

Adam Kowalewski\*

## **Optimization of Distributed Hyperbolic Systems with Multiple Time Delays Given in the Integral Form\*\***

### **1. Introduction**

Various optimization problems associated with the optimal control of distributed parameter systems with time delays appearing in the boundary conditions have been studied recently by Wang (1975) Knowles (1978), Kowalewski (1993a, b, 1995, 1998, 2000) and Kowalewski and Duda (1992).

In Knowles (1978), the time optimal control problem of linear parabolic systems with the Neumann boundary conditions involving constant time delays was considered.

These equations constitute in a linear approximation, a universal mathematical model for many diffusion processes in which time-delayed feedback signals are introduced at the boundary of a system's spatial domain. For example, in the area of plasma control (Wang (1975)), it is of interest to confine a plasma in a given bounded spatial domain  $\Omega$  by introducing a finite electric potential barrier or a "magnetic mirror" surrounding  $\Omega$ . For a collision-dominated plasma, its particle density is describable by a parabolic equation. Due to particle inertia and finiteness of electric potential or the magnetic -mirror field strength, the particle reflection at the domain boundary is not instantaneous. Consequently, the particle flux at the boundary of  $\Omega$  at any time depends on the flux of particles which escaped earlier and reflected back into  $\Omega$  at a later time. This leads to the boundary conditions involving time delays.

Using the results of Wang (1975), the existence of a unique solution of such parabolic systems were discussed. A characterization of the optimal control in terms of the adjoint system was given. Consequently, this characterization was used to derive specific properties of the optimal control (bangbangness, uniqueness, etc.). These results were also extended to certain cases of nonlinear control without convexity and to certain fixed-time problems.

Consequently, in Kowalewski (1993a, b, 1995, 1998) and Kowalewski and Duda (1992) linear quadratic problems for hyperbolic and parabolic systems with time delays given in the different form (constant time delays, time-varying delays, etc.) were presented.

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\* Institute of Automatics, AGH University of Science and Technology

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In particular, in Kowalewski (2000) various fixed-time control problems for distributed hyperbolic systems with time delays appearing in the integral form both in the state equations and in the Neumann boundary conditions were also considered. Such systems constitute a more complex case of distributed parameter systems with time delays given in the integral form.

In this paper, we consider optimal control problems for linear hyperbolic systems in which different multiple time delays appear in the integral form both in the state equations and in the boundary conditions.

Sufficient conditions for the existence of a unique solution of such hyperbolic equations with the Neumann boundary condition involving multiple time delays given in the integral form are proved. The performance functionals have the quadratic form. The time horizon is fixed. Finally, we impose some constraints on the distributed and boundary controls. Necessary and sufficient conditions of optimality with the quadratic performance functionals and constrained controls are derived for the Neumann problem. The optimal boundary control is obtained in the feedback form. Making use of the Schwartz's Kernel Theorem (Schwartz 1950), the representation of the optimal feedback boundary control is given.

## 2. Distributed control of a hyperbolic delay system

### 2.1. Existence and uniqueness of solutions: $f \in H^{0,1}(Q)$

Consider now the distributed-parameter system described by the following hyperbolic delay equation.

$$\frac{\partial^2 y}{\partial t^2} + A(t)y + \sum_{i=1}^m \int_{a_i}^{b_i} y(x, t - h_i) dh_i = f, \quad x \in \Omega, t \in (0, T) \quad (2.1)$$

$$y(x, t') = \Phi_0(x, t'), \quad x \in \Omega, t' \in [-\Delta, 0) \quad (2.2)$$

$$y(x, 0) = y_0(x), \quad x \in \Omega \quad (2.3)$$

$$\frac{\partial y}{\partial t}(x, 0) = y_I(x), \quad x \in \Omega \quad (2.4)$$

$$\frac{\partial y}{\partial \eta_A} = \sum_{s=1}^l \int_{c_s}^{d_s} y(x, t - k_s) dk_s + v, \quad x \in \Gamma, t \in (0, T) \quad (2.5)$$

$$y(x, t') = \Psi_0(x, t'), \quad x \in \Gamma, t' \in [-\Delta, 0) \quad (2.6)$$

where:  $\Omega \subset R^n$  – a bounded, open set with boundary  $\Gamma$ , which is a  $C^\infty$  – manifold of dimension  $(n-1)$ . Locally,  $\Omega$  is totally on one side of  $\Gamma$ .

$$y \equiv y(x, t, f), \quad f \equiv f(x, t), \quad v \equiv v(x, t)$$

$$Q = \Omega \times (0, T), \quad \bar{Q} = \bar{\Omega} \times [0, T], \quad Q_0 = \Omega \times [-\Delta, 0), \quad \Sigma = \Gamma \times (0, T), \quad \Sigma_0 = \Gamma \times [-\Delta, 0)$$

$h_i, k_s$  are time delays (deviating arguments) such that  $h_i \in (a_i, b_i)$  and  $k_s \in (c_s, d_s)$  where  $0 < a_1 < a_2 < \dots < a_m, \quad 0 < b_1 < b_2 < \dots < b_m$  for  $i = 1, \dots, m$

and  $0 < c_1 < c_2 < \dots < c_l, \quad 0 < d_1 < d_2 < \dots < d_l$  for  $s = 1, \dots, l$ ,

$\Phi_0, \Psi_0$  are initial functions defined on  $Q_0$  and  $\Sigma_0$  respectively,

$$\overset{df}{\Delta} = \max\{b_m, d_l\}.$$

The operator  $A(t)$  is given by

$$A(t)y = - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( a_{ij}(x, t) \frac{\partial y(x, t)}{\partial x_j} \right) \tag{2.7}$$

and the functions  $a_{ij}(x, t)$  satisfy the following condition in  $Q = \Omega \times (0, T)$

$$\left. \begin{aligned} \sum_{i,j=1}^n a_{ij}(x, t) \varphi_i \varphi_j &\geq \alpha \sum_{i=1}^n \varphi_i^2, \quad \alpha > 0, \quad \forall (x, t) \in \bar{Q}, \varphi_i \in R \\ a_{ij} &= a_{ji}, \quad \forall i, j \end{aligned} \right\} \tag{2.8}$$

where:  $a_{ij}(x, t)$  – real  $C^\infty$  functions defined on  $\bar{Q}$  (closure of  $Q$ ).

