AUTOMATYKA • 2011 • Tom 15 • Zeszyt 1

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Is a Fractional System a Dynamical System?

1. Introduction

We start from recalling definition of dynamical system and next we present scalar fractional differential equation. Solution of such equation is known and has an analytical form. This form allows to state that the fractional differential equation does not generate dynamical system.

2. Dynamical system on X

Let X be a metric space and $\{S^t\}$ for $t \geq 0$ be a dynamical system (Rudnicki 2004) on X, i. e.

$$\begin{aligned} & S^t : X \to X, \ \text{for} \ t \geq 0 \\ & S^0 = I, S^{t+s} = S^t \circ S^s, \ \text{for} \ t,s \geq 0 \\ & S : [0,\infty) \times X \to X \text{ is a continuous function of } (t,x) \end{aligned} \tag{1}$$

3. Fractional system

Consider the continuous-time fractional scalar system described by the equation

$$\frac{d^{\alpha}x(t)}{dt^{\alpha}} = \lambda x(t), \alpha \in (0,1] \tag{2}$$

where $\lambda \in R$, $\frac{d^{\alpha}x(t)}{dt^{\alpha}}$ is the fractional derivative (Kaczorek 2011). The solution of equation (2) is given by

$$x(t) = E_{\alpha}(\lambda t^{\alpha})x(0) \tag{3}$$

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where $E_\alpha(z)$ is the Mittage-Leffler function (Pillai 1990),

$$E_\alpha(z) = 1 + \frac{z}{\Gamma(1+\alpha)} + \frac{z^2}{\Gamma(1+2\alpha)} \ldots \cong \sum_{k=0}^{kk=\infty} \frac{z^k}{\Gamma(1+k\alpha)} \tag{4}$$

and $\Gamma(\alpha)$ denotes Euler's continuous gamma function

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt = \int_0^1 (\ln(1/t))^{\alpha-1} dt \tag{5}$$

For $\alpha = 1$ we have $E_\alpha(\lambda t^\alpha) = e^{\lambda t}$ and equation (2) generates the dynamical system (1) with

$$S^t = e^{\lambda t} \tag{6}$$

Note that for $\alpha \in (0,1)$

$$E_\alpha(\lambda(t+s)^\alpha) \neq E_\alpha(\lambda t^\alpha) \cdot E_\alpha(\lambda s^\alpha) \tag{7}$$

Therefore (3) is not a dynamical system trajectory.

4. Example

Let $x(0) = 1$. Let $kk = 20$ (see (4)). In Fig. 1 trajectories $x(t)$ for $\lambda = -1$ and $\alpha = 0.5$ (solid line), $\alpha = 1$ (dotted line) are shown.

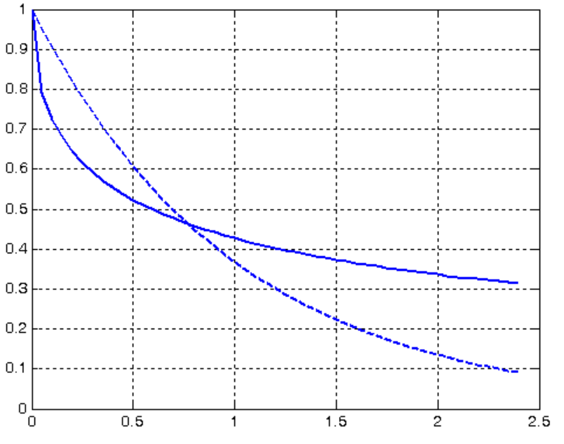

Fig. 1. Trajectories of the system (2) for $\alpha = 0.5$ (solid line), $\alpha = 1$ (dotted line).

For $\lambda = -1$ and $\alpha = 1$ (dotted line) we have $E_\alpha(\lambda t^\alpha) = e^{\lambda t}$ and consequently $e^{-(t+s)} = e^{-t}e^{-s}$.

Let for $\lambda = -1$ and $\alpha = 0.5$ (solid line) $E_\alpha(\lambda t^\alpha) = \Phi(t)$. Let $t = 0.5$ and $s = 1.0$. In this case we have $\Phi(0.5) = 0.5232$, $\Phi(1.0) = 0.4276$ and $\Phi(1.5) = 0.3732$ (see Fig. 1 – solid line). Next, note that $\Phi(0.5)\cdot\Phi(1.0) = 0.2237$. Thus $\Phi(t+s) \neq \Phi(t)\cdot\Phi(s)$.

5. Concluding remarks

- In this work we show that the fractional system is not a dynamical system. Thus the analysis of such systems can not be carried out using properties of dynamical systems. In particular some problems may arise in the analysis of fractional Chua's systems (Petras 2008).
- Solution of the fractional system (1) depends on two parameters λ and α. Let us assume that we obtain some time course as a result of a measurement. Using the least squares method we can choose the appropriate parameters λ and α of the system (1).
- Let $\lambda < 0$. For $\alpha \in (0,1)$ the solution (3) tends to zero asymptotically but not exponentially.

Acknowledgements

Work financed as a research project of the AGH University of Science and Technology, No. 11.11.120.817.

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