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Paweł Skruch*

Stabilization of a Class of SISO Nonlinear Systems by Dynamic Feedback

1. Introduction

1.1. Motivation

Almost all real systems are nonlinear in nature and it is well known that nonlinearity needs complex analysis [2, 6, 11, 12]. On one hand, the dynamics of a nonlinear system is difficult to analyze and gives rise to interesting phenomena such as bifurcations, limit cycles and chaos [1, 10]. On the other hand, nonlinear systems have a wide range of use in mechanics, electronics and robotics. For example, many mechanical systems are subject to nonlinear friction. Nonlinear electronics elements are incorporated into a circuit so as to design electronic devices with specific features like parameters amplifiers, up-converters, mixers, low-power microwave oscillators, electronic tuning devices, etc. Ferromagnetic cores in electrical machines and transformers are often described with nonlinear magnetization curves. Therefore, it is desired and advantageous to consider the nonlinearities directly while analyzing and designing the controllers for such systems.

Moreover, this is an interesting and important field both mathematically and for industrial applications. Uniqueness, stability and existence of solution are important theoretical issues for scientists. Practical implementation issues are critical for realization for engineers and designers. Recent results obtained by Author on stabilization of second-order systems both in linear [13, 14] and nonlinear [15, 16] cases have indeed been the main motivation and inspiration for this paper. The study has shown that similar methods can be applied to other class of nonlinear systems.

1.2. Related work

The stabilization scheme, that is presented in this paper, was initiated in [3] for linear undamped second-order systems. The concept was later advanced and developed in [8] for LC ladder networks and in [9]. In [13], a class of nonlinear controllers was proposed to

* Department of Automatics, AGH University of Science and Technology; pawel.skruch@agh.edu.pl

stabilize linear damped gyroscopic systems. Many fruitful advances and developments on static and dynamic stabilization methods for linear finite and infinite dimensional oscillatory systems were collected in the PhD thesis [14]. More recently, the paper [15] has considered similar stabilization techniques but for a system described by nonlinear second-order differential equations. Then, the results presented in [15] have been generalized for nonlinear second-order systems in [16].

1.3. Contribution

The aim of this paper is to discuss the stabilization problem for a class of single-input single-output (SISO) nonlinear systems using dynamic feedback. It is shown, that an uncontrolled system from such class is already asymptotically stable and the dynamic feedback can improve the dynamic stability performance of the corresponding closed-loop system. Two types of control law are presented. The designed controllers are one-dimensional, system size independent and capable to stabilize the system in a wide range of the controller parameters. Reduced-order design of the controllers and their robustness can be considered as the main advantages of the presented approach.

In this paper, the term (*asymptotic*) *stability* refers to the stability of equilibrium points in the sense of Lyapunov [5]. For determining the stability of equilibrium points of the system Lyapunov second method [5] together with LaSalle's invariance principle [4] are utilized. These methods are now well established subjects as the most powerful techniques for analyzing the stability of the systems whose dynamics is described by non-linear differential equations. The advantage of the method is that it does not require the knowledge of solutions to analyze the stability of the system. However in practical sense, how to find suitable Lyapunov functions for a given system is the most difficult question.

The paper is organized as follows. In Section 2, the class of nonlinear SISO systems is introduced. Section 3 is devoted to the main results regarding construction of dynamic feedback control that asymptotically stabilizes the system. An illustrative example is presented in Section 4. Conclusions are in Section 5.

2. System description

Consider a class of dynamic control systems which is described by the nonlinear differential equation of the following form:

$$\dot{x}(t) + \boldsymbol{G}(x) = \boldsymbol{H}(x)u(t), \quad x(0) = x^0 \tag{1}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $t > 0$, $\boldsymbol{x^0} \in \mathbb{R}^n$, $\boldsymbol{G} : \mathbb{R}^n \supset \Omega \to \mathbb{R}^n$, $\boldsymbol{H} : \mathbb{R}^n \supset \Omega \to \mathbb{R}^n$, Ω is a neighborhood of zero. The functions $\boldsymbol{G}$ and $\boldsymbol{H}$ are of the form:

$$\boldsymbol{G}(\boldsymbol{\xi}) = [g_1(\boldsymbol{\xi})\ g_2(\boldsymbol{\xi})\ \ldots\ g_n(\boldsymbol{\xi})]^{\mathrm{T}} \tag{2}$$

$$\boldsymbol{H}(\boldsymbol{\xi}) = [h_1(\boldsymbol{\xi})\ h_2(\boldsymbol{\xi})\ \ldots\ h_n(\boldsymbol{\xi})]^{\mathrm{T}} \tag{3}$$

