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The Resultants and Discriminants of Real Polynomials and Quasipolynomials

1. Introduction

We shall start with the notion of the resultant

Theorem l

Let $M(s)$ and $L(s)$ be polynomials with real coefficients

$$\left.\begin{array}{ll} M(s) = a_0 s^n + a_1 s^{n-1} + \ldots + a_n & (a_0 \neq 0) \\ L(s) = b_0 s^n + b_1 s^{n-1} + \ldots + b_n & (b_0 \neq 0) \end{array}\right\} \tag{1}$$

The determinant (2) of order $m + n$ is said to be the Sylvester determinant or resultant.

$$R(M, L) = \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} & a_n & 0 & \cdots & 0 & 0 \\ 0 & a_0 & \cdots & a_{n-2} & a_{n-1} & a_n & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & & & & \cdots & \vdots & \\ 0 & 0 & \cdots & a_0 & a_1 & a_2 & \cdots & a_{n-1} & a_n \\ b_0 & b_1 & \cdots & b_{m-1} & b_m & 0 & \cdots & 0 & 0 \\ 0 & b_0 & \cdots & b_{m-2} & b_{m-1} & b_m & \cdots & 0 & 0 \\ & \vdots & \cdots & & & & \cdots & & \\ 0 & 0 & \cdots & b_0 & b_1 & b_2 & \cdots & b_{m-1} & b_m \end{vmatrix} \begin{array}{l} \left.\vphantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right\} m \text{ rows} \\ \left.\vphantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right\} n \text{ rows} \end{array} \tag{2}$$

Relationships between the zeroing of the resultant and the existence of common zeros of the polynomials M and L are the most important issues considered in the paper.

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2. Discriminants of the polynomials

A special case of the resultant is the discriminant.
Consider the polynomial

$$M(s) = a_0 s^n + a_1 s^{n-1} + ... + a_n \quad (a_0 \neq 0) \tag{3}$$

And its derivative with respect to the variable „s"

$$\frac{dM(s)}{ds}(s) = n a_0 s^{n-1} + (n-1) a_1 s^{n-2} + ... + a_{n-1} \tag{4}$$

The discriminant of these polynomials for $n > 2$ is:

$$D[M(s)] = (-1)^{\frac{n(n-1)}{2}} \frac{1}{a_0} R\left[M(s), \frac{dM(s)}{ds}(s) \right] \tag{5}$$

We take also into account that polynomial (3) can be written in the form

$$M(s) = a_0 (s - s_1)(s - s_2) \, ... \, (s - s_n) \tag{6}$$

where s_i – are the roots of $M(s)$.

Using (3), (4) and (5) we can write

$$D(M) = (-1)^{\frac{n(n-1)}{2}} \left| \begin{array}{cccccccc} 1 & a_1 & \cdots & a_n & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_n & \cdots & 0 \\ \hdashline & & a_0 & & \cdots & & a_n \\ n & (n-1) & a_1 & \cdots & a_{n-1} & \cdots & 0 \\ \hdashline & & & & n a_0 & \cdots & a_{n-1} \end{array} \right| \begin{array}{l} \left.\right\} n-1 \\ \\ \\ \left.\right\} n \end{array} \tag{7}$$

Example

Let be $M(s) = a_0 s^3 + a_1 s^2 + a_2 s + a_3$.
Then

$$D(M) = (-1) \begin{vmatrix} 1 & a_1 & a_2 & a_3 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 \\ 3 & 2a_1 & a_2 & 0 & 0 \\ 0 & 3a_0 & 2a_1 & a_2 & 0 \\ 0 & 0 & 3a_0 & 2a_1 & a_2 \end{vmatrix} = a_1^2 a_2^2 - 4 a_1 a_3^3 - 4 a_0 a_2^3 + 18 a_0 a_1 a_2 a_3 - 27 a_0^2 a_3^2 .$$

For the product of the discriminants we have a formula [1]

$$D[M, L] = D[M]\, D[L]\, [R(M, L)]^2 \tag{8}$$

The following generalization of (8) is possible

$$D\left[\prod_{k-1}^{n} M_k\right] = \prod_{k-1}^{n} D(M_k)\cdot \prod_{i>j}^{1,n}\left[R(M_i M_j)\right]^2 \tag{9}$$

If we put $M_k(s) = s - s_k$,

then $D[M_k] = 1$ and $R[M_i, M_j] = \begin{vmatrix} 1 & -s_i \\ 1 & -s_j \end{vmatrix} = s_i - s_j$.

