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THE USE OF PIEZOELECTRIC ELEMENTS TO CONTROL VIBRATIONS OF AX SMA MATERIAL SHAFT

SUMMARY

In the paper an analysis of the influence of structural friction and piezoelectric elements on damping of torsional vibration in a shaft is presented. The structural friction between the piezoelectric actuators and the shaft is taken into account. It is assumed that the shaft is made of SMA material. A dependency describing the equivalent Kirchhoff modulus and the equivalent viscotic damping coefficient is introduced for the assumed material. For the purpose of the analysis a mathematical model of a permanent shaft-sleeve joint is formulated. The Obtained differential equations of motion are solved using the asymptotic Bogoliubov–Krylov–Mitropolsky method via developing relations for derivatives of amplitude and phase. The obtained equations are solved for the case of harmonic excitation with uniformly increasing frequency of the driving torque and the effect of the gain factor magnitude on the damping level is examined. On the grounds of numerical simulations the growth of damping efficiency in the joint as a function of the gain factor is confirmed.

Keywords: *SMA, piezoelectric, control, structural friction, actuators, sensors*

ZASTOSOWANIE ELEMENTÓW PIEZOELEKTRYCZNYCH DO STEROWANIA DRGANIAMI WAŁU WYKONANEGO Z SMA

W pracy przeprowadzono analizę wpływu tarcia wewnętrznego oraz elementów piezoelektrycznych na tłumienie drgań skrętnych wału. Uwzględniono występowanie tarcia konstrukcyjnego pomiędzy wałem a aktuatorem. Przyjęto, że wał jest wykonany ze stopu SMA. Dla przyjętego materiału wału wprowadzono zależność określającą ekwiwalentny moduł Kirchhoffa oraz ekwiwalentny współczynnik tłumienia wiskotycznego. Na potrzeby analizy zbudowano model matematyczny układu z nierozłącznym połączeniem tuleja – wał, z uwzględnieniem tarcia konstrukcyjnego. Otrzymane różniczkowe równania ruchu rozwiązano asymptotyczną metodą Bogoliubowa–Kryłowa–Mitropolskiego, otrzymując różniczkowe równania określające wyrażenia na pochodną amplitudy i pochodną fazy. Otrzymane równania rozwiązano dla przypadku wymuszenia harmonicznego z jednostajnie zmienną częstością momentu wymuszającego oraz dla przypadku drgań swobodnych, badając wpływ wzmocnienia na intensywność tłumienia. Na podstawie przeprowadzonych obliczeń numerycznych otrzymanego rozwiązania stwierdzono zwiększenie efektywności tłumienia w połączeniu, zwiększające się wraz ze wzrostem współczynnika wzmocnienia wzmacniacza.

Słowa kluczowe: *SMA, piezoelektryki, sterowanie, tarcie wewnętrzne, aktuatory, czujniki*

1. INTRODUCTION

The issue of vibration damping is very important in machine dynamics. The decrease of vibration amplitude is an example of mechanical energy dissipation. Environment characteristics, internal material friction and friction in supports and joints influence vibration damping. Thanks to modern technologies materials such as piezoelectric polymers, rheological fluids or shape memory alloys (SMA) – which all react to external stimulation of various nature (electromagnetic waves, mechanical tension, temperature) – have been developed. Such materials can be integral components of composite laminates or be fixed to the construction's surface. One of the basic domains of research and application where they are used is vibration damping and noise reduction.Vibration damping using piezoelectric elements can be either passive or active. Passive methods make use of inertia, rigidity and energy dissipation to correct given system characteristics subject to external perturbations. The effectiveness of those methods is limited to only a narrow range of exploitation parameters. Active damping is realized through counteracting the perturbation or intensifying the energy dissipation and is subject to damping or system rigidity control depending on system's state.

Many researchers have investigated the issue of vibration damping using piezoelectric elements. Among may papers concerned with this subject one can mention [1], where the author investigated active damping in a joint wise supported beam. In [6] a case of damping a laminated beam using piezoelectric elements was described. In both papers the systems described had the piezoelectric elements fixed to the construction element. In [5] the elastic and damping properties of the glue layer between the shaft and the piezoelectric elements were taken into account. In the papers mentioned above the connecting layer was active. Therefore a question arises about the damping quality in case the connecting layer is damaged. We can then take into account the constructional friction between construction's elements and the piezoelectric ones.

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2. THE ESSENCE OF STRUCTURAL FRICTION

One of the natural ways of vibration damping is the friction on the surface of the contacting element.

Energy dissipation in inseparable joints were called the constructional friction. The load-displacement relation is called the constructional hysteresis.

Constructional damping research usually assume many simplifications in joints' models such as:
1. The material of the contacting elements is ideally elastic.
2. The intensity of friction forces on the contact surface of elements is described by Coulomb's law.
3. Normal forces in every point of contact are distributed uniformly.
4. The friction factor is constant along the whole connection length.

Although many simplifications were assumed, the results of analysis were confirmed experimentally.

3. LINEARIZING THE SMA CHARACTERISTICS

Shape memory alloys (SMA) are a new group of construction materials. This group of alloys includes two-component alloys of metals one of which is left and the other one right from Chrome in the table of elements and alloys of precious metals. In certain conditions some three-component alloys of uranium and copper also manifest shape memory characteristics. SMA characteristics are associated with the inverse martensitic transition occurring in environment temperature. SMAs present a variety of unique characteristics which usually are not found in other materials.

Effects in SMA materials:
- pseudo-elasticity,
- one-way shape memory effect,
- two-way shape memory effect.

Characteristics of SMA Alloys:
- change of the Young modulus depending on the temperature and deformation,
- change of internal friction depending on the temperature and deformation amplitude,
- change of plasticity range depending on the temperature,
- electrical resistance change depending on the temperature of deformation.

