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Gauss Quadrature Evaluation for the Signal Analysis

1. Introduction

Signal processing is concerned with the question of interpretation, transmission, reception and storage of the signal. The essential part of the signal processing is converting the signal defined in the time domain into the signal defined in the frequency domain. It allows to obtain a signal spectrum and its possible visualisation. This conversion can be done by the Fourier Transform (for analogue devices) and the Discrete Fourier Transform (DFT) (for discrete signals). It is also possible to apply decomposition of a continuous or a discrete function in the Fourier Series. To construct Fourier Series it is necessary to calculate the coefficients of the Fourier Series. One of the way to calculate the values of the coefficients is applying numerical integration by the means of Quadratures.

This paper provides an analysis of the calculation accuracy of the coefficients of the Fourier Series of five elementary, deterministic signals used in practical applications which should build a challenge by their characteristics for two, quiet differ from each other methods of numerical integration: the most versatile Newton-Cotes Quadrature – Newton-Cotes' Midpoint Rule and the most flexible and sophisticated Gauss Quadrature – Gauss-Legendre Quadrature. Additionally – the Fast Fourier Transform – the FFT was used as the point of reference concerning the accuracy.

By decomposing elementary signals and analyse the accuracy of the calculated coefficients the author wished to find the basic types of real world signals for which the application of Gauss Quadratures may be smarter solution due to better accuracy and lesser computational complexity (by the means of radically reduced amount of the sample points) than the FFT, which is the default tool in this area.

Detailed test guidelines, the reasons of choosing particular signals, test procedure of the operation correctness of the created tools and the straightforwardness as the philosophy of this test will be given in section 3. Before that there will be revealed some important aspects of the construction of the quadratures in section 2. The main tool of the accuracy

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analysis will be the comparison of the tabulated values of the absolute error of each coefficient calculated. Additionally there will be presented plots of the constructed Fourier Series of each signal tested. This will allow the visual evaluation of the series fitting – section 4. Finally – in section 5 – there will be presented the summary of the evaluation followed by the conclusions.

2. Comparison of methods for numerical integration: Gauss Quadrature vs. Newton-Cotes Quadrature

The essence of the numerical integration is to approximate the definite integral over the range $\langle a,b \rangle$ of the function $f(x)$ by the following formula

$$\int_a^b f(x)dx = \sum_{k=1}^{N} A_k f(x_k) + R \tag{1}$$

where:

x_k – quadrature nodes,
A_k – quadrature coefficients (weights),
R – the rest of the approximation.

The right side of the equation is often named quadrature. The formula (1) is shared by all quadratures. The difference lies in algorithms of calculating their nodes and coefficients [1, 2, 3, 5, 6, 10, 11].

The interpretation of quadrature coefficients A_k (so called weights) is the width of the subintervals the range of integration is divided into. In case of Newton-Cotes Quadrature (Rectangle, Trapezoidal and Simpson Rule) the integration range is divided into subintervals, which are of equal width. In case of Newton-Cotes' Midpoint Rule the sample point is taken from the middle of the subinterval. The nodes of Gauss Quadratures are determined by abscissas of the applied polynomials [1, 2, 3, 6, 11].

The Newton-Cotes Quadratures may be applied to almost all types of functions. Trapezoidal and Simpson Rule are the most often used. They are also "closed formulas", which implies that the values of $f(a)$ and $f(b)$ must be determinable. Therefore they can not be applied to integrands which have singularities, asymptotes, etc. In such cases there should be Newton-Cotes' Midpoint Rule applied.

The Gauss Quadratures – other than The Gauss-Legendre Quadrature (weights equal 1) may be applied only to selected kinds of functions. This is due to relationship with their weight functions [1].

The way the coefficients are calculated by the FFT is widely known [7, 8, 9]. It should only be noticed that the way FFT samples the integrand is very similar to Newton-Cotes Rectangular, or Trapezoidal Rules – it depends on how we integrate: (0, N) or (0, N–1).

3. Test procedure, tested functions and selected tools

The accuracy of numerical integration using quadratures depends strongly on the shape of the integrand. Following the philosophy of straightforwardness of this evaluation, the selected functions are elementary and represent typical shapes which are often to meet in real world applications – in this case – in telecommunication and process controlling. This property was needed to determine the basic shapes of the integrands best integrable by Gauss Quadratures.