The equations (2.1)–(2.6) constitute the Neumann problem. The left-hand side of the Neumann boundary condition (2.5) is written in the following form

$$\frac{\partial y}{\partial \eta_A} = \sum_{i,j=1}^n a_{ij}(x, t) \cos(n, x_i) \frac{\partial y(x, t)}{\partial x_j} = q(x, t), \quad x \in \Gamma, t \in (0, T) \tag{2.9}$$

where:  $\frac{\partial y}{\partial \eta_A}$  is a normal derivative at  $\Gamma$ , directed towards the exterior of  $\Omega$ ,  $\cos(n, x_i)$  is an  $i$ -th direction cosine of  $n$ ,  $n$ -being the normal at  $\Gamma$  exterior to  $\Omega$ .

$$q(x, t) = \sum_{s=1}^l \int_{c_s}^{d_s} y(x, t - k_s) dk_s + v(x, t) \tag{2.10}$$

First we shall prove the existence of a unique solution of the mixed initial-boundary value problem (2.1)–(2.6). We shall consider the case where the control  $f$  belongs to  $H^{0,1}(Q)$ .

For this purpose, for any pair of real numbers  $r, s \geq 0$ , we introduce the Sobolev space  $H^{r,s}(Q)$  (Lions and Magenes 1972: Vol. 2, p. 6) defined by

$$\left. \begin{aligned} H^{r,s}(Q) &= H^0(0, T; H^r(\Omega)) \cap H^s(0, T; H^0(\Omega)) \\ &\text{which is a Hilbert space normed by} \\ &\left( \int_0^T \|f(t)\|_{H^r(\Omega)}^2 dt + \|f\|_{H^s(0, T; H^0(\Omega))}^2 \right)^{1/2} \end{aligned} \right\} \quad (2.11)$$

where  $H^s(0, T; H^0(\Omega))$  denotes the Sobolev space of order  $s$  of functions defined on  $(0, T)$  and taking values in  $H^0(\Omega)$ .

Consequently, some properties and central theorems for the functions  $y \in H^{r,s}(Q)$  are given in Lions and Magenes (1972) and Kowalewski (1998).

For simplicity, we introduce the following notations:

$$E_j = ((j-1)\lambda, j\lambda), Q_j = \Omega \times E_j, \Sigma_j = \Gamma \times E_j \text{ for } j = 0, 1, \dots, K, \lambda = \min\{a_1, c_1\}$$

The existence of a unique solution for the mixed initial-boundary value problem (2.1)–(2.6) on the cylinder  $Q$  can be proved using a constructive method, i.e. solving at first equations (2.1)–(2.6) on the subcylinder  $Q_1$  and in turn on  $Q_2$ , etc. until the procedure covers the whole cylinder  $Q$ . In this way, the solution in the previous step determines the next one.

Consequently, using the Theorem 4 of Kowalewski (1998) one may prove the following result.

**Theorem 2.1.** *Let  $y_0, y_I, \Phi_0, \Psi_0, v$  and  $f$  be given with  $y_0 \in H^2(\Omega)$ ,  $y_I \in H^{3/2}(\Omega)$ ,  $\Phi_0 \in H^{2,2}(Q_0)$ ,  $\Psi_0 \in H^{3/2,3/2}(\Sigma_0)$ ,  $v \in H^{3/2,3/2}(\Sigma)$ ,  $f \in H^{0,1}(Q)$  and the compatibility relations*

$$\frac{\partial y_0}{\partial \eta_A}(x, 0) = q_1(x, 0) \quad \text{on } \Gamma \quad (2.12)$$

$$\frac{\partial y_I}{\partial \eta_A}(x, 0) + \left( \frac{\partial}{\partial t} \left( \frac{\partial}{\partial \eta_A} \right) \right) y_0(x, 0) = \frac{\partial}{\partial t} q_1(x, 0) \quad \text{on } \Gamma \quad (2.13)$$

are satisfied.

Then, there exists a unique solution  $y \in H^{2,2}(Q)$  for the time delay hyperbolic equation (2.1)–(2.6) with  $y(\cdot, j\lambda) \in H^2(\Omega)$  and  $y'(\cdot, j\lambda) \in H^{3/2}(Q)$  for  $j = 1, \dots, K$ .

The idea of the proof of Theorem 2.1 is the same as in the case of Theorem 1 in Kowalewski (2000).

## 2.2. Optimal distributed control

We shall formulate the optimal control problem in the context of case where  $f \in H^{0,1}(Q)$ . Let us denote by  $U = H^{0,1}(Q)$  the space of controls. The time horizon  $T$  is fixed in our problem. The the cost function is given by

$$I(u) = \lambda_1 \int_Q |y(x, t; f) - z_d|^2 dxdt + \lambda_2 \|f\|_{H^{0,1}(Q)}^2 \quad (2.14)$$

where:  $\lambda_i \geq 0$  and  $\lambda_1 + \lambda_2 > 0$ ;  $z_d$  is a given element in  $L^2(Q)$ .

Using the formula (2.11), the second term on the right-hand side of (3.1) can be written as

$$\begin{aligned} \|f\|_{H^{0,1}(Q)}^2 &= 2 \int_0^T \langle f(x, t), f(x, t) \rangle_{L^2(\Omega)} dt + \\ &+ \int_0^T \left\langle \frac{\partial f(x, t)}{\partial t}, \frac{\partial f(x, t)}{\partial t} \right\rangle_{L^2(\Omega)} dt = \int_Q \left[ \left( 2 - \frac{\partial^2}{\partial t^2} \right) f \right] f dxdt \end{aligned} \quad (2.15)$$

Moreover,  $f(x, 0) = f(x, T) = 0, x \in \Omega$ .

Finally, we assume the following constraint on controls:

$$f \in U_{Q_{ad}} \text{ is a closed, convex subset of } U \quad (2.16)$$

Let  $y(x, t; f)$  denote the solution of (2.1)–(2.6) at  $(x, t)$  corresponding to a given control  $f \in U_{ad}$ .