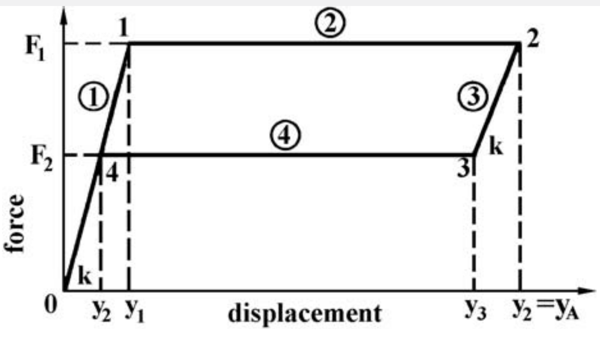

Fig. 1. Simplified force-displacement dependence for SMA materials

This paper is concerned with the influence of constructional friction and piezoelectric element steering on a shaft's torsional vibration. The constructional friction between the shaft and the acutator has been taken into account. It was also assumed that the shaft Is made of an SMA material. The elastic and damping properties of the SMA material described by the hysteresis loop presented on Figure 1 were taken into account.

Basing on the asymptotic Bogoliubov–Krylov–Mitropolsky method, the expressions determining the amplitude derivative and the phase derivative for a mechanical system with an SMA spring look as follows

$$\frac{dx_A}{dt} = \frac{\omega_0}{\pi} x_A \Big[\cos\beta_3 - \cos\beta_4 + \cos\beta_1 - \frac{1}{2}(\cos^2\beta_3 - \cos^2\beta_4 + \cos^2\beta_1 + 1 \Big] \tag{1}$$

and

$$\frac{d\zeta}{dt} = \omega_0 + \frac{\omega_0}{\pi}\Big[\frac{3}{2}(\sin 2\beta_3 - \sin 2\beta_4 - \sin 2\beta_1) - (\beta_3 - \beta_4 - \beta_1)\Big] \tag{2}$$

including

$$\cos\beta_i = \frac{y_i}{y_A} \tag{3}$$

and $y_2 = y_A$ – vibration amplitude.

After transforming (1) and (2) we obtain

$$h_b(y_A) = \frac{\omega_0}{\pi}\Big[\cos\beta_3 - \cos\beta_4 + \cos\beta_1 - \frac{1}{2}(\cos^2\beta_3 - \cos^2\beta_4 + \cos^2\beta_1 + 1)\Big] \tag{4}$$

and

$$\kappa = \frac{k_b(y_A)}{k} = 1 + \frac{1}{\pi}\Big[\frac{3}{2}(\sin 2\beta_3 - \sin 2\beta_4 - \sin 2\beta_1) - (\beta_3 - \beta_4 - \beta_1)\Big] \tag{5}$$

The resulting dependencies describe the equivalent damping coefficient and the coefficient determining the equivalent Kirchhoff modulus of the shaft, which we can present as

$$G_w = \kappa G \tag{6}$$

where:
G – Kirchhoff modulus,
κ – coefficient.

4. THE ESSENCE OF PIEZOELECTRIC ELEMENTS

Most commonly used piezoelectric materials are:
- PZT ceramals, based on zirconium oxide ($PbZrO_3$) and titanium oxide ($PbTiO_3$),
- PVDF organic films, based on viniden polifluoride ($CH_2 - CF_2)_n$.

The materials are widely used because of features such as:
- excellent dynamics characteristics (almost immediate response to applied tension or electrical signal);
- broad frequency range where they can be used;
- dual functions; they can be used both as measurement elements (sensors) and actuators stimulated with electrical impulses;
- extreme tension sensitivity;
- easy steering in closed feedback loops;
- PVDF films are easy to shape (cutting) and attaching to base constructions (beams, plates, shafts);
- easy access and low cost.

The PVDF organic films are often used to damp transverse vibrations of shells and thin beams. Nevertheless, due to the lack of natural shape effects the PVDF films are more commonly used as torsion vibration sensors rather than torsion actuators [1].

The PZT ceramals have much bigger (order of magnitude) electromechanical coupling constants as well as the natural shape effect. Therefore they are excellent as actuators for active damping of transversal as well as torsion vibrations. The PZT ceramals have significantly bigger density than PVDF films. The easiest and most effective way to use a ceramal is to fasten it to the shaft in one of the main plains of the shape strain.

The most important feature of the piezoelectric materials is the possibility of changing is polarization with small changes of external electric field or other external influence such as temperature or mechanical strain.

The actuator law is described by the equations

$$\tau_{pe} = G_A\left(R\frac{d\varphi}{dx} - 2d_{15}\frac{U}{h_e}\right) = G_A\left(R\frac{d\varphi}{dx} - \Psi\right) \qquad (7)$$

where:
- τ_{pe} – piezoelectric element's tangential strain,
- G_A – piezoelectric element's shear modulus (actuator),
- R – piezoelectric element's inner radius,
- φ – piezoelectric element's torsion angle,
- U – control voltage,
- h_e – distance between the electrodes,
- d_{15} – piezoelectric element's coupling constants.

The voltage generated by the sensor can be described with the equation

$$U_s = \frac{G_C d_{15} R}{\varepsilon_0 \varepsilon_{pe}}[\varphi_{pe}(0,t) - \varphi_{pe}(l_{pe},t)] \qquad (8)$$

where:
- ε_0 – permitivity of free space,
- ε_{pe} – permitivity of the piezoelectric element,
- G_C – piezoelectric element's shear modulus (sensor),
- φ_{pe} – piezoelectric element's torsion angle,
- l_{pe} – piezoelectric element's length.

Equation (8) is commonly known as the sensor law.

