7(t,x,z) &= \frac{1}{2}\sum_{k=1}^{i-1}\int_{G\times G} A^{i-k}_k(x,\xi,\eta)z^{i-k}(t,\xi)z^k(t,\eta)d\xi d\eta -\\
&- z^i(t,x)\sum_{k=1}^{\infty}\int_G a^i_k(x,\xi)z^k(t,\xi)d\xi + \\
&+ \sum_{k=1}^{\infty}\int_G B^i_k(x,\xi)z^{i+k}(t,\xi)d\xi - \frac{1}{2}z^i(t,x)\sum_{k=1}^{i-1} b^{i-k}_k(x), \quad \text{for} \quad i = 2,3,\ldots
\end{aligned} \tag{3.42}$$

For system (3.41) *these authors consider the initial condition*

$$z(0,x) = U_0(x) \geq 0 \quad \text{for} \quad x \in G \tag{3.43}$$

and the boundary condition of the Neumann type

$$\frac{\partial z^i}{\partial \nu} = 0, \quad i \in \mathbb{N}, \quad \text{for} \quad (t,x) \in \sigma = (0,T] \times \partial G, \tag{3.44}$$

where $\nu \in C^1$ *is the outward normal vector field to* ∂G*.*

This problem is considered in the real Banach sequence space $\mathcal{X}_r$

$$\mathcal{X}_r := \{w = w(t,x) \colon \sum_{i=1}^{\infty}\int_G i^r \left|w^i(t,\xi)\right| \, \mathrm{d}\xi < \infty\}$$

for $r \geq 0$*, equipped with the weighted norm*

$$\|w\|_r := \sum_{i=1}^{\infty}\int_G i^r \left|w^i(t,\xi)\right| \, \mathrm{d}\xi.$$

They will be search the positive (nonnegative) solutions of this problem, i.e., solutions in the positive cone $\mathcal{X}^+_r$ *of the Banach space* $\mathcal{X}_r$ *defined as follows*

$$\mathcal{X}^+_r := \left\{w \in \mathcal{X}_r \colon w^i(t,x) \geq 0, \quad i \in \mathbb{N}, \quad \text{for} \quad t \in (0,T] \quad \text{and for a.e.} \quad x \in G\right\}.$$

A solution z of problem (3.41)–(3.44) *is constructed from successive approximations $\{z_N\}_{N=1,2,\dots}$, where z_N is defined as a solution of a suitable finite system of N equations of the form*

$$\mathcal{D}_t z_N^j(t,x) - \mathcal{A}^j[z_N^j](t,x) = F_{7,N}^j(t,x,z_N), \quad j=1,2,\dots,N, \tag{3.45}$$

for $(t,x)\in D$, where

$$\begin{aligned} F_{7,N}^1(t,x,z) &:= -z_N^1(t,x)\sum_{k=1}^{N-1}\int_G a_k^1(x,\xi)z_N^k(t,\xi)d\xi + \sum_{k=1}^{N-1}\int_G B_k^1(x,\xi)z_N^{1+k}(t,\xi)d\xi,\\ F_{7,N}^j(t,x,z) &:= \frac{1}{2}\sum_{k=1}^{j-1}\int_{G\times G} A_k^{j-k}(x,\xi,\eta)z_N^{j-k}(t,\xi)z_N^k(t,\eta)d\xi d\eta-\\ &\quad - z_N^j(t,x)\sum_{k=1}^{N-j}\int_G a_k^j(x,\xi)z_N^k(t,\xi)d\xi+\\ &\quad + \sum_{k=1}^{N-j}\int_G B_k^j(x,\xi)z_N^{j+k}(t,\xi)d\xi - \frac{1}{2}z_N^j(t,x)\sum_{k=1}^{j-1}b_k^{j-k}(x), \quad \text{for} \quad j=2,3,\dots \end{aligned} \tag{3.46}$$

with the initial condition

$$z_N(0,x) = U_0(x) \ge 0 \quad \text{for} \quad x\in G \tag{3.47}$$

and the boundary condition

$$\frac{\partial z_N^j}{\partial\nu} = 0, \quad j=1,2,\dots,N \quad \text{for} \quad (t,x)\in\sigma. \tag{3.48}$$