We note from the Theorem 2.1 that for any  $f \in U_{ad}$ , the cost function (2.14) is well-defined since  $y(f) \in H^{2,2}(Q) \subset L^2(Q)$ . The solving of the stated optimal control problem is equivalent to a seeking an  $f_0 \in U_{ad}$  such that  $I(f_0) \leq I(f) \forall f \in U_{Q_{ad}}$ .

The starting point for our considerations will be the following theorem which can be found in (Lions 1971: p. 10):

**Theorem 2.2.** *Assume that the function  $f \rightarrow I(f)$  is strictly convex, differentiable such that  $I(f) \rightarrow +\infty$  as  $\|f\| \rightarrow +\infty, f \in U_{ad}$  (the last hypothesis may be omitted if  $U_{ad}$  is bounded). Then, the unique element  $f_0$  in  $U_{Q_{ad}}$  satisfying  $I(f_0) = \inf_{f \in U_{Q_{ad}}} I(f)$  is characterized by*

$$I(f_0)(f - f_0) \geq 0 \quad \forall f \in U_{Q_{ad}} \quad (2.17)$$

For the above control problem, from the Theorem 2.2, it follows that for  $\lambda_2 > 0$  a unique optimal control  $f_0$  exists; moreover  $f_0$  is characterized by the condition (2.17).

Using the form of the performance functional (2.14), we may express (2.17) in the following form

$$\begin{aligned} &\lambda_1 \int_Q (y(f_0) - z_d)(y(f) - y(f_0)) dxdt + \\ &+ \lambda_2 \langle f_0, f - f_0 \rangle_{H^{0,1}(Q)} \geq 0 \quad \forall f \in U_{Q_{ad}} \end{aligned} \quad (2.18)$$

To simplify (2.18), we introduce the adjoint equation and for every  $f \in U_{Q_{ad}}$  we define the adjoint variable  $p = p(f) = p(x, t; f)$  as the solution of the following equation

$$\begin{aligned} \frac{\partial^2 p(f)}{\partial t^2} + A(t)p(f) + \sum_{i=1}^m \int_{a_i}^{b_i} p(x, t + h_i; f) dh_i = \\ = \lambda_1 (y(f) - z_d) \quad x \in \Omega, t \in (0, T - \Delta) \end{aligned} \quad (2.19)$$

$$\begin{aligned} \frac{\partial^2 p(f)}{\partial t^2} + A(t)p(f) + \sum_{i=1}^m \int_{a_i}^{T-t} p(x, t + h_i; f) dh_i = \\ = \lambda_1 (y(f) - z_d) \quad x \in \Omega, t \in (T - \Delta, T - \lambda) \end{aligned} \quad (2.20)$$

$$\frac{\partial^2 p(f)}{\partial t^2} + A(t)p(f) = \lambda_1 (y(f) - z_d) \quad x \in \Omega, t \in (T - \lambda, T) \quad (2.21)$$

$$p(x, T; f) = 0 \quad x \in \Omega \quad (2.22)$$

$$p'(x, T; f) = 0 \quad x \in \Omega \quad (2.23)$$

$$\frac{\partial p(f)}{\partial \eta_A}(x, t) = \sum_{s=1}^l \int_{c_s}^{d_s} p(x, t + k_s; f) dk_s \quad x \in \Gamma, t \in (0, T - \Delta) \quad (2.24)$$

$$\frac{\partial p(f)}{\partial \eta_A}(x, t) = \sum_{s=1}^l \int_{c_s}^{T-t} p(x, t + k_s; f) dk_s \quad x \in \Gamma, t \in (T - \Delta, T - \lambda) \quad (2.25)$$

$$\frac{\partial p(f)}{\partial \eta_A}(x, t) = 0 \quad x \in \Gamma, t \in (T - \lambda, T) \quad (2.26)$$

where

$$\frac{\partial p(f)}{\partial \eta_A}(x, t) = \sum_{ij=1}^n a_{ij}^{df}(x, t) \cos(n, x_i) \frac{\partial p(f)}{\partial x_j}(x, t)$$

Using the Theorem 2.1, the following result can be proved.

**Lemma 2.1.** *Let assumptions of Theorem 2.1 be satisfied. Then, for given  $z_d \in L^2(Q)$  and any  $f \in H^{0,1}(Q)$ , there exists a unique solution  $p(f) \in H^{2,2}(Q)$  for (2.19)–(2.26).*

Using the adjoint equation (2.19)–(2.26), we simplify the first component of the left-hand side of (2.18). Consequently, after transformations we get

$$\lambda_1 \int_Q (y(f_0) - z_d)(y(f) - y(f_0)) dxdt = \int_Q p(f_0)(f - f_0) dxdt \quad (2.27)$$

Using the formula (2.15) and substituting (2.27) into (2.18) gives

$$\int_Q \left[ p(f_0) + \lambda_2 \left( 2 - \frac{\partial^2}{\partial t^2} \right) f_0 \right] (f - f_0) dxdt \geq 0 \quad \forall u \in U_{Q_{ad}} \quad (2.28)$$

**Theorem 2.3.** *For the problem (2.1)–(2.6) with the performance functional (2.14) with  $z_d \in L^2(\Omega)$  and  $\lambda_2 > 0$  and with constraints on controls (2.16), there exists a unique optimal control  $f_0$  which satisfies the condition (2.28).*

### 3. Boundary control of a hyperbolic delay system

#### 3.1. Existence and uniqueness of solutions: $v \in L^2(\Sigma)$

Consider now the distributed-parameter system described by the following hyperbolic delay equation

$$\frac{\partial^2 y}{\partial t^2} + A(t)y + \sum_{i=1}^m \int_{a_i}^{b_i} y(x, t - h_i) dh_i = f \quad x \in \Omega, t \in (0, T) \quad (3.1)$$

$$y(x, t') = \Phi_0(x, t') \quad x \in \Omega, t' \in [-\Delta, 0) \quad (3.2)$$

$$y(x, 0) = y_0(x) \quad x \in \Omega \quad (3.3)$$

$$y'(x, 0) = y_I(x) \quad x \in \Omega \quad (3.4)$$

$$\frac{\partial y}{\partial \eta_A} = \sum_{s=1}^l \int_{c_s}^{d_s} y(x, t - k_s) dk_s + Gv \quad x \in \Gamma, t \in (0, T) \quad (3.5)$$

$$y(x, t') = \Psi_0(x, t') \quad x \in \Gamma, t' \in [-\Delta, 0) \quad (3.6)$$

where:  $\Omega$  has the same properties as in the problem (2.1)–(2.6)

$$y \equiv y(x, t; v), \quad f \equiv f(x, t), \quad v \equiv v(x, t), \quad Q = \Omega \times (0, T)$$

$$\bar{Q} = \bar{\Omega} \times [0, T], \quad Q_0 = \Omega \times [-\Delta, 0), \quad \Sigma = \Gamma \times (0, T), \quad \Sigma_0 = \Gamma \times [-\Delta, 0)$$