The following functions were selected:

1) Odd function – rectangular signal.
2) Even function – saw signal.
   The choice of odd and even functions in which the coefficients a_n standing at cosine and b_n standing at sine – respectively equal zero was not by accident. The problem of computer calculations in which the correct result should be exactly zero is one of the toughest the good algorithm should take care of. This problem may be solved, for example by defining so called “computer zero” – treating every result calculated that is smaller than arbitral chosen value as a zero. When applying such a way one should not forget about the danger of reset the exact values calculated which are “smaller” than defined “computer zero”.
3) Neither even nor odd function – signal expressed by the following formula:

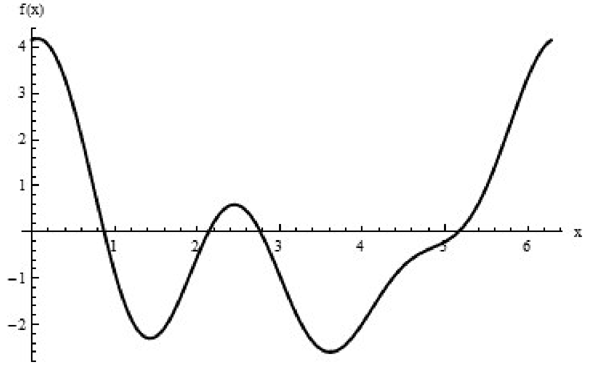
$$f(x)\begin{cases} \pi/2 & \text{for } x=-\pi \\ 0 & \text{for } -\pi<x\le 0 \\ x & \text{for } 0<x\le\pi \\ \pi/2 & \text{for } x=\pi \end{cases}$$

   It is a mixed type function. The challenge is its diversity and fast-changing characteristics. Additionally, half of the function values equal zero.
4) Exponential signal
   Divers and typical exponential function in one. Often used in practical, technical applications – in process controlling.

$$f(x)=e^{-x},\ x\in(0,8),$$

5) Simulated structure signal
   Very challenging function – it has only three harmonics, which can be exactly calculated analytically. The rest equals zero. The function is divers and fast-changing.

$$f(x) = 2\cos x + 1.5\cos\left(2x + \pi/6\right) + 1.2\cos\left(3x - \pi/4\right), x \in (0,6)$$



The next, important reason why the above functions were selected was because of the possibility of calculation of the exact values of their coefficients of the Fourier Series analytically. This allowed to determine proper operation of written tools, which in turn assured that, theoretically, coefficients of all existing signals will be calculated correctly.

The tools to analyse tested signals by Fourier Series were programs written by the author of this paper in procedural C++ language. They utilized Newton-Cotes' Midpoint Rule (NCM) and Gauss-Legendre Quadrature (GaLEG).

The idea of Gauss Quadratures dates back to 1820 when K.F. Gauss was determined to solve the problem of placing the nodes the quadrature the way they could be exact for the polynomials of the highest possible degree. The greatest progress in the field of quadrature development is dated back to the 60's and 70's of the last century, the time of building big aeroplanes and shuttles, where the idea of quadratures was perfectly applied in aerodynamics research. That's why the most essential literature about construction of quadratures and their practical applications dates to this period. The newest studies deals mostly with strategies of applications of quadratures and modifications of classical types of quadratures. That is why the author of this paper build his own tools upon classical sources of information about quadratures [1, 6, 10]. These sources are generally recognized as the didactically adequate sources of building classical quadratures [11]. Mathematica incorporates many excellent methods of numerical integration, which were used as one of the sources of exact values for this test. The Mathematica methods were grounded partly on the same literature.

The classical, acknowledged literature fits perfectly into „classical" character of this experiment: the main goal was to obtain the smallest as possible absolute errors using the smallest possible number of sample points. The amount of sample points should be exactly determinable. That is why there was not any advanced techniques of integration as for example adaptive strategies applied.

The criterion of the calculation accuracy was the absolute error as the difference between exact, analytically calculated values of coefficients and their numerically obtained counterparts. That's why there is no need to make such simple test complicated by incorporating extensive error estimation of every method applied. Additionally, theoretically estimation error formulas relate to a value of derivative of appropriate orders (for Newton--Cotes' Midpoint Rule – 2^{nd} order derivative and Gauss Quadrature – $2n^{nd}$ order derivative (n – degree of the polynomial applied in approximation)) inside the integration range. Now, the value of derivative does depend on the shape of the integrand!