5. THE STRUCTURE OF THE MATHEMATICAL MODEL

One of typical models is an inseparable forced sleeve – shaft connection. It often appears in mechanisms as a link between the engine and the machine. It is one of the elements of the kinematical chain. Taking the symmetry of the connection into account and assuming that all axial forces in every connection point have an uniform distribution and that friction factor has a constant valve along the whole connection, we will examine only one half of it. We assume that the sleeve is made of PZT piezoelectric, the sensor is made of weightless piezoelectric PVDF foil. The scheme of such dynamic system is presented in Figure 2.

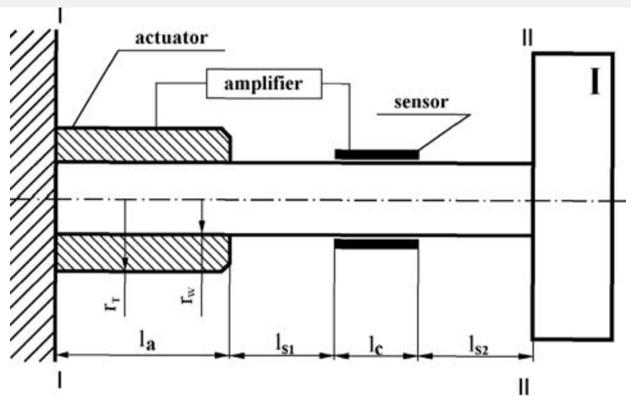

Fig. 2. Scheme of the considered dynamical system

For easier calculation, we assume that lengths $l_{s1} = l_{s2} = 0$. Dry friction exists on the surface of the connection of the actuator and the shaft. The Coulomb's law defines an elementary friction force which is proportional to pressure

$$q_T = p \cdot f \qquad (9)$$

where:
- p – piezoelectric element's shear modulus,
- f – friction factors.

An elementary friction force generates an elementary moment of friction, with is constant in the slip zone and is equal to

$$m_T = 2\pi r_w\, pf = 2\pi r_w\, q \qquad (10)$$

where r_w – radius of the shaft.

In the last slip zone, the moment of friction is directed oppositely to the movement of the shaft. There is no friction without any slip. Simultaneously the actuator will cause a torsion moment in slip zones because of the piezoelectric effect. This moment will counteract the movement. Using equation from [2] and concerning the shaft's stiffness the generalized relationship between the torsion angle section II – II and I – I, and the friction forces can be presented as

$$\frac{(1-k)^2(\alpha_{im}+\alpha_{i+1})^2}{4m_T G_W I_W} M^2 \operatorname{sign}\dot{\varphi} + k\frac{\alpha_{im}-\alpha_{i+1}}{G_W I_W} l_a M + \frac{\alpha_{im}-\alpha_{i+1}}{G_W I_W} l_c M - (\varphi_{im}-\varphi_{i+1}) = 0 \tag{11}$$

where:

$k = \frac{G_W I_W}{G_W I_W + G_A I_A}$ – factor of participation of stiffness of the shaft in stiffness of the connection,

α_{im} – load factor at the and of i part of the movement,

α_{i+1} – load factor in i+1 part of the movement,

I_W – geometric moment of inertia of the shaft,

G_A – modulus of rigidity of the actuator,

I_A – geometric moment of inertia of the actuator,

M – moment obciążający,

l_a – length of the actuator,

l_c – length of the sensor,

φ_{im} – twisting angle the of each part of the movement,

φ_{i+1} – twisting angle in the i+1 part of the movement.

After appropriate grouping of elements in (11), we get what follows

$$\frac{(1-k)^2(\alpha_{im}-\alpha_{i+1})^2}{4m_T G_W I_W} M^2 \operatorname{sign}\dot{\varphi} + k\frac{\alpha_{im}-\alpha_{i+1}}{G_W I_W}\left(l_a + \frac{l_c}{k}\right) M - (\alpha_{im}-\alpha_{i+1}) = 0 \tag{12}$$

designating $l_a + \frac{l_c}{k} = l_z$ (replacement length of connection), we get

$$\frac{(1-k)^2(\alpha_{im}-\alpha_{i+1})^2}{4m_T G_W I_W} M^2 \operatorname{sign}\dot{\varphi} + k\frac{\alpha_{im}-\alpha_{i+1}}{G_W I_W} l_z M - (\alpha_{im}-\alpha_{i+1}) = 0 \tag{13}$$

After inverse transformation we get a dependence describing the actual torsion moment that comes from elementary friction forces

where:

$k_0 = \sqrt{m_T G_W I_W}$ – coefficient of elasticity of friction,

$\mu_z = \frac{m_T l_z^2}{G_W I_W}$ – coefficient that defines length of the connection,

$\kappa = \frac{1-3k+3k^2}{1-k}$ – coefficient,

φ_p – vibration amplitude of replacement connection.

Taking into account the torsion moment from friction forces and the torsion moment generated by the actuator, we get the equation of motion

$$I\frac{d^2\varphi}{dt^2} + M_{i+1}(\varphi_p) - M_A(\varphi_a)\sin\varphi = M(t) \tag{15}$$

where:

I – mass moment of inertia of the disk,

$M_A(\varphi_a)$ – moment generated by actuator,

φ_a – vibration amplitude of actuator,

$M(t)$ – external forcing moment.

The moment generated by the actuator can be defined using this equation

$$M_A(\varphi_a) = \tau\int_{r_W}^{r_T} r\,dA \tag{16}$$

using the actuator law given by equation (7), the integrated equation for torsion is as follows

$$M_A(\varphi_a) = \frac{4}{3}\pi\frac{d_{15}^a}{l_a}U(r_T^3 - r_W^3)G_A \tag{17}$$

where:

d_{15}^a – coupling constant of actuator,

r_T – outer radius of actuator.