$h_i, k_s$  – are time delays (deviating arguments) such that  $h_i \in (a_i, b_i)$  and  $k_s \in (c_s, d_s)$ ,  $h_i$  and  $k_s$  have the same properties as in problem (2.1)–(2.6),  $\Phi_0, \Psi_0$  – are initial functions defined on  $Q_0$  and  $\Sigma_0$  respectively,  $\Delta = \max_{df} \{b_m, d_l\}$ ,  $G$  – is a linear continuous operator on  $L^2(\Sigma)$  into

$$(H^{5/2} \Xi^{5/2}(\Sigma))' \text{ with } v \in L^2(\Sigma) \text{ and } Gv \in H^{-5/2} \Xi^{-5/2}(\Sigma).$$

The operator  $A(t)$  is given by the formula (2.7).

The equations (3.1)–(3.6) constitute the Neumann problem. The left-hand side of the Neumann boundary condition (3.5) can be written in the following form

$$\frac{\partial y}{\partial \eta_A} = q(x, t) \quad x \in \Gamma, t \in (0, T) \quad (3.7)$$

where

$$q(x, t) = \sum_{s=1}^l \int_{c_s}^{d_s} y(x, t - k_s) dk_s + Gv(x, t) \quad x \in \Gamma, t \in (0, T), k_s \in (c_s, d_s) \quad (3.8)$$

We shall prove the existence of a unique solution of the mixed initial-boundary value problem (3.1)–(3.6) defined by transposition, i.e.

$$\langle y, u'' + Au \rangle = L(u) \quad \forall u \in X^1(Q) \quad (3.9)$$

where

$$L(u) = \langle l, u \rangle + \langle q, u \rangle + \langle y_I, u(0) \rangle - \langle y_0, u'(0) \rangle \quad (3.10)$$

and

$$l = \left[ f - \sum_{i=1}^m \int_{a_i}^{b_i} y(x, t - h_i) dh_i \right]_Q \quad (3.11)$$

and  $X^1(Q)$  is the space described by the solutions  $u$  of the adjoint problem (3.7) in Kowalewski (2000).

Using the Theorem 4 of Kowalewski (2000) the following theorem can be proved.

**Theorem 3.1.** *Let  $y_0, y_I, \Phi_0, \Psi_0, v$  and  $f$  be given with*

$$y_0 \in \Xi^{-3/2}(\Omega), y_I \in \Xi^{-5/2}(\Omega), \Phi_0 \in H^{-1,-2}(Q_0),$$

$$\Psi_0 \in H^{-5/2} \Xi^{-5,2}(\Sigma_0), v \in L^2(\Sigma) \text{ and } f \in \Xi^{-3,-3}(Q).$$

*Then, there exists a unique solution  $y \in D_{A+D_I}^{-1}(Q)$  for the problem (3.1)–(3.6) defined by transposition (3.9). Moreover,  $y(\cdot, j\lambda) \in \Xi^{-3/2}(\Omega)$  and  $y'(\cdot, j\lambda) \in \Xi^{-5/2}(\Omega)$  for  $j = 1, \dots, K$ .*



The sketch of the proof of Theorem 3.1 is similar as in the case of Theorem 5 in Kowalewski (2000).

### 3.2. Optimal boundary control

Now we shall formulate the optimal control problem in the context of the case where  $v \in L^2(\Sigma)$ . Let us denote by  $U = L^2(\Sigma)$  the space of controls. The time horizon  $T$  is fixed in our problem. The cost function is given by

$$I(v) = \lambda_1 \|y(v) - z_d\|_{H^{-1,-2}(Q)}^2 + \lambda_2 \langle Nv, v \rangle_{L^2(\Sigma)} \quad (3.12)$$

where:  $\lambda_i \geq 0$  and  $\lambda_1 + \lambda_2 > 0$ ;  $z_d$  is a given element in  $H^{-1,-2}(Q)$ , and  $N$  is a positive linear operator on  $L^2(\Sigma)$  into  $L^2(\Sigma)$ .

Finally, we assume the following constraint on controls

$$v \in U_{\Sigma_{ad}} \quad (3.13)$$

where:  $U_{\Sigma_{ad}}$  is a closed, convex subset of  $U$ .

Let  $y(x, t; v)$  denote the solution of (3.1)–(3.6) at  $(x, t)$  corresponding to a given control  $v \in U_{\Sigma_{ad}}$ . We note from Theorem 3.1 that for any  $v \in U_{\Sigma_{ad}}$ , the performance functional (3.12) is well-defined since  $y \in D_{A+D^2}^{-1}(Q) \subset H^{-1,-2}(Q)$ . The solving of the formulated optimal control problem is equivalent to seeking a  $v_0 \in U_{\Sigma_{ad}}$  such that  $I(v_0) \leq I(v) \forall v \in U_{\Sigma_{ad}}$ .

Then, from Theorem 2.2 (replacing  $f$  by  $v$ ), it follows that for  $\lambda_2 > 0$  a unique optimal control  $v_0$  exists, moreover  $v_0$  is characterized by

$$\begin{aligned} & \lambda_1 \langle \Lambda_1(y(v_0) - z_d), y(v) - y(v_0) \rangle_{H^{-1,-2}(Q)} + \\ & + \lambda_2 \langle Nv_0, v - v_0 \rangle_{L^2(\Sigma)} \geq 0 \quad \forall v \in U_{\Sigma_{ad}} \end{aligned} \quad (3.14)$$

where:  $\Lambda_1$  is a canonical isomorphism of  $H^{-1,-2}(Q)$  onto  $H_{0,0}^{1,2}(Q)$ .