Using (9) we obtain that

$$D\left[\prod_{k-1}^{n} (s - s_k)\right] = \prod_{i>j}^{1,n} (s_i - s_j)^2 \tag{10}$$

Taking into account the relation (6) we have

$$D = \prod_{i>j}^{1,n} (s_i - s_j)^2 = (-1)^{\frac{n(n-1)}{2}} \left.\begin{vmatrix} 1 & a_1 & \cdots & a_n & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_n & \cdots & 0 \\ \hdashline & & a_0 & & \cdots & & a_n \\ n & (n-1)a_1 & & \cdots & a_{n-1} & \cdots & 0 \\ \hdashline & & & & na_0 & \cdots & a_{n-1} \end{vmatrix}\right. \begin{matrix} \left.\right\} n-1 \\ \left.\right\} n \end{matrix} \tag{11}$$

We know also that Vandermond's determinant is equal

$$V = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ s_1 & s_2 & \cdots & s_n \\ s_1^2 & s_2^2 & \cdots & s_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ s_1^{n-1} & s_2^{n-1} & \cdots & s_n^{n-1} \end{vmatrix} = \prod_{i>j}^{1,n} (s_i - s_j) \quad \begin{matrix} \left.\right\} n-1 \\ \left.\right\} n \end{matrix} \tag{12}$$

From (11) and (12) we obtain the important relation

$$D = V^2 \tag{13}$$

In control theory we meet also other resultants [2].

Let us consider the characteristic equation

$$M(s) = 0 \tag{14}$$

We put $s = \alpha + j\omega$ in (14) after reordering we obtain two equations for the real and imaginary parts of the equation (14)

$$\left.\begin{aligned} P(\omega) &= \operatorname{Re} M(\alpha + j\omega) \\ Q(\omega) &= \operatorname{Im} M(\alpha + j\omega) \end{aligned}\right\} \tag{15}$$

where the coefficients of the polynomials $P(\omega)$ and $Q(\omega)$ depend on „α"

$$P(\omega) = \sum_{0 \le 2i \le n} (-1)^i A_{2i}(\alpha)\omega^{2i} \tag{16}$$

$$Q(\omega) = \sum_{1 \le 2i+1 \le n} (-1)^i A_{2i}(\alpha)\omega^{2i+1} \tag{17}$$

We obtain

Theorem 2

If

$$M(s) = a_0 \prod_{i=0}^{n} (s - s_i) \quad a_0 \neq 0,$$

then

$$R[P, Q] = a_0^{2n} (-1) \frac{n(n+1)}{2} 2^{n(n-1)} \prod_{i,k=1}^{n} \left(\alpha - \frac{s_i + s_k}{2} \right) \tag{18}$$

It is possible [2] to obtain simpler formulae

$$R[P, Q] = a_0^2 M(s) R_1^2(\alpha) \tag{19}$$

where

$$R_1^2(\alpha) = \left[a_0^{n-1} 2^{\frac{n(n-1)}{2}} \prod_{\substack{i,k=1 \\ i<k}}^{n} (\alpha - \alpha_{lk}) \right]^2 \tag{20}$$

and $\alpha_{ik} = \dfrac{s_i + s_k}{2}$.

It is worth noting that the free term of the polynomial $R_1(\alpha)$ is identical with the Hurwitz determinant Δ_{n-1}. From this we conclude that

$$\Delta_{n-1} = a_0^{n-1}(-1)^{\frac{n(n-1)}{2}} \prod_{\substack{i,k=1 \\ i<k}}^{n} (s_i + s_k) \tag{21}$$

which represents the well known Orlando formula.