We determinate the control voltage applied to the actuator from

$$U = \lambda U_C \tag{18}$$

where:

λ – constant of the amplifier,

U_C – outer radius of actuator.

$$M_{i+1}(\varphi) = \frac{k_0}{\kappa}\left[-\sqrt{\mu_z k^2} - \sqrt{\mu_z k^2 - 2\kappa\varphi_p \operatorname{sign}\dot{\varphi}} + 2\sqrt{\mu_z k^2 + \kappa(\varphi_p - \varphi_{i+1})\operatorname{sign}\dot{\varphi}}\right]\operatorname{sign}\dot{\varphi} \tag{14}$$

Using the sensor law given by equation (8), we can express the voltage generated by sensor as

$$U_C = \frac{G_C d_{15}^c r_W}{\varepsilon_0 \varepsilon_c} \varphi_c(l_c) = \frac{G_C d_{15}^c r_W}{\varepsilon_0 \varepsilon_c} \frac{l_c}{G_W I_W} M_C \tag{19}$$

where:

M_C – forcing moment of the sensor,
ε_c – permittivity of the sensor,
d_{15}^c – coupling constant of the sensor.

We define the torsion moment M_c as

$$M_c = \frac{k_0}{\kappa}\left(-\sqrt{\mu_z k^2} + \sqrt{\mu_z k^2 - 2\kappa\varphi_P \operatorname{sign}\dot{\varphi}}\right)\operatorname{sign}\dot{\varphi} \tag{20}$$

Eventually, after including equations (17), (18) and (19), the equation defining the amplitude of torsion moment generated by the actuator can be written as follows:

$$\begin{aligned} M_A(\varphi_A) &= \frac{4}{3}\pi\frac{d_{15}^a}{l_a}(r_A^3 - r_W^3)G_T \lambda \frac{G_C d_{15}^c}{\varepsilon_0 \varepsilon_c}\frac{l_c r_W}{G_W I_W}\frac{k_0}{\kappa}\left(-\sqrt{\mu_z k^2} + \sqrt{\mu_z k^2 - 2\kappa\varphi_A \operatorname{sign}\dot{\varphi}}\right)\operatorname{sign}\dot{\varphi} = \\ &= \chi\left(-\sqrt{\mu_z k^2} + \sqrt{\mu_z k^2 - 2\kappa\varphi_A \operatorname{sign}\dot{\varphi}}\right)\operatorname{sign}\dot{\varphi} \end{aligned} \tag{21}$$

where $\chi = \frac{4}{3}\pi\frac{d_{15}^a}{l_a}(r_A^3 - r_W^3)G_T \lambda \frac{G_C d_{15}^c}{\varepsilon_0 \varepsilon_c}\frac{l_c r_W}{G_W I_W}\frac{k_0}{\kappa}$ – coefficient.

6. SOLVING THE EQUATION OF MOTION

We will solve the equation of motion (15) for the case of forced harmonic vibration with an uniformly changing angular frequency and constant driving momentum, that is

$$\left.\begin{aligned} M(t) &= M\sin\Theta \\ \Theta &= \frac{\varepsilon t^2}{2} + \varphi_0 \\ \omega(t) &= \frac{d\Theta}{dt} = \varepsilon t \end{aligned}\right\} \tag{22}$$

where:

M – driving momentum amplitude,
ε – angular acceleration,
φ_0 – initial phase,
Θ – movement phase,
$\omega(t)$ – driving force frequency.

Including (14) and (22), the equation of motion (15) can be written as follows:

$$\begin{aligned} I\ddot{\varphi} + \frac{k_0}{\kappa}\left[-\sqrt{\mu_z k^2} + \sqrt{\mu_z k^2 - 2\kappa\varphi_A \operatorname{sign}\dot{\varphi}} + 2\sqrt{\mu_z k^2 + \kappa(\varphi - \varphi_p)\operatorname{sign}\dot{\varphi}}\right]\operatorname{sign}\dot{\varphi} = \\ = M_A(\varphi_A)\sin\varphi + M\sin\left(\frac{\varepsilon t^2}{2} + \varphi_0\right) \end{aligned} \tag{23}$$

We can easily see that in equation (23) a component proportional to the displacement is present. To obtain such a component we develop the square root containing the deflection into a power series.

After the developing it into a power series and the necessary transformations, we obtain the following equation of motion

$$I\ddot{\varphi}+\frac{k_0}{\sqrt{\mu_z k^2-\kappa\varphi_A \operatorname{sign}\dot{\varphi}}}\varphi=\frac{k_0}{\kappa}\left\{\sqrt{\mu_z k^2}+\sqrt{\mu_z k^2-2\kappa\varphi_A \operatorname{sign}\dot{\varphi}}-\right.$$
$$\left.-2\sqrt{\mu_z k^2-\kappa\varphi_A \operatorname{sign}\dot{\varphi}}\left[1+\sum_{n=2}^{N}(-1)^{n-1}\frac{[2(n-1)]!}{2^{2n-1}n!(n-1)!}\left(\frac{\kappa\varphi}{\mu_p k^2-\kappa\varphi_A \operatorname{sign}\dot{\varphi}}\right)^n\right]\right\}\operatorname{sign}\dot{\varphi}+ \tag{24}$$
$$+\chi\left(-\sqrt{\mu_z k^2}+\sqrt{\mu_z k^2-2\kappa\varphi_A \operatorname{sign}\dot{\varphi}}\right)\operatorname{sign}\dot{\varphi}\sin\varphi$$