There are few first terms the most important in process of the Fourier Series construction. That is why for the function 1–3 there were 32 ($N = 32$), 4–5 – 16 ($N = 16$) – respectively – subintervals used – in case of Newton-Cotes' Midpoint Rule and degree of the polynomial used in approximation – in Gauss-Legendre Quadrature. In tables below are presented results for most significant, major role playing in the process of series fitting, terms only.

The FFT program utilised a popular algorithm Radix2.

The above mentioned programs utilising the calculated coefficients constructed the Fourier series of the approximated signals. They also calculated the values of amplitude C_n – and phase φ_n, spectrums. Generated Mathematica code were used to visually evaluate the series fitting.

The visual approximation should be treated as a additional tools of the evaluation only. The correct visual Series fitting depends not only on accuracy of the calculated coefficients and amount of terms used. Additionally the Gibbs Effect should be taken under the consideration. The Gibbs Effect, the kind of convergence error causes that the correctness of the fitting of the Fourier Series depends strongly on the continuity of the particular function or/and its derivatives [11, 12, 13]. This means in reality that the fitting of the Fourier Series for example for 3 terms, of the saw function will be visually better evaluated as in case of rectangular function. Despite of the same level of absolute error of the calculated coefficients for both functions.

4. The results

Table 1
The values of absolute error – odd function

ΔC_n	**FFT**	**GaLEG**	**NCM**	**NCM**	**GaLEG**	**FFT**	$\Delta\varphi_n$
0	3.125 e-02	0.0	0.0	–	–	–	0
1	2.555 e-03	2.455 e-03	2.048 e-03	0.0	0.0	4.921 e-02	1
2	6.250 e-02	4.951 e-03	0.0	0.0	π/2	0.0	2
3	7.631 e-03	7.527 e-03	6.200 e-03	0.0	0.0	1.509 e-01	3
4	6.250 e-02	1.023 e-02	0.0	0.0	π/2	0.0	4
5	1.258 e-02	1.312 e-02	1.052 e-02	0.0	0.0	2.612 e-01	5
6	6.250 e-02	1.626 e-02	0.0	0.0	π/2	0.0	6
7	1.725 e-02	1.971 e-02	1.515 e-02	0.0	0.0	3.894 e-01	7
8	6.250 e-02	2.360 e-02	0.0	0.0	π/2	0.0	8
9	2.135 e-02	2.804 e-02	2.023 e-02	0.0	0.0	5.472 e-01	9

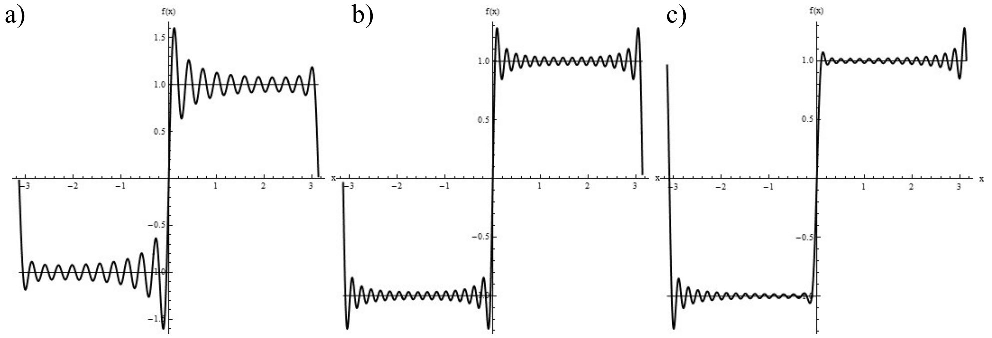


Fig. 1. The Fourier Series fitting of the rectangular signal (odd function);
a) Gauss-Legendre; b) Newton-Cotes' Midpoint Rule; c) the FFT

Table 2
The values of absolute error – saw signal

ΔC_n	**FFT**	**GaLEG**	**NCM**	**NCM**	**GaLEG**	**FFT**	$\Delta\varphi_n$
0	0.0	1.224 e-03	0.0	–	–	–	0
1	4.099 e-03	2.468 e-03	2.052 e-03	0.0	0.0	0.0	1
2	0.0	2.529 e-03	0.0	0.0	0.0	0.0	2
3	4.163 e-03	7.527 e-03	2.108 e-03	0.0	0.0	0.0	3
4	0.0	1.023 e-02	0.0	0.0	0.0	0.0	4
5	4.296 e-02	1.312 e-02	2.225 e-02	0.0	0.0	0.0	5
6	0.0	1.626 e-02	0.0	0.0	0.0	0.0	6
7	4.508 e-02	1.971 e-02	2.413 e-02	0.0	0.0	0.0	7
8	0.0	2.360 e-02	0.0	0.0	0.0	0.0	8
9	4.818 e-02	2.804 e-02	2.690 e-02	0.0	0.0	0.0	9