The output function for the system is evaluated using the formula:

$$y(t) = \int_0^{\boldsymbol{x}(t)} \boldsymbol{H}(\xi)^{\mathrm{T}} d\xi \tag{4}$$

where $y(t) \in \mathbb{R}$, $t > 0$, $\int_0^{x(t)} (...) d\xi$ is a line integral along the straight line in the space $\mathbb{R}^n$ from the beginning point $\mathbf{0}$ to the ending point $\boldsymbol{x}(t)$. The physical interpretation of this integral is the work done by the force field $\boldsymbol{H}$ in the space $\mathbb{R}^n$ on a particle as the particle moves along the path that starts from $\mathbf{0}$ and ends in $\boldsymbol{x}$. In the case when the function $\boldsymbol{H} \in \mathbb{R}^{n\times 1}$ is a matrix with constant elements, the formula (4) reduces to:

$$y(t) = \boldsymbol{H}^{\mathrm{T}}\boldsymbol{x}(\mathrm{t}) \tag{5}$$

The objective of the paper is to study the nonlinear system (1) under the following conditions:

Assumption 1. *The functions* $g_i(\boldsymbol{\xi})$, $h_i(\boldsymbol{\xi})$, i, j = 1, 2, ..., *n are continuous with continuous derivatives with respect to each variable in the set* Ω.

Assumption 2.

$$\langle \xi, \boldsymbol{G}(\xi) \rangle > 0 \quad \text{for} \quad \xi \in \Omega \backslash \{\mathbf{0}\} \quad \text{and} \quad \boldsymbol{G}(\mathbf{0}) = \mathbf{0} \tag{6}$$

Theorem 1. *Suppose Assumptions 1 and 2 hold. Then the uncontrolled system (1) (i.e. with* $u(t) \equiv 0$) *is locally asymptotically stable (in the Lyapunov sense).*

Proof. The asymptotic stability of the uncontrolled system (1):

$$\dot{x}(t) = -\boldsymbol{G}(\boldsymbol{x}), \quad \boldsymbol{x}(0) = \boldsymbol{x}^{\mathbf{0}} \tag{7}$$

can be established using the Lyapunov function:

$$V(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{x}(t) \tag{8}$$

The right hand side of the differential equation (7) is continuous and satisfies Lipschitz condition, what means that the equation (7) has one and only one solution is some neigh-borhood of zero. As it is well known (see for example [7]), a sufficient condition for the stability is that the time derivative of $V(\boldsymbol{x})$ is negative along the solutions of (7). In this case

$$\dot{V}(\boldsymbol{x}) = \boldsymbol{x}(t)^{\mathrm{T}}\dot{\boldsymbol{x}}(t) = -\boldsymbol{x}(t)^{\mathrm{T}}\boldsymbol{G}(\boldsymbol{x}) = -\langle \boldsymbol{x}, \boldsymbol{G}(\boldsymbol{x}) \rangle < 0 \tag{9}$$

for $\boldsymbol{x} \neq \mathbf{0}$ and $V(\mathbf{0}) = 0$. Thus the system (7) is asymptotically stable. Q.E.D. □

3. Main results

3.1. Linear dynamic feedback control law

Consider the following dynamic feedback controller:

$$\dot{w}(t) + aw(t) = bu(t), \quad w(0) = w^0 \tag{10}$$

$$u(t) = -k(w(t) + y(t)) \tag{11}$$

where $w(t) \in \mathbb{R}$ for $t > 0$, $a > 0$, $b > 0$, $k > 0$, $w^0 \in \mathbb{R}$ is the initial condition for the dynamic part. The resulting closed-loop system becomes:

$$\dot{\boldsymbol{x}}(t) + \boldsymbol{G}(\boldsymbol{x}) + k\boldsymbol{H}(\boldsymbol{x})\int_0^{x(t)} \boldsymbol{H}(\xi)^{\mathrm{T}} d\xi + k\boldsymbol{H}(\boldsymbol{x})w(t) = 0 \tag{12}$$

$$\dot{w}(t) + (a + bk)w(t) + bk\int_0^{x(t)} \boldsymbol{H}(\xi)^{\mathrm{T}} d\xi = 0 \tag{13}$$

for the initial conditions $\boldsymbol{x}(0) = \boldsymbol{x}^{\mathbf{0}}$, $w(0) = w^0$.

Assumption 3. *There exists a neighborhood* $\Omega_{xw} \subset \mathbb{R}^n \times \mathbb{R}$ *of zero in which the closed-loop system (12), (13) has only one equilibrium point* $(\boldsymbol{x}_e, w_e) = (0, 0)$.

Practially, the above assumption is quite easy to check. It means that in some neigborhood Ω_{xw}, the system of the algebraic equations:

$$\boldsymbol{G}(\boldsymbol{x}^{\mathbf{0}}) - \frac{a}{b}\boldsymbol{H}(\boldsymbol{x}^{\mathbf{0}})w^0 = 0 \tag{14}$$

$$(a + bk)w^0 + bk\int_0^{\boldsymbol{x}^{\mathbf{0}}} \boldsymbol{H}(\xi)^{\mathrm{T}} d\xi = 0 \tag{15}$$

has only zero solution $(x_e, w_e) = (0, 0)$.