Introducing the motion:

$$k_z=\frac{k_0}{\sqrt{\mu_z k^2-\kappa\varphi_A \operatorname{sign}\dot{\varphi}}} \tag{25}$$

$$vF(\varphi,\dot{\varphi},t)=\frac{k_0}{\kappa}\left\{\sqrt{\mu_z k^2}+\sqrt{\mu_z k^2-2\kappa\varphi_A\, sign\,\dot{\varphi}}-\right.$$
$$\left.-2\sqrt{\mu_z k^2-\kappa\varphi_A \operatorname{sign}\dot{\varphi}}\left[1+\sum_{n=2}^{N}(-1)^{n-1}\frac{[2(n-1)]!}{2^{2n-1}n!(n-1)!}\left(\frac{\kappa\varphi}{\mu_z k^2-\kappa\varphi_A \operatorname{sign}\dot{\varphi}}\right)^n\right]\right\}\operatorname{sign}\dot{\varphi}+ \tag{26}$$
$$+\chi\left[-\sqrt{\mu_z k^2}+\sqrt{\mu_z k^2-2\kappa\varphi_A \operatorname{sign}\dot{\varphi}}\right]\operatorname{sign}\dot{\varphi}\sin\varphi$$

where:

k_z – rigidity for torsion,
v – parameter.

We acquire an equation of motion

$$I\ddot{\varphi}+k_z\varphi=M\sin\left(\frac{\varepsilon t^2}{2}+\varphi_0\right)+vF(\varphi,\dot{\varphi},t) \tag{27}$$

This equation will be solved using the asymptotic Bogoliubov–Krylov–Mitropolsky method presented [7]. We assume the first approximation of solution

$$\varphi=\varphi_A\cos(\Theta+\xi) \tag{28}$$

where φ_A i ξ are determined from the system of differential equation of the first approximation

$$\left.\begin{aligned}\frac{d\varphi}{dt}&=vA_1(t,\varphi_A,\xi)\\ \frac{d\xi}{dt}&=\omega_0-\omega(t)+vB_1(t,\varphi_A,\xi)\end{aligned}\right\} \tag{29}$$

where $\omega_0=\sqrt{\frac{k_z}{I}}$ – the frequency of free vibration.

After appropriate calculations, the first approximation looks as follows

$$\frac{d\varphi_A}{dt}=-\frac{v}{2\pi I\omega_0}\int_0^{2\pi}F_0(\varphi_A,\vartheta)\sin\vartheta d\vartheta \tag{30}$$

$$\frac{d\xi}{dt}=\omega_0-\omega(t)-\frac{v}{2\pi I\omega_0}\int_0^{2\pi}F_0(\varphi_A,\vartheta)\cos\vartheta d\vartheta \tag{31}$$

where:

$$\left.\begin{array}{r}\vartheta=\Theta+\xi\\ F_0(\varphi_A,\vartheta)=F(\varphi_A\cos\vartheta-\varphi_A\sin\vartheta)\end{array}\right\}\tag{32}$$

We calculate the integral in equation (30) assuming the following notation:

$$\begin{aligned}
&C_1(\varphi_P)=v\int_0^{2\pi}F_0(\varphi_A,\vartheta)\sin\vartheta d\vartheta=\frac{k_0}{\kappa}\int_0^{2\pi}\Bigg\{\sqrt{\mu_z k^2}+\sqrt{\mu_z k^2-2\kappa\varphi_A\operatorname{sign}\dot\varphi}-\\
&-\sqrt{\mu_z k^2-\kappa\varphi_A\operatorname{sign}\dot\varphi}\left[1+\sum_{n=2}^{N}(-1)^{n-1}\frac{[2(n-1)]!}{2^{2n-1}n!(n-1)!}\left(\frac{\kappa\varphi}{\mu_z k^2-\kappa\varphi_A\operatorname{sign}\dot\varphi}\right)^n\right]+\\
&+\chi\left(-\sqrt{\mu_z k^2}+\sqrt{\mu_z k^2-2\kappa\varphi_A\operatorname{sign}\dot\varphi}\right)\sin\vartheta\Bigg\}\operatorname{sign}\dot\varphi\sin\vartheta d\vartheta=\frac{k_0}{\kappa}\int_0^{2\pi}\left(\sqrt{\mu_p k^2}+\sqrt{\mu_p k^2-2\kappa\varphi_A\operatorname{sign}\dot\varphi}\right)\operatorname{sign}\dot\varphi\sin\vartheta d\vartheta-\\
&-2\frac{k_0}{\kappa}\int_0^{2\pi}\sqrt{\mu_p k^2-\kappa\varphi_A\operatorname{sign}\dot\varphi}\left\{1-\sum_{j=1}^{\operatorname{int}(N/2)}\frac{[2(2j-1)]!}{2^{4j-1}(2j)!(2j-1)!}\left(\frac{\kappa\varphi}{\mu_p k^2-\kappa\varphi_A\, sign\dot\varphi}\right)^{2j}\right\}\operatorname{sign}\dot\varphi\sin\vartheta d\vartheta-\\
&-2\frac{k_0}{\kappa}\int_0^{2\pi}\sqrt{\mu_p k^2-2\kappa\varphi_A\operatorname{sign}\dot\varphi}\left\{\sum_{j=1}^{\operatorname{int}(N/2)}\frac{[2(2j)]!}{2^{4j-1}(2j+1)!(2j)!}\left(\frac{\kappa\varphi}{\mu_p k^2-\kappa\varphi_A\operatorname{sign}\dot\varphi}\right)^{2j+1}\right\}\sin\vartheta d\vartheta+\\
&+\chi\int_0^{2\pi}\left(-\sqrt{\mu_z k^2}+\sqrt{\mu_z k^2-2\kappa\varphi_A\operatorname{sign}\dot\varphi}\right)\operatorname{sign}\dot\varphi\sin^2\vartheta d\vartheta
\end{aligned}\tag{33}$$

where $j=\dfrac{n}{2}$.