To simplify (3.14) we introduce the adjoint equation and for every  $v \in U_{\Sigma_{ad}}$ , we define the adjoint variable  $p = p(v) = p(x, t; v)$  as the solution of the following equation

$$\begin{aligned} & \frac{\partial^2 p(v)}{\partial t^2} + A(t)p(v) + \sum_{i=1}^m \int_{a_i}^{b_i} p(x, t + h_i; v) dh_i = \\ & = \lambda_1 \Lambda_1(y(v_0) - z_d) \quad x \in \Omega, t \in (0, T - \Delta) \end{aligned} \quad (3.15)$$

$$\begin{aligned} \frac{\partial^2 p(v)}{\partial t^2} + A(t)p(v) + \sum_{i=1}^m \int_{a_i}^{T-t} p(x, t+h_i; v) dh_i &= \\ = \lambda_1 \Lambda_1 (y(v_0) - z_d) \quad x \in \Omega, t \in (T-\Delta, T-\lambda) \end{aligned} \quad (3.16)$$

$$\frac{\partial^2 p(v)}{\partial t^2} + A(t)p(v) = \lambda_1 \Lambda_1 (y(v_0) - z_d) \quad x \in \Omega, t \in (T-\lambda, T) \quad (3.17)$$

$$p(x, T; v) = 0 \quad x \in \Omega \quad (3.18)$$

$$p'(x, T; v) = 0 \quad x \in \Omega \quad (3.19)$$

$$\frac{\partial p(v)}{\partial \eta_A}(x, t) = \sum_{s=1}^l \int_{c_s}^{d_s} p(x, t+k_s; v) dk_s \quad x \in \Gamma, t \in (0, T-\Delta) \quad (3.20)$$

$$\frac{\partial p(v)}{\partial \eta_A}(x, t) = \sum_{s=1}^l \int_{c_s}^{T-t} p(x, t+k_s; v) dk_s \quad x \in \Gamma, t \in (T-\Delta, T-\lambda) \quad (3.21)$$

$$\frac{\partial p(v)}{\partial \eta_A}(x, t) = 0 \quad x \in \Gamma, t \in (T-\lambda, T) \quad (3.22)$$

where

$$\frac{\partial p(v)}{\partial \eta_A}(x, t) = \sum_{i,j=1}^n a_{ij}(x, t) \cos(n, x_i) \frac{\partial p(v)}{\partial x_j}(x, t)$$

Then,  $p(v)$  is defined by transposition, i.e.

$$\langle p, y'' + Ay \rangle = M(y) \quad \forall y \in D_{A+D_t}^{-1}(Q) \quad (3.23)$$

where

$$M(y) = \langle p'' + Ap, y \rangle + \langle p, l \rangle - \langle p, q \rangle - \langle p(0), y_I \rangle + \langle p'(0), y_0 \rangle$$

and  $y$  satisfies (3.1)–(3.6).

**Lemma 3.1.** *Let assumptions of Theorem 3.1 be satisfied. Then for given  $z_d \in H^{-1,-2}(Q)$  and any  $v \in L^2(\Sigma)$ , there exists a unique solution  $p(v) \in H^{3,3}(Q) \subset \Xi^{3,3}(Q)$  to the problem (3.15)–(3.22), defined by transposition (3.23).*

Using the adjoint equation (3.15)–(3.22), we transform the first component of the left-hand side of (3.14). Consequently, after transformations we get

$$\begin{aligned} & \lambda_1 \langle \Lambda_1(y(v_0) - z_d), y(v) - y(v_0) \rangle_{H^{-1,-2}(Q)} = \\ & = \langle p(x, t; v_0), G(v - v_0) \rangle_{H^{-5/2}\Xi^{-5/2}(\Sigma)} = \langle G^* p, v - v_0 \rangle_{L^2(\Sigma)} \end{aligned} \quad (3.24)$$

Substituting (3.24) into (3.14) gives

$$\langle G^* p(v_0) + \lambda_2 N v_0, v - v_0 \rangle_{L^2(\Sigma)} \geq 0 \quad \forall v \in U_{\Sigma_{ad}} \quad (3.25)$$

The foregoing result is now summarized.

**Theorem 3.2.** *For the problem (3.1)–(3.6) with the cost function (3.12) with  $z_d \in H^{-1,-2}(Q)$  and  $\lambda_2 > 0$  and with constraints on controls (3.13) there exists a unique optimal control  $v_0$  which satisfies maximum condition (3.25).*

We can also consider analogous optimal control problem with the cost function given by

$$\hat{I}(v) = \lambda_1 \|y(v)|_{\Sigma} - z_{\Sigma_d}\|_{H^{-5/2}\Xi^{-5/2}(\Sigma)}^2 + \lambda_2 \langle N v, v \rangle_{L^2(\Sigma)} \quad (3.26)$$

where:  $z_{\Sigma_d}$  is a given element in  $H^{-5/2}\Xi^{-5/2}(\Sigma)$ ; we assume the space  $H^{-5/2}\Xi^{-5/2}(\Sigma)$  such that  $y(v)|_{\Sigma} \in H^{-5/2}\Xi^{-5/2}(\Sigma)$ .

Then the optimal control  $v_0$  is characterized by

$$\begin{aligned} & \lambda_1 \langle \Lambda_2(y(v_0)|_{\Sigma} - z_{\Sigma_d}), y(v)|_{\Sigma} - y(v_0)|_{\Sigma} \rangle_{H^{-5/2}\Xi^{-5/2}(\Sigma)} + \\ & + \lambda_2 \langle N v_0, v - v_0 \rangle_{L^2(\Sigma)} \geq 0 \quad \forall v \in U_{\Sigma_{ad}} \end{aligned} \quad (3.27)$$

where:  $\lambda_2$  a canonical isomorphism of  $H^{-5/2}\Xi^{-5/2}(\Sigma)$  into  $H^{5/2}\Xi^{5/2}(\Sigma)$ .