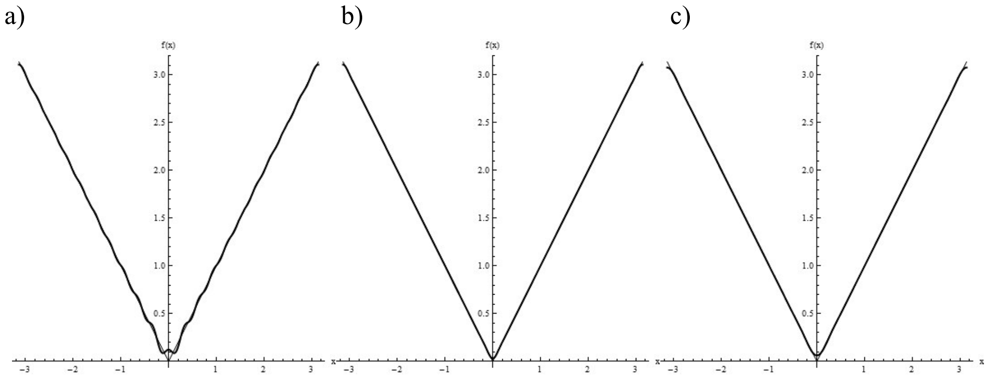


Fig. 2. The Fourier Series fitting of the saw signal (even function):
a) Gauss-Legendre; b) Newton-Cotes' Midpoint Rule; c) the FFT

Table 3
The values of absolute error – neither even nor odd function

ΔC_n	FFT	GaLEG	NCM	NCM	GaLEG	FFT	$\Delta\varphi_n$
0	0.0	6.121 e-04	0.0	–	–	–	0
1	1.606 e-03	6.623 e-04	8.068 e-04	1.580 e-03	8.78 e-04	2.919 e-03	1
2	6.442 e-03	1.534 e-06	3.227 e-03	0.0	2.52 e-03	0.0	2
3	9.027 e-03	2.710 e-04	4.550 e-03	5.913 e-03	3.78 e-03	1.220 e-02	3
4	1.299 e-02	3.788 e-06	6.543 e-03	0.0	5.76 e-03	0.0	4
5	1.588 e-02	1.840 e-04	8.068 e-03	1.024 e-03	7.39 e-02	2.258 e-02	5
6	1.974 e-02	7.990 e-06	1.004e-02	0.0	9.85 e-03	0.0	6
7	2.285 e-02	1.539 e-04	1.176 e-02	1.469 e-02	1.27 e-03	3.607 e-02	7
8	2.683 e-02	1.709 e-05	1.384 e-02	0.0	1.66 e-02	0.0	8
9	3.017 e-02	1.431 e-04	1.578 e-02	1.937 e-02	2.14 e-02	5.615 e-02	9

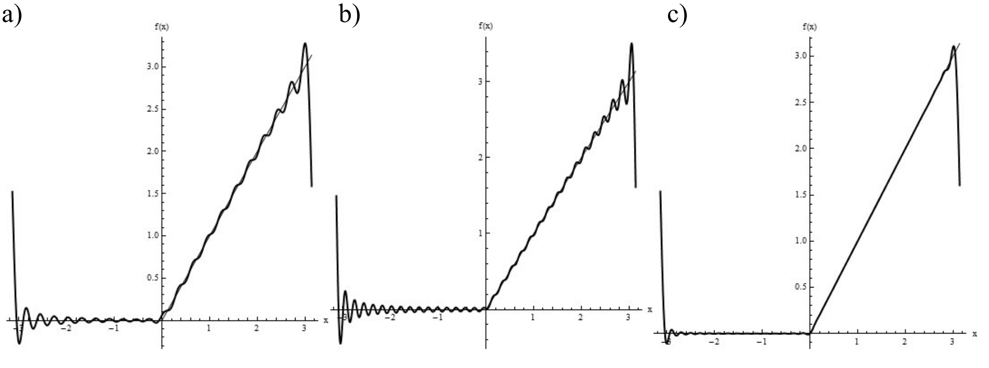


Fig. 3. The Fourier Series fitting of the mixed type signal (neither even nor odd function): a) Gauss-Legendre; b) Newton-Cotes' Midpoint Rule; c) the FFT