Including (28) and (32) into equation (33) we get

$$\begin{aligned}
&C_1(\varphi_P)=\frac{k_0}{\kappa}\int_0^{2\pi}\left(\sqrt{\mu_z k^2}+\sqrt{\mu_z k^2-2\kappa\varphi_A\operatorname{sign}\dot\varphi}\right)\operatorname{sign}\dot\varphi\sin\vartheta d\vartheta-\\
&-2\frac{k_0}{\kappa}\int_0^{2\pi}\sqrt{\mu_z k^2-\kappa\varphi_A\operatorname{sign}\dot\varphi}\left\{\sin\vartheta-\sum_{j=1}^{\operatorname{int}(n/2)}\frac{[2(2j-1)]!}{2^{4j-1}(2j)!(2j-1)!}\left(\frac{\kappa\varphi_A}{\mu_z k^2-\kappa\varphi_A\operatorname{sign}\dot\varphi}\right)^{2j}\cos^{2j}\vartheta\sin\vartheta\right\}\operatorname{sign}\dot\varphi\, d\vartheta-\\
&-2\frac{k_0}{\kappa}\int_0^{2\pi}\sqrt{\mu_z k^2-\kappa\varphi_A\operatorname{sign}\dot\varphi}\cdot\left\{\sum_{j=1}^{\operatorname{int}(N/2)}\frac{[2(2j)]!}{2^{4j-1}(2j+1)!(2j)!}\left(\frac{\kappa\varphi}{\mu_p k^2-\kappa\varphi_A\operatorname{sign}\dot\varphi}\right)^{2j+1}\cos^{2j+1}\vartheta\sin\vartheta\right\}d\vartheta+\\
&+\chi\int_0^{2\pi}\left(-\sqrt{\mu_z k^2}+\sqrt{\mu_z k^2-2\kappa\varphi_A\operatorname{sign}\dot\varphi}\right)\operatorname{sign}\dot\varphi\sin^2\vartheta d\vartheta
\end{aligned}\tag{34}$$

Including dependence

$$\operatorname{sign}\dot\varphi=\begin{Bmatrix}-1\\+1\end{Bmatrix}\quad\text{for}\quad\begin{Bmatrix}0\le\vartheta\le\pi;\ \varphi_A>0\\ \pi\le\vartheta\le2\pi;\ \varphi_A<0\end{Bmatrix}\tag{35}$$

and

$$\int_0^{\pi}\sin\vartheta d\vartheta=-\int_\pi^{2\pi}\sin\vartheta d\vartheta\tag{36}$$

we will bring equation (34) to

$$
\begin{aligned}
C_1(\varphi_P) &= 2\frac{k_0}{\kappa}\int_0^{2\pi}\left(\sqrt{\mu_z k^2} + \sqrt{\mu_z k^2 + 2\kappa\varphi_A}\right)\sin\vartheta d\vartheta - \\
&- 4\frac{k_0}{\kappa}\int_0^{2\pi}\sqrt{\mu_z k^2 + \kappa\varphi_A}\left\{\sin\vartheta - \sum_{j=1}^{\mathrm{int}(n/2)}\frac{[2(2j-1)]!}{2^{4j-1}(2j)!(2j-1)!}\left(\frac{\kappa\varphi_P}{\mu_z k^2 + \kappa\varphi_A}\right)^{2j}\cos^{2j}\vartheta\sin\vartheta\right\}d\vartheta - \\
&- \frac{k_0}{\kappa}\int_0^{2\pi}\sqrt{\mu_z k^2 + \kappa\varphi_A}\left\{\sum_{j=1}^{\mathrm{int}(N/2)}\frac{(4j)!}{2^{4j+1}(2j+1)!(2j)!}\left(\frac{\kappa\varphi}{\mu_z k^2 + \kappa\varphi_A}\right)^{2j+1}\cos^{2j+1}\vartheta\sin\vartheta\right\}d\vartheta + \\
&+ 2\chi\int_0^{\pi}\left(\sqrt{\mu_z k^2} - \sqrt{\mu_z k^2 + 2\kappa\varphi_A}\right)\sin^2\vartheta d\vartheta
\end{aligned}
\tag{37}
$$

because

$$
\int_0^{\pi}\cos^{2j+1}\vartheta\sin\vartheta d\vartheta = 0 \tag{38}
$$

and

$$
\int_0^{\pi}\cos^{2j}\vartheta\sin\vartheta d\vartheta = -\frac{1}{2j+1}\int_0^{\pi}\cos^{2j+1}\vartheta d\vartheta = -\frac{2}{2j+1} \tag{39}
$$

we get dependence (33) as

$$
\begin{aligned}
C_1(\varphi_P) &= 4\frac{k_0}{\kappa}(\sqrt{\mu_z k^2} + \sqrt{\mu_z k^2 + 2\kappa\varphi_A}) + 8\frac{k_0}{\kappa}\sqrt{\mu_z k^2 + \kappa\varphi_A}\left\{1 - \sum_{j=1}^{\mathrm{int}(N/2)}\frac{[2(2j-1)]!}{2^{4j-1}(2j)!(2j-1)!}\left(\frac{\kappa\varphi_P}{\mu_z k^2 + \kappa\varphi_A}\right)^{2j}\right\} + \\
&+ \chi\pi(\sqrt{\mu_z k^2} - \sqrt{\mu_z k^2 + 2\kappa\varphi_A})
\end{aligned}
\tag{40}
$$