We note that these authors obtain truncated system (3.45), (3.46) *of N equations from system* (3.41), (3.42) *by setting (cp. [55], p. 52)*

$$a_k^i \equiv 0, \quad \text{and} \quad b_k^i \equiv 0, \quad \text{for} \quad i+k>N.$$

On the other hand, we have $0 \le f^i(t,x,0) = 0$ in D, $0 \le U_0(x)$ on G and therefore $u_0 = u_0(t,x) = 0$ in $\overline{D}$ is the lower solution of this problem. Therefore, from (3.38) *it follows that system* (3.45), (3.46) *may be obtained from system* (3.41), (3.42) *by substituting*

$$z_{N,0}^j(t,x) = 0, \quad j=N+1,N+2,\dots, \quad \text{for} \quad (t,x)\in\overline{D}.$$

and we will have

$$f^j\left(t,x,z_{N,0}^1,z_{N,0}^2,\dots,z_{N,0}^N,0,0,\dots\right) := F_{N,0}^j\left(t,x,z_{N,0}^1,z_{N,0}^2,\dots,z_{N,0}^N\right).$$

NOTES AND COMMENTS

Monotone iterative methods for partial differential equations of parabolic and elliptic type have been studied by numerous authors: H. Amann [1, 2], D. Bange [10], S. Brzychczy [16–18], [20–28], A. Bychowska [32], O. Diekman and N. M. Temme [39], T. Kusano [54], V. Lakshmikantham [59, 60], H. Leszczyński [63, 64], I. Łojczyk-Królikiewicz [70], S. G. Mikhlin and H. L. Smolickiĭ [72], W. Mlak [73], W. Mlak and C. Olech [74], I. P. Mysovskikh [75], M. Nowotarska [79], C. V. Pao [80,81], D. H. Sattinger [106], J. Smoller [108], P. K. Zeragia [131–133] and for problems with nonlocal boundary condition e.g. S. Carl and S. Heikkilä [33], C. V. Pao [84, 85]. Abstract monotone iterative methods in ordered Banach spaces are studied by E. Liz [66], E. Liz and J. J. Nieto [67]. It is also to be said that the monographs G. S. Ladde, V. Lakshmikantham and A. S. Vatsala [56] and C. V. Pao [83] plays a crucial role in this field.

Equations of reaction–diffusion–convection type arise naturally in numerous models. These describe e.g. the heat transfer process [34], the prediction of groundwater level (A. M. Nakhushev and V. N. Borisov [76]), the fluid flows with fading memory through fissured media (M. Peszyńska [87]) and other phenomena pertaining to the memory (H. Bellout [13]). Numerous examples of these systems arising in applications are given by O. Diekmann and N. M. Temme [39], P. C. Fife [40], A. Leung [65], J. D. Logan [68], C. V. Pao [82,83], F. Rothe [104], J. Smoller [108], J. Wu [130].

In the theory of differential inequalities the monographs by V. Lakshmikantham and S. Leela [58], J. Szarski [111] and W. Walter [122] play a crucial role. In the papers W. Walter [123, 124] given a review of the problems in the theory of parabolic equations In the case of finite systems of differential–functional inequalites, the fundamental results were obtained by J. Szarski [112–115] and, under somewhat different assumptions, by K. Nickel [77, 78], R. Redheffer and W. Walter [97–101], R. Redlinger [102, 103] (cp. also A. Bartłomiejczyk and H. Leszczyński [11]). In the case of infinite systems of inequalities, the fundamental results were obtained by J. Szarski [116, 117], D. Jaruszewska-Walczak [42], B. Kraśnicka [49, 50] and S. Brzychczy [31].

Infinite systems of differential equations, partial differential equations, integral and differential–functional equations have been studied by numerous authors: J. Banaś and M. Lecko [6–9], S. Brzychczy [26–28, 30], J. Chandra, V. Lakshmikantham and S. Leela [35], K. Deimling [37], D. Jaruszewska-Walczak [42], Z. Kamont and S. Kozieł [43], Z. Kamont [45], B. Kraśnicka [49, 50], W. Mlak and C. Olech [74], W. Pogorzelski [90, 92], A. Pudełko [94, 95], B. Rzepecki [105].

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