Table 4
The values of absolute error – exponential signal

ΔC_n	FFT	GaLEG	NCM	NCM	GaLEG	FFT	$\Delta\varphi_n$
0	3.383 e-02	0.0	1.292 e-03	–	–	–	0
1	5.481 e-02	1.0 e-12	7.927 e-04	1.634 e-02	7.0 e-12	1.800 e-1	1
2	4.074 e-02	1.0 e-12	2.044 e-03	3.292 e-02	4.0 e-12	3.598 e-1	2
3	3.381 e-02	2.0 e-12	4.736 e-03	5.003 e-02	1.1 e-11	5.390 e-1	3
4	3.104 e-02	2.0 e-12	7.405 e-03	6.798 e-02	1.7 e-10	7.174 e-1	4

a) b) c)

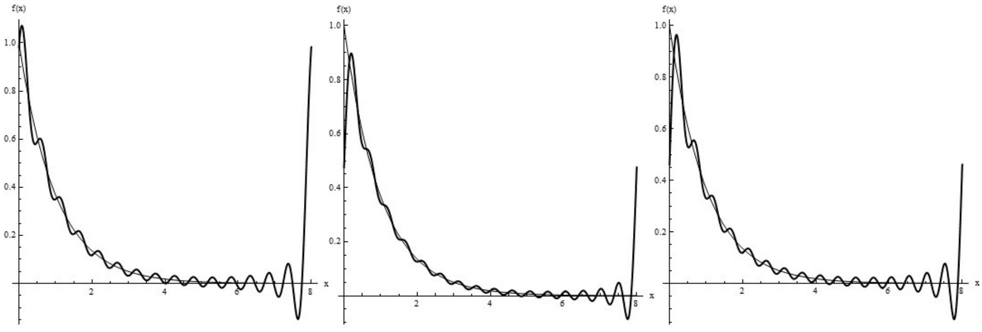


Fig. 4. The Fourier Series fitting of the exponential signal:
a) Gauss-Legendre; b) Newton-Cotes' Midpoint Rule; c) the FFT

Table 5
The values of absolute error – simulated structure signal

ΔC_n	**FFT**	**GaLEG**	**NCM**	**NCM**	**GaLEG**	**FFT**	$\Delta\varphi_n$
0	0.0	5.634 e-9	0.0	–	–	–	0
1	0.0	1.269 e-7	0.0	0.0	2.938 e-9	0.0	1
2	8.331 e-4	8.330 e-4	8.331 e-4	3.205 e-4	3.204 e-4	3.205 e-04	2
3	2.082 e-3	2.081 e-3	2.082 e-3	0.0	1.476 e-7	0.0	3
4	0.0	3.164 e-7	0.0	0.0	6.370 e-2	0.0	4

a) b) c)

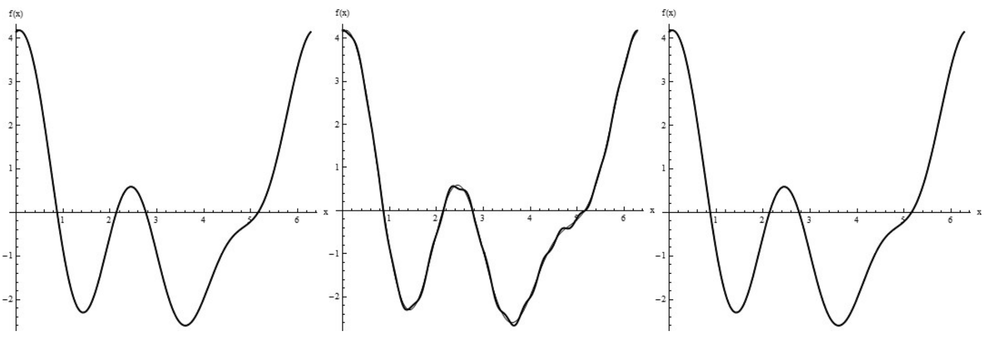


Fig. 5. The Fourier Series fitting of the simulated structure signal:
a) Gauss-Legendre; b) Newton-Cotes' Midpoint Rule; c) the FFT

5. Conclusions

Comparative analysis of errors of calculated values of coefficients (the values of integrals) and obtained on the basis of them spectral components of the signal: amplitude and phase shift allowed the following conclusions:

There indeed exist some elementary functions which can be exact integrated by Gauss Quadratures! More exact than by the reference tool. The results showed in section 4 proved that exponential, mixed and simulated structure type signals are best integrated using Gaussian-style quadratures. To be more specific – „smooth" and exponential-shape signals are best integrated with this kind of quadratures. In contrast to that – odd and even functions caused problems for this method mainly because of their fast-changing characteristics.