After calculating the integral in equation (31) we get

$$
\begin{aligned}
C_2(\varphi_A) &= v\int_0^{2\pi}F_0(\varphi_A,\vartheta)\cos\vartheta d\vartheta = \frac{k_0}{\kappa}\int_0^{2\pi}\left\{\sqrt{\mu_z k^2} + \sqrt{\mu_z k^2 - 2\kappa\varphi_A\,\mathrm{sign}\,\dot\varphi} - \right. \\
&- 2\sqrt{\mu_z k^2 - \kappa\varphi_A\,\mathrm{sign}\,\dot\varphi}\left[1 + \sum_{n=2}^{N}(-1)^{n-1}\frac{[2(n-1)]!}{2^{2n-1}n!(n-1)!}\left(\frac{\kappa\varphi}{\mu_z k^2 - \kappa\varphi_A\,\mathrm{sign}\,\dot\varphi}\right)^n\right]\mathrm{sign}\,\dot\varphi + \\
&+ \chi\left(-\sqrt{\mu_z k^2} + \sqrt{\mu_z k^2 - 2\kappa\varphi_A\,\mathrm{sign}\,\dot\varphi}\right)\mathrm{sign}\,\dot\varphi\sin\vartheta\left.\right\}\cos\vartheta d\vartheta = \frac{k_0}{\kappa}\int_0^{2\pi}\left(\sqrt{\mu_z k^2} + \sqrt{\mu_z k^2 - 2\kappa\varphi_A\,\mathrm{sign}\,\dot\varphi}\right)\mathrm{sign}\,\dot\varphi\cos\vartheta d\vartheta - \\
&- 2\frac{k_0}{\kappa}\int_0^{2\pi}\sqrt{\mu_z k^2 - \kappa\varphi_A\,\mathrm{sign}\,\dot\varphi}\left\{1 - \sum_{j=1}^{\mathrm{int}(N/2)}\frac{[2(2j-1)]!}{2^{4j-1}(2j)!(2j-1)!}\left(\frac{\kappa\varphi}{\mu_z k^2 - \kappa\varphi_A\,sign\,\dot\varphi}\right)^{2j}\right\}\mathrm{sign}\,\dot\varphi\cos\vartheta d\vartheta - \\
&- 2\frac{k_0}{\kappa}\int_0^{2\pi}\sqrt{\mu_z k^2 - \kappa\varphi_A\,\mathrm{sign}\,\dot\varphi}\left\{\sum_{j=1}^{\mathrm{int}(N/2)}\frac{[2(2j)]!}{2^{4j-1}(2j+1)!(2j)!}\left(\frac{\kappa\varphi}{\mu_z k^2 - \kappa\varphi_A\,\mathrm{sign}\,\dot\varphi}\right)^{2j+1}\right\}\cos\vartheta d\vartheta + \\
&+ \chi\int_0^{2\pi}\left(-\sqrt{\mu_z k^2} + \sqrt{\mu_z k^2 - 2\kappa\varphi_A\,\mathrm{sign}\,\dot\varphi}\right)\mathrm{sign}\,\dot\varphi\sin\vartheta\cos\vartheta d\vartheta
\end{aligned}
\tag{41}
$$

where $j = \frac{n}{2}$.

Including (32) and (35) in equation (41) we get

$$
\begin{aligned}
C_2(\varphi_A) &= \frac{k_0}{\kappa}\int_0^{2\pi}\left(\sqrt{\mu_z k^2} + \sqrt{\mu_z k^2 - 2\kappa\varphi_A \operatorname{sign}\dot{\varphi}}\right) sign\dot{\varphi}\cos\vartheta d\vartheta - \\
&- 2\frac{k_0}{\kappa}\int_0^{2\pi}\sqrt{\mu_z k^2 - \kappa\varphi_A \operatorname{sign}\dot{\varphi}}\left\{\cos\vartheta - \sum_{j=1}^{\operatorname{int}(n/2)}\frac{[2(2j-1)]!}{2^{4j-1}(2j)!(2j-1)!}\left(\frac{\kappa\varphi_P}{\mu_z k^2 - \kappa\varphi_A \operatorname{sign}\dot{\varphi}}\right)^{2j}\cos^{2j+1}\right\}\operatorname{sign}\dot{\varphi} d\vartheta - \\
&- 2\frac{k_0}{\kappa}\int_0^{2\pi}\sqrt{\mu_z k^2 - \kappa\varphi_A \operatorname{sign}\dot{\varphi}}\left\{\sum_{j=1}^{\operatorname{int}(N/2)}\frac{[2(2j)]!}{2^{4j-1}(2j+1)!(2j)!}\left(\frac{\kappa\varphi_P}{\mu_z k^2 - \kappa\varphi_A \operatorname{sign}\dot{\varphi}}\right)^{2j+1}\cos^{2j+1}\vartheta\right\} d\vartheta + \\
&+ \chi\int_0^{2\pi}\left(-\sqrt{\mu_z k^2} + \sqrt{\mu_z k^2 - 2\kappa\varphi_A \operatorname{sign}\dot{\varphi}}\right)\operatorname{sign}\dot{\varphi}\sin\vartheta\cos\vartheta d\vartheta
\end{aligned} \tag{42}
$$