Theorem 2. *Suppose Assumptions 1, 2 and 3 hold. Then the closed-loop system (12), (13) is locally asymptotically stable (in the Lyapunov sense).*

Proof. The proof is approached by imposing a suitable Lyapunov function for the closed-loop system (12), (13). It is considered the following candidate:

$$V(z) = \int_0^{x(t)} \boldsymbol{G}(\xi)^{\mathrm{T}} d\xi + \frac{a}{2b}w(t)^2 + \frac{k}{2}\left(w(t) + \int_0^{x(t)} \boldsymbol{H}(\xi)^{\mathrm{T}} d\xi\right)^2 \tag{16}$$

where $z(t) = \mathrm{col}\,(\boldsymbol{x}(t), w(t))$, $\int_0^{x(t)}(\ldots)d\xi$ is a line integral along the straight line in the space $\mathbb{R}^n$ from the beginning point $\mathbf{0}$ to the ending point $\boldsymbol{x}(t)$. According to Lemma 1 this integral is positive for $\boldsymbol{x} \neq \mathbf{0}$ and equals zero for $\boldsymbol{x} = \mathbf{0}$. The functional V can also take the form:

$$V(z) = \int_0^{x(t)} \boldsymbol{G}(\xi)^{\mathrm{T}} d\xi + \frac{a}{2b} w(t)^2 + \frac{1}{2k} u(t)^2 \tag{17}$$

Evaluating the time derivative of V gives:

$$\begin{aligned}\dot{V}(z) &= \nabla_x \left(\int_0^x \boldsymbol{G}(\xi)^{\mathrm{T}} d\xi \right) \dot{\boldsymbol{x}}(t) + \frac{a}{b} w(t)\dot{w}(t) + \\ &+ k \left(w(t) + \int_0^{x(t)} \boldsymbol{H}(\xi)^{\mathrm{T}} d\xi \right) \left(\dot{w}(t) + \nabla_x \left(\int_0^x \boldsymbol{H}(\xi)^{\mathrm{T}} d\xi \right) \dot{\boldsymbol{x}}(t) \right)\end{aligned} \tag{18}$$

and next:

$$\begin{aligned}\dot{V}(z) &= \boldsymbol{G}(\boldsymbol{x})\dot{\boldsymbol{x}}(t) + \frac{a}{b} w(t)\dot{w}(t) + \\ &+ k \left(w(t) + \int_0^{x(t)} \boldsymbol{H}(\xi)^{\mathrm{T}} d\xi \right) \left(\dot{w}(t) + \boldsymbol{H}(\boldsymbol{x})^{\mathrm{T}} \dot{\boldsymbol{x}}(t) \right)\end{aligned} \tag{19}$$

Along the trajectory of the system (12), (13) it holds that:

$$\begin{aligned}\dot{V}(z) &= \boldsymbol{G}(\boldsymbol{x})^{\mathrm{T}} \left(-\boldsymbol{G}(\boldsymbol{x}) + \boldsymbol{H}(\boldsymbol{x})u(t) \right) + \frac{a}{b} w(t) \left(-aw(t) + bu(t) \right) + \\ &- u(t) \left(-aw(t) + bu(t) \right) - u(t)\boldsymbol{H}(\boldsymbol{x})^{\mathrm{T}} \left(-\boldsymbol{G}(\boldsymbol{x}) + \boldsymbol{H}(\boldsymbol{x})u(t) \right)\end{aligned} \tag{20}$$

After some elementary calculations:

$$\begin{aligned}\dot{V}(z) &= -\boldsymbol{G}(\boldsymbol{x})^{\mathrm{T}} \boldsymbol{G}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x})^{\mathrm{T}} \boldsymbol{H}(\boldsymbol{x})u(t) - \frac{a^2}{b} w(t)^2 + 2aw(t)u(t) - bu(t)^2 + \\ &+ u(t)\boldsymbol{H}(\boldsymbol{x})^{\mathrm{T}} \boldsymbol{G}(\boldsymbol{x}) - u(t)\boldsymbol{H}(\boldsymbol{x})^{\mathrm{T}} \boldsymbol{H}(\boldsymbol{x})u(t)\end{aligned} \tag{21}$$

it can be seen that:

$$\dot{V}(z) = -\left(\boldsymbol{G}(\boldsymbol{x}) - \boldsymbol{H}(\boldsymbol{x})u(t) \right)^{\mathrm{T}} \left(\boldsymbol{G}(\boldsymbol{x}) - \boldsymbol{H}(\boldsymbol{x})u(t) \right) - b \left(\frac{a}{b} w(t) - u(t) \right)^2 \tag{22}$$

and finally:

$$\dot{V}(z) = -\dot{\boldsymbol{x}}(t)^{\mathrm{T}} \dot{\boldsymbol{x}}(t) - \frac{1}{b} \dot{w}(t)^2 \tag{23}$$