The above facts allowed to fulfil the goal of this comparison, which was to show that applying Gauss Quadratures allow to obtain low or error-free results despite the drastic reduction of sample points. The analysis of the results showed in tables 3, 4, 5 proves that applying the Gauss-Legendre Quadrature allows to obtain the values of amplitude and phase spectrum with absolute error smaller than 10e-12 using only 16 sample points! For comparison – Newton-Cotes' Midpoint Rule needed 1.000.000 sample points to reach the same level of accuracy! Using only 16 sample points this method calculated the values charged with one-digit % error, the FFT – with two-digit % errors. This means to reduce the complexity of computer calculations!

Above conclusions can also be assured by visual analysis of the approximated waveforms on the basis of their graphs presented (Figs. 1–5).

The second best tools the Newton-Cotes' Midpoint Rule proved to be the most versatile – the method of sampling used in this method allows it to be applied to signals of various characteristics with similar second best accuracy

The reference tool in this experiment – the FFT achieved slightly worse results than Newton-Cotes' Midpoint Rule. This is understandable bearing in mind the fact that both techniques work similar. Though the FFT's versality is weakened by inclusion of "closed" formulas (Trapezoidal or Rectangular Rule), which determines that the values at the ends of the integration range of processed function must be determinable. Therefore it may not be applied to the waveforms which have irremovable singularities. It also should not be forgotten that in case of the FFT, Radix2 number of sample points should be the power of 2. This feature can cause the problems when we have only limited number of samples sampled at specified places.

One important fact should not be forgotten – the tested functions are elementary shapes in adversity to the real-world waveforms, which usually do not have such perfect characteristics. Nevertheless, the results presented in this paper indicate huge potential of accuracy of the Gauss methods when applied to the signals with precisely chosen characteristics. Best results can be then achieved with only handful of sample points.

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References

[1] Stroud A.H., Secrest D., *Gaussian Quadrature Formulas*. Prentice-Hall, Englewood Cliffs, NJ, 1966.
[2] Krommer R.A., Ueberhuber Ch.W., *Computational Integration*. SIAM, Philadelphia 1986.
[3] Kythe P.K., Schäferkotter M.R., *Handbook of Computational Methods For Integration*. Chapman & Hall/CRC, 2005.
[4] Love C.H., *Abscissas and Weights for Gaussian Quadrature For N = 2 to 100, and N = 125, 150, 175, 200*. United States Department Of Commerce, National Bureau of Standards Monograph 98, (issued December 28, 1966).
[5] Desmarais R.N., *Programs For Computing Abscissas and Weights For Classicaland Nonclassical Gaussian Quadrature Formulas*. NASA Technical Note, NASA TN D-7924, NASA, Washington D.C., 05.October 1975.
[6] Krylov V.I., *Priblizhennoe vychislenie integralov*. 2e izd. Nauka, 1967.
[7] Zieliński T.P., *Cyfrowe przetwarzanie sygnałów*. WKiŁ, Warszawa 2007.
[8] Lyons R.G., *Wprowadzenie do cyfrowego przetwarzania danych*. WKiŁ, Warszawa 2000.
[9] Szabatin J., *Podstawy teorii sygnałów*. WKiŁ, Warszawa 2003.
[10] Abramowitz M., Stegun I.A., *Handbook of Mathematical Functions, Applied Mathematics Series*. Dover Publications, NY, 1968.
[11] Wolfram MathWorld, http://mathworld.wolfram.com.
[12] Sean Mauch, *Introduction To Methods Of Applied Mathematics* or *Advanced Mathematical Methods For Scientists And Engineers*. April 8, 2002.
[13] Leksiński W., Nabiałek I., Żakowski W., *Matematyka. Definicje, twierdzenia, przykłady, zadania*. Podręczniki akademickie, Elektronika, informatyka, telekomunikacja, WNT, Warszawa 2001.