Including (34) and

$$
\int_0^{\pi}\cos\vartheta d\vartheta = -\int_{\pi}^{2\pi}\cos\vartheta d\vartheta \tag{43}
$$

we will bring equation (41) to

$$
\begin{aligned}
C_2(\varphi_A) &= 2\frac{k_0}{\kappa}\int_0^{2\pi}\left[\sqrt{\mu_z k^2} + \sqrt{\mu_z k^2 + 2\kappa\varphi_A}\right]\cos\vartheta d\vartheta - \\
&- 4\frac{k_0}{\kappa}\int_0^{2\pi}\sqrt{\mu_z k^2 + \kappa\varphi_A}\left\{\cos\vartheta - \sum_{j=1}^{\operatorname{int}(n/2)}\frac{[2(2j-1)]!}{2^{4j-1}(2j)!(2j-1)!}\left(\frac{\kappa\varphi_A}{\mu_z k^2 + \kappa\varphi_{\mathrm{A}}}\right)^{2j}\cos^{2j+1}\vartheta\right\} d\vartheta - \\
&-2\frac{k_0}{\kappa}\int_0^{2\pi}\sqrt{\mu_z k^2 + \kappa\varphi_A}\left\{\sum_{j=1}^{\operatorname{int}(N/2)}\frac{(4j)!}{2^{4j+1}(2j+1)!(2j)!}\left(\frac{\kappa\varphi_A}{\mu_z k^2 + \kappa\varphi_{\mathrm{A}}}\right)^{2j+1}\cos^{2(j=1)}\vartheta\right\} d\vartheta + \\
&+ 2\frac{k_0}{\kappa}\int_0^{2\pi}\left(\sqrt{\mu_z k^2} - \sqrt{\mu_z k^2 + 2\kappa\varphi_A}\right)\sin\vartheta\cos\vartheta d\vartheta
\end{aligned} \tag{44}
$$

due to

$$
\int_0^{\pi}\cos^{2j+1}\vartheta = 0 \tag{45}
$$

and

$$
\int_0^{\pi}\cos^{2(j+1)}\vartheta = \frac{2j+1}{2(j+1)}\int_0^{\pi}\cos^{2j}\vartheta d\vartheta = \frac{(2j+1)!!}{[2(j+1)]!!}\pi \tag{46}
$$

so we will get equation (44) as follows:

$$
C_2(\varphi_A) = 4\pi\frac{k_0}{\kappa}\left\{\sum_{j=1}^{\operatorname{int}(N/2)}\frac{(4j)!}{2^{4j+1}(2j)!(2j+1)!}\frac{(2j+1)!!}{[2(j+1)]!!}\left(\frac{\kappa\varphi_A}{\mu_z k^2 + \kappa\varphi_A}\right)^{2j+1}\right\} \tag{47}
$$

Including (40) and (47) in equations (30) i (31) we will get the differential equations of the first approximation

$$\frac{d\varphi_A}{dt} = \\ = \frac{2k_0}{\pi I \omega_0 \kappa}\left(\sqrt{\mu_z k^2} + \sqrt{\mu_z k^2 + 2\kappa\varphi_A} - 2\sqrt{\mu_z k^2 + 2\kappa\varphi_A}\right)\left\{1 - \sum_{j=1}^{\mathrm{int}(N/2)} \frac{[2(2j-1)]!}{2^{4j+1}(2j-1)!(2j)!}\left(\frac{\kappa\varphi_A}{\mu_z k^2 + \kappa\varphi_A}\right)^{2j}\right\} - \\ - \frac{M}{I[\omega_0 - \omega(t)]}\cos\xi + \frac{k_0}{I\omega_0\kappa}\left(\sqrt{\mu_z k^2} - \sqrt{\mu_z k^2 + 2\kappa\varphi_A}\right) \tag{48}$$

$$\frac{d\xi}{dt} = \omega_0 - \omega(t) - \frac{2k_0}{I\omega_0\varphi_A\kappa}\left\{\sum_{j=1}^{\mathrm{int}(N/2)} \frac{(4j)!}{2^{4j1}(2j)!(2j+1)!} \cdot \frac{(2j+1)!!}{[2(j+1)]!!}\left(\frac{\kappa\varphi_A}{\mu_z k^2 + \kappa\varphi_A}\right)^{2j+1}\right\} + \frac{M}{I[\omega_0 - \omega(t)]}\sin\xi \tag{49}$$

7. THE RESULTS OF THE CALCULATIONS

Numerical simulations were conducted using the acquired equations for amplitude derivative (48) and phase amplitude (49).

The simulations were carried out for the case of harmonically driven vibration with a uniformly changing frequency of the driving toque. The results of calculations for those simulations are presented on Figure 3.

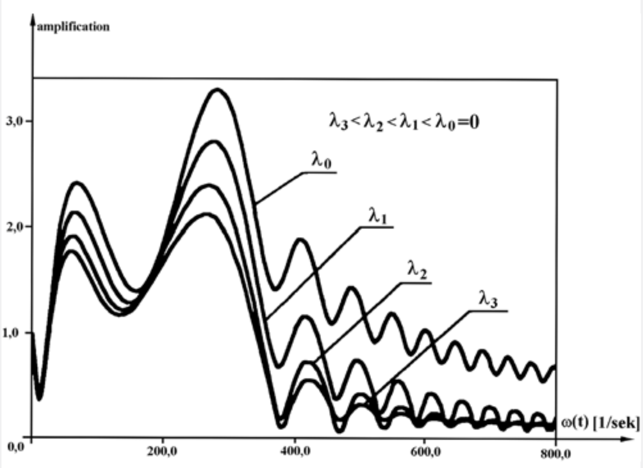

Fig. 3. Harmonically driven vibration

The analysis shows that including constructional friction between the shaft and the actuator results in much higher damping effectively. This effect is more visible with bigger gain factors of the amplifier. Bigger amplification also results in slight decrease of the system self vibration frequency. Result analysis confirms damping efficiency growth with gain factor increase.

8. CONCLUSION

1. Including the friction the between actuator and the shaft increases the efficiency of damping in the permanent shaft – sleeve joint.
2. Damping factor growth increases vibration damping intensiveness.
3. Damping factor growth decreases system's free vibration frequency.
4. Actuators assembled without glue layers exhibit damping only when strictly connected to the shaft.

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