The adjoint equation has the form

$$\frac{\partial^2 p(v_0)}{\partial t^2} + A(t)p(v_0) + \sum_{i=1}^m \int_{a_i}^{b_i} p(x, t + h_i, v_0) dh_i = 0 \quad x \in \Omega, t \in (0, T - \Delta) \quad (3.28)$$

$$\frac{\partial^2 p(v_0)}{\partial t^2} + A(t)p(v_0) + \sum_{i=1}^m \int_{a_i}^{T-t} p(x, t + h_i, v_0) dh_i = 0 \quad x \in \Omega, t \in (T - \Delta, T - \lambda) \quad (3.29)$$

$$\frac{\partial^2 p(v_0)}{\partial t^2} + A(t)p(v_0) = 0 \quad x \in \Omega, t \in (T - \lambda, T) \quad (3.30)$$

$$p(x, T; v) = 0 \quad x \in \Omega \quad (3.31)$$

$$p'(x, T; v) = 0 \quad x \in \Omega \quad (3.32)$$

$$\frac{\partial p(v_0)}{\partial \eta_A}(x, t) = \sum_{s=1}^l \int_{c_s}^{d_s} p(x, t + k_s; v_0) dk_s + \quad (3.33)$$

$$+ \lambda_1 \Lambda_2(y(v_0))|_{\Sigma - z_{\Sigma_d}}(x, t) \quad x \in \Gamma, t \in (0, T - \Delta)$$

$$\frac{\partial p(v_0)}{\partial \eta_A}(x, t) = \sum_{s=1}^l \int_{c_s}^{T-t} p(x, t + k_s; v_0) dk_s + \quad (3.34)$$

$$+ \lambda_1 \Lambda_2(y(v_0))|_{\Sigma - z_{\Sigma_d}}(x, t) \quad x \in \Gamma, t \in (T - \Delta, T - \lambda)$$

$$\frac{\partial p(v_0)}{\partial \eta_A}(x, t) = \lambda_1 \Lambda_2(y(v_0))|_{\Sigma - z_{\Sigma_d}}(x, t) \quad x \in \Gamma, t \in (T - \lambda, T) \quad (3.35)$$

Using the Theorem 3.1, one may prove the following result.

**Lemma 3.2.** *Let assumptions of Theorem 3.1 be satisfied. Then for given  $z_{\Sigma_d} \in H^{-5/2} \Xi^{-5/2}(\Sigma)$  and any  $v \in L^2(\Sigma)$ , there exists a unique solution  $p(v) \in H^{3,3}(Q) \subset \Xi^{3,3}(Q)$  to the problem (3.28)–(3.35) defined by transposition (3.23).*

In this case the condition (3.27) can be also rewritten in the form (3.25). The following theorem is now satisfied.

**Theorem 3.3.** *For the mixed initial-boundary value problem (3.1)–(3.6) with the cost function given by (3.26) with  $z_{\Sigma_d} \in H^{-5/2} \Xi^{-5/2}(\Sigma)$  and  $\lambda_2 > 0$  and with constraints on controls (3.13), there exists a unique optimal control  $v_0$  which satisfies maximum condition (3.25).*

The optimality conditions derived above (Theorems 2.3, 3.2 and 3.3) do not provide any analytical formulae for the optimal control. Thus, we turn from the exact determining of the optimal control and we have to use approximation methods.

In the case of performance functionals (2.14), (3.12) and (3.26) with  $\lambda_1 > 0$  and  $\lambda_2 = 0$ , the optimal control problem reduces to the minimizing of the functional on a closed and convex subset in a Hilbert space. Then, the optimization problem is equivalent to a quadra-



with boundary conditions

$$\frac{\partial y}{\partial \eta_A}(x, t) = \begin{cases} \sum_{s=1}^l \int_{c_s}^{d_s} y(x, t - k_s) dk_s - \lambda_2 N^{-1} G^* p & \text{for } t - k_s \geq \varepsilon \\ \sum_{s=1}^l \int_{c_s}^{d_s} \Psi_\varepsilon(x, t - k_s) dk_s - \lambda_2 N^{-1} G^* p & \text{for } t - k_s < \varepsilon \\ (x, t) \in \Gamma \times (\varepsilon, T) \end{cases} \quad (3.39)$$

$$\frac{\partial p}{\partial \eta_A}(x, t) = \begin{cases} \sum_{s=1}^l \int_{c_s}^{d_s} p(x, t + k_s) dk_s & \text{for } (x, t) \in \Gamma \times (\varepsilon, T - \Delta) \\ \sum_{s=1}^l \int_{c_s}^{T-t} p(x, t + k_s) dk_s & \text{for } (x, t) \in \Gamma \times (T - \Delta, T - \lambda) \\ 0 & \text{for } (x, t) \in \Gamma \times (T - \lambda, T) \end{cases} \quad (3.40)$$

and with the following conditions

$$\begin{cases} y(x, \varepsilon) = y_\varepsilon(x) & x \in \Omega \\ y'(x, \varepsilon) = y'_\varepsilon(x) & x \in \Omega \\ p(x, T) = 0 & x \in \Omega \\ p'(x, T) = 0 & x \in \Omega \end{cases} \quad (3.41)$$

where:  $y_\varepsilon \in \Xi^{-3/2}(\Omega)$  and  $y'_\varepsilon \in \Xi^{-5/2}(\Omega)$ ,  $y_\varepsilon \in H^1(\Omega)$ ,  $\Phi_\varepsilon$  and  $\Psi_\varepsilon$  are given function defined on  $\Omega \times [\varepsilon - \Delta, \varepsilon)$  and  $\Gamma \times [\varepsilon - \Delta, \varepsilon)$  respectively, that is  $\Phi_\varepsilon \in H^{-1,-2}(\Omega \times [\varepsilon - \Delta, \varepsilon))$  and  $\Psi_\varepsilon \in H^{-5/2} \Xi^{-5/2}(\Gamma \times [\varepsilon - \Delta, \varepsilon))$ .