Let Ω_c be a compact set defined as:

$$\Omega_c = \{ z \in \Omega_{xw} \subset \mathbb{R}^{n+1} : V(z) < c \} \tag{24}$$

where $c > 0$ is a real positive number. It is easy to check that $V(z) > 0$ for $z \in \Omega_c \backslash \{\mathbf{0}\}$, $V(\mathbf{0}) = 0$ and $\dot{V}(z) \leq 0$ for $z \in \Omega_c$. As a consequence of LaSalle's invariance principle [4], the trajectories of the closed-loop system (12), (13) enter the largest invariant set in S, where:

$$S = \left\{ z \in \Omega_c : \dot{V}(z) = 0 \right\} \tag{25}$$

From $\dot{V}(z) = 0$ it follows that:

$$\dot{\boldsymbol{x}}(t) = 0 \quad \text{and} \quad \dot{w}(t) = 0 \tag{26}$$

This means that S contains only equilibrium points of the system (12), (13). Since Assumption 3 holds, thus $S = \{0\}$ and according to LaSalle's principle, the origin $0 \in \mathbb{R}^{n+1}$ is asymptotically stable (in the Lyapunov sense). Q.E.D. □

Lemma 1. *If Assumption 2 holds, then the line integral* $\int_0^x \boldsymbol{G}(\xi)^{\mathrm{T}} d\xi$ *along the straight line in the space* $\mathbb{R}^n$ *from the beginning point 0 to the ending point* $\boldsymbol{x}$ ($\boldsymbol{x} \neq 0$) *is positive.*

Proof. The line integral extends the concept of the Riemann integral in one dimension to integration along a curved path in n-dimensional space:

$$\int_0^x \boldsymbol{G}(\xi)^{\mathrm{T}} d\xi = \lim_{\|\boldsymbol{dl}_i\| \to 0} \sum_i \langle \boldsymbol{G}(\xi_i), \boldsymbol{dl}_i \rangle = \lim_{\|\boldsymbol{dl}_i\| \to 0} \sum_i \|\boldsymbol{G}_i\| \|\boldsymbol{dl}_i\| \cos \alpha_i \tag{27}$$

where $\boldsymbol{dl}_i = [d\boldsymbol{\xi}_{i1}\ d\boldsymbol{\xi}_{i2} \ldots d\boldsymbol{\xi}_{in}]^{\mathrm{T}}$ is an increment of length along this path, $\boldsymbol{G}_i = \boldsymbol{G}(\boldsymbol{\xi}_i)$, α_i is the angle between the vectors $\boldsymbol{G}_i$ and $\boldsymbol{dl}_i$. Since the integration path is along the straight line, then the direction of the vector $\boldsymbol{dl}_i$ is the same as $\boldsymbol{\xi}_i$ (see Fig. 1). Following Assumption 2:

$$\langle \xi, \boldsymbol{G}(\xi_i) \rangle = \|\xi\| \|\boldsymbol{G}(\xi)\| \cos \alpha > 0 \quad \text{for} \quad \xi \neq 0 \tag{28}$$

it can be concluded that $\cos \alpha_i > 0$. This means that the line integral (27) is positive. It is also obvious that for $\boldsymbol{x} = \mathbf{0}$ the integral (27) equals zero. Q.E.D. □

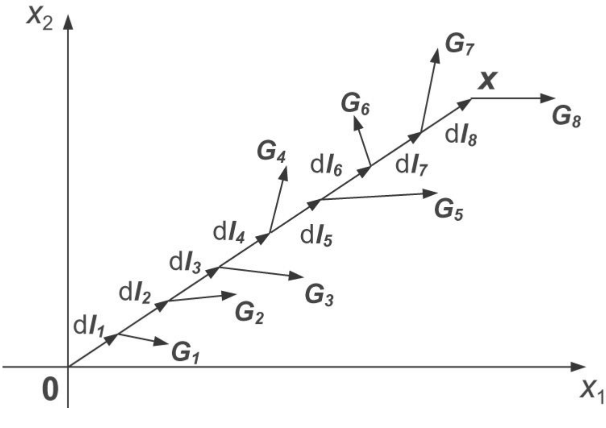

Fig. 1. Concept of a line integral along the straight line

3.2. Nonlinear dynamic feedback control law

Consider the system (1) with feedback control which is described by the following formula:

$$\dot{w}(t) + aw(t) = bu(t), \quad w(0) = w^0 \tag{29}$$

$$u(t) = -\frac{1}{K(\boldsymbol{x}, w)}\big(w(t) + y(t)\big) \tag{30}$$

$$K(\boldsymbol{x}, w) = K_0 + \gamma\big(w(t) + y(t)\big)^2 = K_0 + \gamma\left(w(t) + \int_0^{\boldsymbol{x}(t)} \boldsymbol{H}(\xi)^{\mathrm{T}} d\xi\right)^2 \tag{31}$$

where $w(t) \in \mathbb{R}$ for $t > 0$, $w^0 \in \mathbb{R}$, $a > 0$, $\mathrm{b} > 0$, $K_0 > 0$, $\gamma > 0$. The resulting closed-loop system satisfies the equations:

$$\dot{\boldsymbol{x}}(t) + \boldsymbol{G}(\boldsymbol{x}) + \frac{1}{K(\boldsymbol{x}, w)}\boldsymbol{H}(\boldsymbol{x})\int_0^{\boldsymbol{x}(t)} \boldsymbol{H}(\xi)^{\mathrm{T}} d\xi + \frac{1}{K(\boldsymbol{x}, w)}\boldsymbol{H}(\boldsymbol{x})w(t) = 0 \tag{32}$$

$$\dot{w}(t) + \left(a + \frac{b}{K(\boldsymbol{x}, w)}\right)w(t) + \frac{b}{K(\boldsymbol{x}, w)}\int_0^{\boldsymbol{x}(t)} \boldsymbol{H}(\xi)^{\mathrm{T}} d\xi = 0 \tag{33}$$

for the initial conditions $\boldsymbol{x}(0) = \boldsymbol{x}^{\mathbf{0}}$, $w(0) = w^0$.

Assumption 4. *There exists a neighborhood* $\Omega_{xw} \subset \mathbb{R}^n \times \mathbb{R}$ *of zero in which the closed-loop system (32), (33) has only one equilibrium point* $(\boldsymbol{x_e}, w_e) = (\mathbf{0}, 0)$.

The assumption is equivalent to the statement that in some neighborhood Ω_{xw} of zero the system of the algebraic equations:

$$\boldsymbol{G}(\boldsymbol{x}^{\mathbf{0}}) - \frac{a}{b}\boldsymbol{H}(\boldsymbol{x}^{\mathbf{0}})w^0 = 0 \tag{34}$$

$$\left(a + \frac{b}{K(\boldsymbol{x}^{\mathbf{0}}, w^0)}\right)w^0 + \frac{b}{K(\boldsymbol{x}^{\mathbf{0}}, w^0)}\int_0^{\boldsymbol{x}^{\mathbf{0}}} \boldsymbol{H}(\xi)^{\mathrm{T}} d\xi = 0 \tag{35}$$

has only zero solution $(\boldsymbol{x_e}, w_e) = (0, 0)$.

Theorem 3. *Suppose Assumptions 1, 2 and 4 hold. Then the closed-loop system (32), (33) is locally asymptotically stable (in the Lyapunov sense).*

Proof. Choose a Lyapunov functional as:

$$V(z) = \int_0^{\boldsymbol{x}(t)} \boldsymbol{G}(\xi)^{\mathrm{T}} d\xi + \frac{a}{2b}w(t)^2 + \frac{1}{2\gamma}\ln\frac{K(\boldsymbol{x}, w)}{K_0} \tag{36}$$

where $z(t) = \mathrm{col}(\boldsymbol{x}(t), w(t))$, $\int_0^{\boldsymbol{x}(t)} (\ldots)\, d\xi$ denotes a line integral along the straight line in the space $\mathbb{R}^n$ from the beginning point $\mathbf{0}$ to the ending point $\boldsymbol{x}(t)$. According to Lemma 1 this integral is positive for $\boldsymbol{x} \neq \mathbf{0}$ and equals zero for $\boldsymbol{x} = \mathbf{0}$. The time derivative of the functional (36) becomes:

$$\begin{aligned}\dot{V}(z) &= \nabla_{\boldsymbol{x}} \left(\int_0^{\boldsymbol{x}} \boldsymbol{G}(\xi)^{\mathrm{T}} d\xi \right) \dot{\boldsymbol{x}}(t) + \frac{a}{b} w(t)\dot{w}(t) + \frac{1}{2\gamma K(\boldsymbol{x}, w)} \dot{K}(\boldsymbol{x}, w) = \\ &= \nabla_{\boldsymbol{x}} \left(\int_0^{\boldsymbol{x}} \boldsymbol{G}(\xi)^{\mathrm{T}} d\xi \right) \dot{\boldsymbol{x}}(t) + \frac{a}{b} w(t)\dot{w}(t) + \\ &+ \frac{1}{K(\boldsymbol{x}, w)} \big(w(t) + y(t)\big) \left(\dot{w}(t) + \nabla_{\boldsymbol{x}} \left(\int_0^{\boldsymbol{x}} \boldsymbol{H}(\xi)^{\mathrm{T}} d\xi \right) \dot{\boldsymbol{x}}(t) \right) = \\ &= \boldsymbol{G}(\boldsymbol{x})^{\mathrm{T}} \dot{\boldsymbol{x}}(t) + \frac{a}{b} w(t)\dot{w}(t) - u(t)\left(\dot{w}(t) + \boldsymbol{H}(\boldsymbol{x})^{\mathrm{T}} \dot{\boldsymbol{x}}(t) \right)\end{aligned} \tag{37}$$