Following this way with polynomials $U(\alpha)$ and $V(\alpha)$ having coefficients dependent on „ω", where

$$\left.\begin{aligned} U(\alpha) &= \operatorname{Re} M(j\omega+\alpha) \\ V(\alpha) &= \operatorname{Im} M(j\omega+\alpha) \end{aligned}\right\} \tag{22}$$

we have

$$U(\alpha) = \frac{1}{2}\Big[\big(B_0(\omega)+C_0(\omega)\big)+\big(B_1(\omega)+C_1(\omega)\alpha\big)+\ldots+\big(B_n(\omega)+C_n(\omega)\big)\alpha^n\Big] \tag{23}$$

$$V(\alpha) = \frac{1}{2j}\Big[\big(B_0(\omega)+C_0(\omega)\big)+\big(B_1(\omega)-C_1(\omega)\alpha\big)+\ldots+\big(B_n(\omega)-C_n(\omega)\big)\alpha^n\Big] \tag{24}$$

where

$$\left.\begin{aligned} B_k(\omega) &= \frac{1}{k!}M^{(k)}_{(j\omega)} \\ C(\omega) &= \frac{1}{k!}M^{(k)}_{(j\omega)} \end{aligned}\right\} \quad k = 0, 1, \ldots, n \tag{25}$$

represent (k)-th derivatives with respect to (ω).

We can state

Theorem 3

If $M(s) = a_0 \prod_{i=0}^{n} (s - s_i)$,

then the resultant

$$R[U,V] = a_0^{2n}(-4)^{\frac{n(n-1)}{2}} \prod_{i,k=1}^{n} \left(\omega - \frac{s_i - s_k}{2j}\right) \tag{26}$$

It is also possible to obtain a simpler formula because

$$R[U,V] = a_0(-1)^{\frac{n(n-1)}{2}}\omega^n R_2(\beta)\Big|_{\beta=\omega^2} \tag{27}$$

where

$$R_2[\beta] = a_0^{2n-1} 4^{\frac{n(n-1)}{2}} \prod_{\substack{i,k=1 \\ i<k}}^{n} (\beta - \beta_{i\omega}) \tag{28}$$

and $\beta_{ik} = -\left[\frac{s_i - s_k}{2}\right]^2$.

Finally we have

$$R_2[U,V] = a_0^{2n}(-4)^{\frac{n(n-1)}{2}}\omega^n \prod_{\substack{i,k=1 \\ i<k}}^{n} \left[\omega^2 + \left(\frac{s_i - s_k}{2}\right)^2\right] \tag{29}$$

The free term $R_2(\beta)$ is identical to Meerov determinant Γ_{2n-1} which is given in the criterion of aperiodic stability.

On the basis of the theorem 3 we obtain an analogue of the Orlando formula

$$\Gamma_{2n-1} = a_0^{2n} \prod_{\substack{i,k=1 \\ i<k}}^{n} \left(\frac{s_i - s_k}{2}\right)^2 \tag{30}$$

which for $a_0 = 1$ is also connected with the well known Vandermonde determinant.

$$V_n^2 = \Gamma_{2n-1} = \begin{vmatrix} 1 & a_1 & \cdots & a_n, 0 & \cdots & 0 \\ \hdashline 0 & & \cdots & a_1 & \cdots & a_n \\ n(n-1)a_1 & & \cdots & a_{n-1} & \cdots & 0 \\ \hdashline 0 & & \cdots & 0\; n(n-1)a_1 & \cdots & a_{n-1} \end{vmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\0\end{matrix}}\right\} n-1 \\ \left.\vphantom{\begin{matrix}1\\0\end{matrix}}\right\} n \end{matrix} \tag{31}$$

We conclude that the obtained resultants allow us to unify the study of different kinds of stability of linear stationary dynamie systems. In particular the determinant Δ_{n-1} determined by (21) represents the limit of the oscillatory stability and the determinant Γ_{2n-1} (formula (30)) the limit of the aperiodic stability.

3. Resultansts of quasipolynomials[1)]

Let us consider the differential equation which describes a linear, stationary dynamic system with constant parameters $a_0, ..., a_n$

$$a_0 \frac{d^n x(t)}{dt^n} + a_1 \frac{d^{n-1} x(t)}{dt^{n-1}} + ... + a_{n-1} \frac{dx(t)}{dt} + a_n x(t) = 0 \qquad (32)$$

We assume that not all initial conditions are equal to zero, ie

$$x^{(i-1)}(0) = c_i \neq 0 \quad \text{for some} \quad i = 1, 2, ..., n.$$

The solution of equation (32) takes the form

$$x(t) = \sum_{k=1}^{n} A_k e^{s_k t} \qquad (33)$$

where s_k are different roots of the characteristic equation

$$a_0 s^n + a_1 s^{n-1} + ... + a_{n-1} s + a_n = 0 \qquad (34)$$

We will also use the derivatives of $x(t)$

$$x^{(p)}(t) = \sum_{k \neq 1}^{n} s_k^n A_k e^{s_k t} \quad p = 1, 2, ..., n-1 \qquad (35)$$

It is not possible to use discriminants directly, because we cannot eliminate „n" exponential terms from two equations (33) and (35) for $p = 1$.