We shall consider problem (3.37)–(3.41) subject to (3.1) for  $t \in (\varepsilon, T)$  and  $U_{\Sigma_{ad}} = L^2(\Sigma)$ . The performance functional is given by

$$I_\varepsilon(u) = \lambda_1 \|y(v) - z_d\|_{H^{-1,-2}(\Omega \times (\varepsilon, T))}^2 + \lambda_2 \int_{\varepsilon}^T \int_{\Gamma} (Nv)v \, dxdt \quad (3.42)$$

Then the problem (3.37)–(3.41) with  $\lambda_2 > 0$  has a unique optimal control in the form (3.36). Also it is easy to verify that (3.37)–(3.41) has a unique solution

$$\{y, p\} \in H^{-1,-2}(\Omega \times (\varepsilon, T)) \times \Xi^{3,3}(\Omega \times (\varepsilon, T)).$$

**Proposition 3.1.** *Let  $\{y, p\}$  be solution of (3.37)–(3.41) with  $\varepsilon = 0$ . We define  $\sigma_\varepsilon$ , the system "state" at time  $\varepsilon$ , by the triplet  $(y(\cdot, \varepsilon), \Phi_\varepsilon, \Psi_\varepsilon)$ , where*

$$\Phi_\varepsilon(\cdot, t') = \begin{cases} \Phi_0(\cdot, t') & \text{for } t' \in \hat{E}_\varepsilon = [-\Delta, 0) \cap [\varepsilon - \Delta, \varepsilon) \\ y(\cdot, t')|_\Omega & \text{for } t' \in [\varepsilon - \Delta, \varepsilon) - \hat{E}_\varepsilon \end{cases} \quad (3.43)$$

$$\Psi_\varepsilon(\cdot, t') = \begin{cases} \Psi_0(\cdot, t') & \text{for } t' \in \hat{E}_\varepsilon = [-\Delta, 0) \cap [\varepsilon - \Delta, \varepsilon) \\ y(\cdot, t')|_\Gamma & \text{for } t' \in [\varepsilon - \Delta, \varepsilon) - \hat{E}_\varepsilon \end{cases} \quad (3.44)$$

Then, for all triplets  $\varepsilon \leq t$  in  $(0, T)$ ,

$$p(\cdot, t) = P(t, \varepsilon)\sigma_\varepsilon + r_\varepsilon(\cdot, t) \quad (3.45)$$

where  $P(t, \varepsilon)$  and  $r_\varepsilon(\cdot, t)$  are determined by the following procedure:

First we solve the set of equations

$$\left\{ \begin{array}{l} \frac{\partial^2 \alpha}{\partial t^2} + A(t)\alpha + \sum_{i=1}^m \int_{a_i}^{b_i} \alpha(x, t - h_i) dh_i = 0 \quad \text{for } t - h_i \geq \varepsilon \\ \frac{\partial^2 \alpha}{\partial t^2} + A(t)\alpha + \sum_{i=1}^m \int_{a_i}^{b_i} \Phi_\varepsilon(x, t - h_i) dh_i = 0 \quad \text{for } t - h_i < \varepsilon \\ (x, t) \in \Omega \times (\varepsilon, T) \end{array} \right. \quad (3.46)$$

$$\left\{ \begin{array}{l} \frac{\partial^2 \beta}{\partial t^2} + A(t)\beta + \sum_{i=1}^m \int_{a_i}^{b_i} \beta(x, t + h_i) dh_i - \lambda_1 \Lambda_1 \alpha = 0 \\ \text{for } (x, t) \in \Omega \times (\varepsilon, T - \Delta) \\ \frac{\partial^2 \beta}{\partial t^2} + A(t)\beta + \sum_{i=1}^m \int_{a_i}^{T-t} \beta(x, t + h_i) dh_i - \lambda_1 \Lambda_1 \alpha = 0 \\ \text{for } (x, t) \in \Omega \times (T - \Delta, T - \lambda) \\ \frac{\partial^2 \beta}{\partial t^2} + A(t)\beta - \lambda_1 \Lambda_1 \alpha = 0 \quad \text{for } (x, t) \in \Omega \times (T - \lambda, T) \end{array} \right. \quad (3.47)$$

with boundary conditions

$$\frac{\partial \alpha}{\partial \eta_A}(x, t) = \begin{cases} \sum_{s=1}^l \int_{c_s}^{d_s} \alpha(x, t - k_s) dk_s - \lambda_2^{-1} N^{-1} G^* \beta & \text{for } t - k_s \geq \varepsilon \\ \sum_{s=1}^l \int_{c_s}^{d_s} \Psi_\varepsilon(x, t - k_s) dk_s - \lambda_2^{-1} N^{-1} G^* \beta & \text{for } t - k_s < \varepsilon \\ (x, t) \in \Gamma \times (\varepsilon, T) \end{cases} \quad (3.48)$$

$$\frac{\partial \beta}{\partial \eta_{A^*}}(x, t) = \begin{cases} \sum_{s=1}^l \int_{c_s}^{d_s} \beta(x, t + k_s) dk_s & \text{for } (x, t) \in \Gamma \times (\varepsilon, T - \Delta) \\ \sum_{s=1}^l \int_{c_s}^{T-t} \beta(x, t + k_s) dk_s & \text{for } (x, t) \in \Gamma \times (T - \Delta, T - \lambda) \\ 0 & \text{for } (x, t) \in \Gamma \times (T - \lambda, T) \end{cases} \quad (3.49)$$

and with the following conditions

$$\begin{cases} \alpha(x, \varepsilon) = y(x, \varepsilon) & x \in \Omega \\ \alpha'(x, \varepsilon) = y'(x, \varepsilon) & x \in \Omega \\ \beta(x, T) = 0 & x \in \Omega \\ \beta'(x, T) = 0 & x \in \Omega \end{cases} \quad (3.50)$$

then

$$P(t, \varepsilon) \sigma_\varepsilon = \beta(\cdot, t) \quad (3.51)$$

Next we solve the set of equations

$$\begin{cases} \frac{\partial^2 \kappa}{\partial t^2} + A(t) \kappa + \sum_{i=1}^m \int_{a_i}^{b_i} \kappa(x, t - h_i) dh_i = u(x, t) & \text{for } t - h_i \geq \varepsilon \\ \frac{\partial^2 \kappa}{\partial t^2} + A(t) \kappa = u(x, t) & \text{for } t - h_i < \varepsilon \\ (x, t) \in \Omega \times (\varepsilon, T) \end{cases} \quad (3.52)$$