Evaluating $\dot{V}(z)$ along the trajectory of (32), (33) gives:

$$\begin{aligned}\dot{V}(z) &= \boldsymbol{G}(\boldsymbol{x})^{\mathrm{T}} \big(-\boldsymbol{G}(\boldsymbol{x}) + \boldsymbol{H}(\boldsymbol{x})u(t)\big) + \frac{a}{b} w(t)\big(-aw(t) + bu(t)\big) + \\ &- u(t)\big(-aw(t) + bu(t)\big) - u(t)\boldsymbol{H}(\boldsymbol{x})^{\mathrm{T}} \big(-\boldsymbol{G}(\boldsymbol{x}) + \boldsymbol{H}(\boldsymbol{x})u(t)\big)\end{aligned} \tag{38}$$

Further calculations show that:

$$\begin{aligned}\dot{V}(z) &= -\boldsymbol{G}(\boldsymbol{x})^{\mathrm{T}} \boldsymbol{G}(\boldsymbol{x}) + \boldsymbol{G}(\boldsymbol{x})^{\mathrm{T}} \boldsymbol{H}(\boldsymbol{x})u(t) - \frac{a^2}{b} w(t)^2 + \\ &+ 2aw(t)u(t) - bu(t)^2 + u(t)\boldsymbol{H}(\boldsymbol{x})^{\mathrm{T}} \boldsymbol{G}(\boldsymbol{x}) - u(t)\boldsymbol{H}(\boldsymbol{x})^{\mathrm{T}} \boldsymbol{H}(\boldsymbol{x})u(t)\end{aligned} \tag{39}$$

and:

$$\dot{V}(z) = -\big(\boldsymbol{G}(\boldsymbol{x}) - \boldsymbol{H}(\boldsymbol{x})u(t)\big)^{\mathrm{T}} \big(\boldsymbol{G}(\boldsymbol{x}) - \boldsymbol{H}(\boldsymbol{x})u(t)\big) - b\left(\frac{a}{b} w(t) - u(t) \right)^2 \tag{40}$$

The last equation can be reformulated as follows:

$$\dot{V}(z) = -\dot{\boldsymbol{x}}(t)^{\mathrm{T}} \dot{\boldsymbol{x}}(t) - \frac{1}{b} \dot{w}(t)^2 \tag{41}$$

It should be noted that $V(z) > 0$ for $z \in \Omega_c \backslash \{\mathbf{0}\}$, $V(\mathbf{0}) = 0$ and $\dot{V}(z) \leq 0$ for $z \in \Omega_c$, where Ω_c is a compact set defined as follows:

$$\Omega_c = \{z \in \Omega_{xw} \subset \mathbb{R}^{n+1} : V(z) < c\} \tag{42}$$

and c is a real positive number. According to LaSalle's theorem [4], the trajectories enter the largest invariant set in S, where:

$$S = \left\{ z \in \Omega_c : \dot{V}(z) = 0 \right\} \tag{43}$$

To prove that all solutions starting from Ω_c tend to zero, it is sufficient to show that S contains only the zero solution. The condition $\dot{V}(z) = 0$ holds if and only if $\dot{\boldsymbol{x}}(t) = 0$ and $\dot{w}(t) = 0,$ therefore S contains all equilibrium points of the system (32), (33). Since Assumption 4 is valid, thus $S = \{\mathbf{0}\}$. Q.E.D. □

4. Illustrative example

Consider a nonlinear analog circuit that is shown in Figure 2. The circuit consists of a power source, two resistors and two nonlinear capacitors. In this example, it is assumed that the voltage drops across the resistors and capacitors can be written as:

$$V_{R_k}(t) = R_k i_k(t) = R_k \dot{p}_k(t), \quad R_k > 0 \tag{44}$$

$$V_{C_k}(t) = \frac{1}{C_k(q_k)} \int_0^t j_k(t) dt = \frac{q_k(t)}{C_k(q_k)}, \quad C_k(q_k) > 0 \tag{45}$$

where i_k, j_k denote the currents in the circuit, P_k, q_k represent the electric charges, $i_k(t) = \dot{p}_k(t)$, $j_k(t) = \dot{q}_k(t)$, R_k, C_k stand for the resistances and capacitances, respectively, $k = 1, 2$. Units are omitted for simplification: unless noted, voltage is measured in volts [V], currents are measured in amperes [A], electric charges are measured in coulombs [C], resistances are in ohms [Ohm] and capacitances are in farads [F].