The additional equations we obtain by an appropriate change of only one initial condition from c_i to c_i^*.

We adjust the value c_1^* in such a way that both $x(t_0)$ and $x^*(t_0)$ are equal to zero. All the values of other derivatives are the same.

Remark

In the case of the triple roots we can use de L'Hospital theorem.

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1) quasipolynomials it is the expression of the form $a_0 s^{\alpha_n} + a_1 s^{\alpha_{n-1}} + ... + a_{n-1} s^{\alpha_1} + a_n$ where $\{\alpha_r\}$ denote real or complex numbers

Theorem 4

It is proved in [4] that the resultant of the equations (33) and (35) for $p = 1, 2, ..., n-1$ is equal to

$$R = \frac{\left(\prod_{i=1}^{n} s_i\right)^{n-2}}{\prod_{\substack{i,j=1\\ i<j}} (s_i - s_j)} \left[(c_1 - c_1^*)\right]^{n-2} \left(c_2^2 - c_1^* c_3\right) \quad n \geq 2 \tag{36}$$

or in an equivalent form

$$R = \frac{a_n^{n-2}}{V}\left[(c_1 - c_1^*)^{n-2}\left(c_2^2 - c_1^* c_3\right)\right] \tag{37}$$

where V is Vandermonde's determinant whose square is equal according to (31), the determinant responsible for aperiodicity.

The resultant is equal to zero if

$$c_1^* = \frac{c_2^2}{c_3} \tag{38}$$

Example

Consider the differential equation of the 3rd order

$$\frac{d^3 x}{dt^3} + a_1 \frac{d^2 x}{dt^2} + \frac{dx}{dt} + a_3 x = 0$$

with initial conditions:

$$x(0) = c_1,$$
$$\dot{x}(0) = c_2,$$
$$\ddot{x}(0) = c_3.$$

The solution of this equation is $x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + A_3 e^{s_3 t}$ or in detail

$$x(t) = \frac{c_3 - (s_2 + s_3)c_2 + s_2 s_3 c_1}{(s_1 - s_2)(s_1 - s_3)} e^{s_1 t} + \frac{c_3 - (s_1 + s_3)c_2 + s_1 s_3 c_1}{(s_2 - s_1)(s_2 - s_3)} e^{s_2 t} +$$
$$+ \frac{c_3 - (s_1 + s_2)c_2 + s_1 s_2 c_1}{(s_3 - s_1)(s_3 - s_2)} e^{s_3 t}$$

where $s_i \neq s_i$ are the roots of the characteristic equation

$$s_3 + a_1 s^2 + a_2 s^{+} a_3 = 0.$$

We can write that for the extremum at the point t_e it is required

$$A_1^* e^{s_1 te} + A_2^* e^{s_2 te} + A_3^* e^{s_3 te} = 0,$$

$$s_1 A_1^* e^{s_1 te} + s_2 A_2^* e^{s_2 te} + s_3 A_3^* e^{s_3 te} = 0,$$

$$s_1 A_1 e^{s_1 te} + s_2 A_2 e^{s_2 te} + s_3 A_3 e^{s_3 te} = 0.$$

Where A_1^*, A_2^*, A_3^* denote the coefficients with the new condition $x^*(0) = c_1^*$, and others remain the same.

The resultant of this equation is

$$R = \begin{vmatrix} A_1^* & A_2^* & A_3^* \\ s_1 A_1^* & s_2 A_2^* & s_3 A_3^* \\ s_1 A_1 & s_2 A_2 & s_3 A_3 \end{vmatrix} = 0.$$

After calculation we obtain

$$R = \frac{s_1 s_2 s_3}{(s_1 - s_2)(s_2 - s_3)(s_3 - s_1)} \left[\left(c_1^*\right)^2 c_3 - c_1^* \left(c_2^2 + c_1 c_3\right) + c_1 c_2^2 \right] = 0.$$

From this we obtain that $c_1^* = \dfrac{c_2^*}{c_3}$ or $c_1^* = c_1$.

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