$$\left\{ \begin{array}{l} \frac{\partial^2 \delta}{\partial t^2} + A(t)\delta + \sum_{i=1}^m \int_{a_i}^{b_i} \delta(x, t + h_i) dh_i - \lambda_1 \Lambda_1 \kappa = -\lambda_1 \Lambda_1 z_d \\ \text{for } (x, t) \in \Omega \times (\varepsilon, T - \Delta) \\ \frac{\partial^2 \delta}{\partial t^2} + A(t)\delta + \sum_{i=1}^m \int_{a_i}^{T-t} \delta(x, t + h_i) dh_i - \lambda_1 \Lambda_1 \kappa = -\lambda_1 \Lambda_1 z_d \\ \text{for } (x, t) \in \Omega \times (T - \Delta, T - \lambda) \\ \frac{\partial^2 \delta}{\partial t^2} + A(t)\delta - \lambda_1 \Lambda_1 \kappa = -\lambda_1 \Lambda_1 z_d \quad \text{for } (x, t) \in \Omega \times (T - \lambda, T) \end{array} \right. \quad (3.53)$$

with boundary conditions

$$\frac{\partial \kappa}{\partial \eta_A}(x, t) = \left\{ \begin{array}{l} \sum_{s=1}^l \int_{c_s}^{d_s} \kappa(x, t - k_s) dk_s - \lambda_2^{-1} N^{-1} G^* \delta \quad \text{for } t - k_s \geq \varepsilon \\ -\lambda_2^{-1} N^{-1} G^* \delta \quad \text{for } t - k_s < \varepsilon \\ (x, t) \in \Gamma \times (\varepsilon, T) \end{array} \right. \quad (3.54)$$

$$\frac{\partial \delta}{\partial \eta_A}(x, t) = \left\{ \begin{array}{l} \sum_{s=1}^l \int_{c_s}^{d_s} \delta(x, t + k_s) dk_s \quad \text{for } (x, t) \in \Gamma \times (\varepsilon, T - \Delta) \\ \sum_{s=1}^l \int_{c_s}^{T-t} \delta(x, t + k_s) dk_s \quad \text{for } (x, t) \in \Gamma \times (T - \Delta, T - \lambda) \\ 0 \quad \text{for } (x, t) \in \Gamma \times (T - \lambda, T) \end{array} \right. \quad (3.55)$$

and with the following conditions

$$\left\{ \begin{array}{l} \kappa(x, \varepsilon) = y(x, \varepsilon) \quad x \in \Omega \\ \kappa'(x, \varepsilon) = y'(x, \varepsilon) \quad x \in \Omega \\ \delta(x, T) = 0 \quad x \in \Omega \\ \delta'(x, T) = 0 \quad x \in \Omega \end{array} \right. \quad (3.56)$$

then

$$r_\varepsilon(x, t) = \delta(x, t) \quad (3.57)$$

Consequently, from Theorem 3 of Kowalewski (1998) and Lemma 3.1  $\beta \in X^1(Q) \subset H^{3,3}(Q)$  and  $\delta \in X^1(Q) \subset H^{3,3}(Q)$  imply that  $\beta \rightarrow \beta|_\Sigma$  and  $\delta \rightarrow \delta|_\Sigma$  are linear continuous mappings of  $H^{3,3}(Q) \rightarrow H^{5/2,5/2}(\Sigma)$  respectively.

Setting  $\varepsilon = t$  in (3.45) and substituting the result into (3.36) we obtain

$$v_0(\cdot, t) = -\lambda_2^{-1} N^{-1} G^* (P(t, t) \sigma_t + r_t(\cdot, t)) \Big|_\Gamma, \quad x \in \Gamma, t \in (0, T) \quad (3.58)$$

Let us assume that  $N$  is the identity operator on  $L^2(\Sigma)$ . Then, using of Schwartz's Kernel Theorem [Schwartz (1950)], it can be proved that the optimal feedback control (3.58) can be expressed in the following form

$$\begin{aligned} v_0(x, t) = & -\lambda_2^{-1} G^* \left\{ \int_{\Omega} K_0(x, x', t) y(x', t) dx' + \right. \\ & + \int_{t-\Delta}^t \int_{\Omega} K_1(x, x', t, t') \Phi_t(x', t') dx' dt' + \\ & \left. + \int_{t-\Delta}^t \int_{\Gamma} K_2(x, x', t, t') \Psi_t(x', t') d\Gamma dt' + r_t(x, t) \right\} \quad x \in \Gamma, t \in (0, T) \end{aligned} \quad (3.59)$$

where  $\{K_0, K_1, K_2\}$  is the kernel of  $P(t, t)$ .

The explicit expressions for the kernels of the optimal feedback controls such as (3.59) are generally quite complex. Consequently, this motivates the consideration of suboptimal feedback controls with prescribed kernels having simple forms.

We can also consider analogous optimal feedback control problem with the performance functional (3.26) strictly defined at the boundary  $\Gamma$ .

## 4. Final remarks

The results presented in the paper can be treated as a generalization of the results obtained by Wang (1975) and Kowalewski (1993a, b, 2000) onto the case of distributed hyperbolic systems with different multiple time delays appearing in the integral form both in the state equations and in the Neumann boundary conditions.

We have considered a new type of time delays, namely different multiple time delays given in the integral form both in the state equation and in the boundary condition.

Sufficient conditions for the existence of a unique solution of such hyperbolic equations with the Neumann boundary conditions were presented – Theorems 2.1 and 3.1.

The optimal distributed and boundary controls were characterized by the adjoint equations – Lemmas 2.1, 3.1 and 3.2. By using this characterization, necessary and sufficient conditions of optimality were proved for linear quadratic problems with the Neumann boundary conditions – Theorems 2.3, 3.2 and 3.3.

The optimal control was obtained in the feedback form – the formula (3.36) and Proposition 3.1. Making use of Schwartz's Kernel Theorem (Schwartz 1950), the representation of the optimal feedback control was given – the formula (3.59).

In this paper we have considered optimal control problems for time delay hyperbolic systems with the Neumann boundary conditions.

We can also consider analogous optimization problems for such systems with the non-homogeneous Dirichlet boundary conditions.

Consequently, we can consider a more complex case of distributed hyperbolic systems with different multiple time delays given in the integral form both in the state equations and in the boundary conditions such that  $h_i \in (0, b_i)$  for  $i = 1, \dots, m$  and  $k_s \in (0, d_s)$  for  $s = 1, \dots, l$ .

The ideas mentioned above will be developed in forthcoming papers.

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