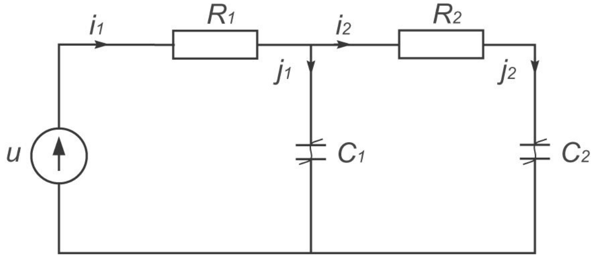

Fig. 2. Electrical RC circuit

The dynamics of electric charge flow in the circuit can be described by the following equations (based on Kirchhoff's current and voltage laws):

$$R_1 \dot{p}_1(t) + \frac{1}{C_1(q_1)} \left(p_1(t) - p_2(t) \right) = u(t) \tag{46}$$

$$R_2 \dot{p}_2(t) - \frac{1}{C_1(q_1)} \left(p_1(t) - p_2(t) \right) + \frac{1}{C_2(q_2)} p_2(t) = 0 \tag{47}$$

for the given initial conditions $p_1(0)$ and $p_2(0)$. Introduce the state variable $\boldsymbol{x}(t) = [x_1(t)\ x_2(t)]^T$, where $x_1(t) = p_1(t)$ and $x_2(t) = p_2(t)$. Without loss of generality it can be considered that:

$$C_1(q_1) = C_1(p_1, p_2) = C_1(\boldsymbol{x}), \quad C_2(q_2) = C_2(p_2) = C_2(\boldsymbol{x}) \tag{48}$$

Then, the equations (46), (47) can be rewritten to the form (1) assuming that:

$$\boldsymbol{G}(\boldsymbol{x}) = \begin{bmatrix} R_1^{-1} & 0 \\ 0 & R_2^{-1} \end{bmatrix} \begin{bmatrix} C_1(\boldsymbol{x})^{-1}x_1 - C_1(\boldsymbol{x})^{-1}x_2 \\ -C_1(\boldsymbol{x})^{-1}x_1 + C_1(\boldsymbol{x})^{-1}x_2 + C_2(\boldsymbol{x})^{-1}x_2 \end{bmatrix} \tag{49}$$

$$\boldsymbol{H} = \begin{bmatrix} R_1^{-1} & 0 \\ 0 & R_2^{-1} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} R_1^{-1} \\ 0 \end{bmatrix} \tag{50}$$

Assumption 1 is fulfilled for the following resistances and capacitances:

$$R_1 = 0.02, \quad R_2 = 0.03 \tag{51}$$

$$C_1(p) = 250\mathrm{e}^{-0.1(p1-p2)^2}, \quad C_2(p) = 100\mathrm{e}^{-0.5p_2^2} \tag{52}$$

In order to check Assumption 2 of this paper, it is useful to notice that:

$$\langle \xi, \boldsymbol{G}(\xi) \rangle = \langle \xi, \tilde{\boldsymbol{G}}(\xi)\xi \rangle \tag{53}$$

where:

$$\tilde{\boldsymbol{G}}(\xi) = \begin{bmatrix} R_1^{-1}C_1(\xi)^{-1} & -R_1^{-1}C_1(\xi)^{-1} \\ R_2^{-1}C_1(\xi)^{-1} & R_2^{-1}C_1(\xi)^{-1} + R_2^{-1}C_2(\xi)^{-1} \end{bmatrix} \tag{54}$$

The matrix $\tilde{\boldsymbol{G}}(\xi)$ for every $\boldsymbol{\xi}$ is positive definite because the leading principal minors M_1 and M_2 for this matrix are positive [17]:

$$M_1 = \frac{1}{R_1 C_1(\xi)} > 0, \quad M_2 = \frac{1}{R_1 R_2 C_1(\xi) C_2(\xi)} > 0 \tag{55}$$

since the values of the circuit parameters R_i, C_i, $i = 1, 2$ are positive. This means that:

$$\langle \xi, \boldsymbol{G}(\xi) \rangle = \langle \xi, \tilde{\boldsymbol{G}}(\xi)\xi \rangle > 0 \tag{56}$$

for $\boldsymbol{\xi} \neq 0$ and additionally $\boldsymbol{G}(\boldsymbol{0}) = \boldsymbol{0}$, so Assumption 2 holds.

Consider the controller (10), (11) with the following parameters: $k = 2$, $a = 0.5$, $b = 1$, $w^0 = 0$, i.e.:

$$u(t) = -2(w(t) + 50x_1(t)) \tag{57}$$

$$\dot{w}(t) + 0.5w(t) = u(t), \quad w(0) = 0 \tag{58}$$

The corresponding closed-loop system is described by the equations:

$$\dot{x}_1(t)+0.2\mathrm{e}^{0.1(x_1(t)-x_2(t))^2}\left(x_1(t)-x_2(t)\right)+ \\ +5000x_1(t)+100w(t)=0 \tag{59}$$

$$\dot{x}_2(t)-\frac{4}{30}\mathrm{e}^{0.1(x_1(t)-x_2(t))^2}\left(x_1(t)-x_2(t)\right)+\frac{1}{3}\mathrm{e}^{0.5x_2(t)^2}x_2(t)=0 \tag{60}$$

$$\dot{w}(t)+2.5w(t)+100x_1(t)=0 \tag{61}$$

It is not difficult to check that Assumption 3 is valid, so according to Theorem 2 the trajectories of the closed-loop system (59), (60), (61) tend asymptotically to zero as time goes to infinity. This conclusion is illustrated in Figure 3 by solid line.

Design now the dynamic feedback (29), (30), (31) with the following parameters: $a = 0.5$, $b = 1$, $w^0 = 0$, $K_0 = 50$, $\gamma = 100$, i.e.:

$$u(t)=-\frac{1}{K(\boldsymbol{x},w)}\left(w(t)+50x_1(t)\right) \tag{62}$$

$$\dot{w}(t)+0.5w(t)=u(t),\quad w(0)=0 \tag{63}$$

$$K(\boldsymbol{x}, w) = 50 + 100\,(w(t) + 50x_1(t))^2 \tag{64}$$

The closed-loop system can be described by the equations:

$$\dot{x}_1(t)+0.02\mathrm{e}^{0.1(x_1(t)-x_2(t))^2}\left(x_1(t)-x_2(t)\right)+\frac{w(t)+100x_1(t)}{1+2\left(w(t)+50x_1(t)\right)^2}=0 \tag{65}$$

$$\dot{x}_2(t)-\frac{4}{30}\mathrm{e}^{0.1(x_1(t)-x_2(t))^2}\left(x_1(t)-x_2(t)\right)+\frac{1}{3}\mathrm{e}^{0.5x_2(t)^2}x_2(t)=0 \tag{66}$$

$$\dot{w}(t)+0.5w(t)+\frac{w(t)+100x_1(t)}{50+100\left(w(t)+50x_1(t)\right)^2}=0 \tag{67}$$

It can be checked that Assumption 4 is also fulfilled. This means that the closed-loop system (65), (66), (67) is asymptotically stable as shown in Figure 3 by dashed line. For comparison purposes, the norm $\|x(t)\|$ of the trajectories of the open-loop system (dotted line) is presented together with the trajectories of the closed-loop systems: with linear dynamic controller (solid line) and with nonlinear dynamic controller (dashed line). The norm in all cases decreases to zero, but the controllers introduce to the system additional damping and improve the dynamic performance. The trajectories start from the initial conditions: $x_1(0) = 0.1$, $x_2(t) = -0.05$.

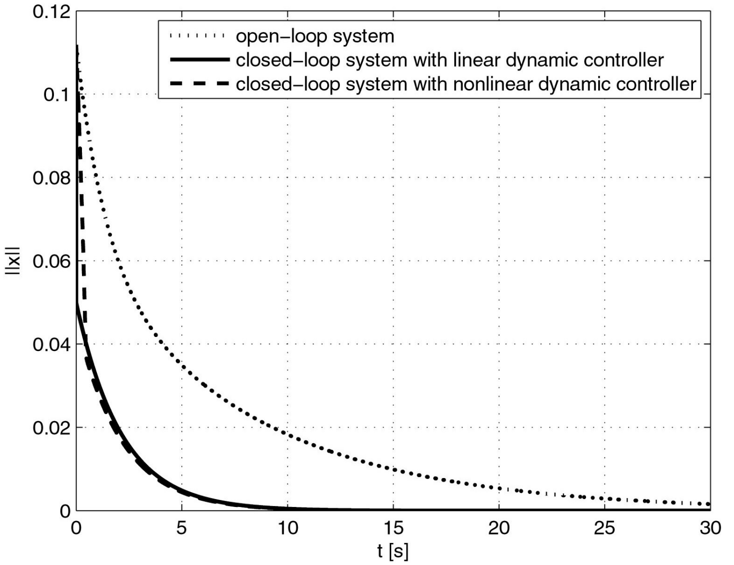

Fig. 3. Norm $||x(t)||$ of the trajectories of the open-loop system (dotted line), closed-loop system with linear dynamic controller (solid line) and closed-loop system with nonlinear dynamic controller (dashed line)

5. Conclusions

The paper has addressed the stabilization problem for a class of nonlinear SISO systems. Dynamic feedback control laws have been proposed to make the state asymptotically stable. The asymptotic stability of the closed-loop systems has been proved by the use of Lyapunov functionals and concluded by LaSalle's invariance principle. Results of numerical simulations have shown the effectiveness of the proposed controllers. The computer simulations have been performed in the MathWorks™ MATLAB®/Simulink® environment.

The main advantages of the presented approach are summarized below:

- Reduced-order design of the controllers – the designed dynamic controllers are one-dimensional, system size independent.
- Controller robustness – stabilization in a wide range of the controller parameters.
- The controllers provide excellent damping and dynamic performance improvement in comparison with an open-loop system.

The following weaknesses in the stabilization strategy should be emphasized:

- The controllers are designed for a special class of nonlinear systems.
- Assumptions 3 and 4 can be sometimes hard to verify.
- Damping and dynamic performance improvement is not theoretically shown.

The following points can be considered as future research topics:

- Extension of the results to nonlinearMIMO systems (multiple-input multiple-output).
- The proof showing that the designed controllers provide better dynamic performance.
- Optimization of the controller parameters.

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