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State Estimation in Linear Multi-Output Systems – Design Example and Discussion of Optimality**

1. Introduction

Most of literature positions on the subject of linear observer design, are focused on systems with single output. Typical approach is to determine whether the system is observable in respect to the output considered and then to design a Luenberger observer, based on the following classical result

Theorem 1 (see [7], p. 146)
For linear time invariant system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

where A, B and C are matrices of appropriate dimensions, one can design a full-rank Luenberger observer

$$\dot{\hat{x}}(t) = (A - GC)\hat{x} + Bu(t) + Gy(t) \tag{2}$$

Where dynamics of estimation error $e(t) = x(t) - \hat{x}(t)$ is described by differential equation

$$\dot{e}(t) = (A - GC)e(t) \tag{3}$$

If $(C; A)$ is observable, one can choose such matrix gain G that characteristic polynomial of matrix $(A–GC)$ has any desired real coefficients.

Theorem 1 presents a design tool allowing construction of observers by pole placement. Pole placement is along with Kalman filter (see [5]) a fundamental method of observer design and is important part of this paper. Typical approach with such designs is to set

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the eigenvalues of matrix ($A–GC$) negative, real and with moduli greater than maximum of moduli of eigenvalues of matrix A (see for example [1]). Reasoning of such design is quite obvious, negativity is required for exponential convergence of estimation error to zero, realness is required for lack of oscillatory behaviour and modulus requirement is needed for assuring, that the dynamics of observer is 'faster' than the one of observed system. Moreover, one usually requires the eigenvalues of matrix ($A–GC$) to be all equal.

Pole placement design, however does not distinguish the systems observable with respect to more than one output. The goal of this paper is to show what are the benefits of such property of the system. Especially it will be shown, that for such systems, one can design the observer that joins the pole placement with additional optimality with respect to chosen performance index. Two of such performance indexes will be presented as well with design examples.

2. Considered example

In this paper the following example of multi-output system will be considered.

Let's consider the RC ladder network as depicted in Figure 1. Such system was considered in many papers [7, 8] and it's properties are fairly well known. To focus on properties important in this paper, let $n = 3$ and $RC = 1$.

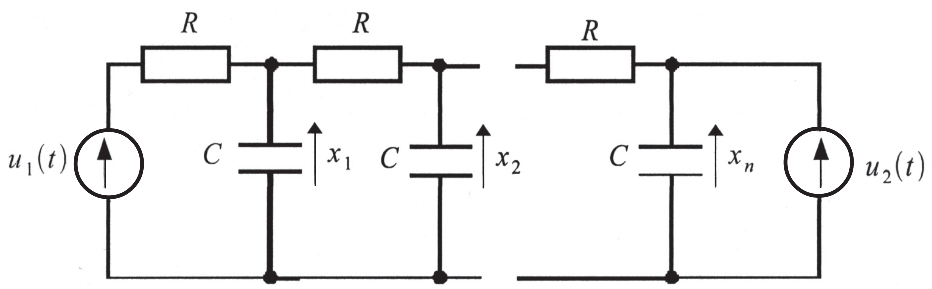

Fig. 1. The RC ladder network

In that case, dynamics of voltages x_i at resistances can be described by differential equation (1) where

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \tag{4}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{5}$$

It can be shown (see [7]), that eigenvalues of matrix A are equal

$$\lambda_i = -2\left(1 - \cos\left(\frac{i\pi}{4}\right)\right) \tag{6}$$

for $i = 1, 2, 3$.

Remark 1

It can be easily shown that RC ladder system for $n = 3$ is observable in respect to and $x_n = x_3$ both separately and at the same time. Observability of the system for separate outputs x_1 and x_3 is equivalent to observability of pairs $(C_1; A)$ and $(C_2; A)$ where A is defined by equality (4) and C_1 and C_2 are as follows:

$$\begin{aligned} C_1 &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \\ C_2 &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{7}$$

The observability can be verified by applying well known Kalman algebraic criterion (see for example [3]). For observability with respect to both of x_1 and x_3 one needs to show that pair $(C; A)$ is observable, where $C = [C_1\ C_2]^T$. This property can be proven for any n.

3. Single output designs

Following, there will be presented observer designs while using only one of the outputs. Both observers will be designed by pole placement, setting eigenvalues of $(A–GC)$ all equal –4 with multiplicity of 3 (observability of system, allows setting of the eigenvalues to any values available for roots of a polynomial with real coefficients).

3.1. Observer using x_1

For this design, the following system is considered:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= C_1 x(t) \end{aligned} \tag{8}$$

Where A, B and C_1 are given by respectably (3), (4) and (7). The task is to compute matrix gain G_1 in such way, that eigenvalues of matrix $(A–G_1C_1)$ were equal –4 (with multiplicity of 3). Because $G_1 = [g_1\ g_2\ g_3]^T$ characteristic polynomial of $(A–G_1C_1)$ takes form

$$\lambda^3 + (6 + g_1)\lambda^2 + (10 + 4g_1 + g_2)\lambda + 4 + 3g_1 + 2g_2 + g_3 \tag{9}$$

By expanding $(\lambda+4)^3$ desired coefficients of characteristic polynomial are obtained, which leads to set of equations

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} G_1 = \begin{bmatrix} 6 \\ 38 \\ 60 \end{bmatrix} \tag{10}$$

With solution

$$G_1 = \begin{bmatrix} 6 \\ 14 \\ 14 \end{bmatrix} \tag{10a}$$

3.2. Observer using x_3

Analogous procedure can be performed for output x_3, in that case considered system takes form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= C_2 x(t) \end{aligned} \tag{11}$$

Where A, B and C_2 are given by respectably (3), (4) and (7). Pole placement leads to equations

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} G_2 = \begin{bmatrix} 60 \\ 38 \\ 6 \end{bmatrix} \tag{12}$$

With solution

$$G_2 = \begin{bmatrix} 14 \\ 14 \\ 6 \end{bmatrix} \tag{13}$$

4. Two output design

In comparison to classical single output design, the one using both outputs (x_1 and x_3) opens many possibilities. Considering pole placement design, with the goal of setting eigenvalues of matrix (A–GC) to –4 with multiplicity of 3, where

$$G = \begin{bmatrix} k_1 & k_4 \\ k_2 & k_5 \\ k_3 & k_6 \end{bmatrix} \tag{14}$$

characteristic polynomial leads to following three equations:

$$
\begin{aligned}
&6 + k_1 + k_6 = 12 \\
&4k_1 + k_2 - k_3k_4 + k_5 + 4k_6 + k_1k_6 = 48 \\
&4 + 3k_1 + 2k_2 + k_3 + 3k_6 + 2k_5 + k_4 + 2k_1k_6 + \\
&+k_1k_5 + k_2k_6 - k_2k_4 - 2k_3k_4 - k_3k_5 = 64
\end{aligned}
\tag{15}
$$

Equations (15) are nonlinear and depend on 6 variables. There is an infinite number of solutions (among them are for example $G = [G_1\ 0]$ and $G = [0\ G_2]$). However, because all those solutions are satisfying requirement of pole placement, one can attempt to find an optimal one with respect to some performance index. In that case equations (15) can be used as equality constraints. In the following sections two performance indexes are proposed. Each of them is applied in the observer design for considered system.

5. Minimal disturbance augmentation

One can see that matrix gains (11) and (13) have considerably large coefficients. In the presence of disturbances, large gains increase the risk of large disturbance impact on state estimation. One can consider the following performance index to be minimized

$$
J(G) = \sum_{i=1}^{6} k_i^2 \tag{16}
$$

The interpretation of (16) is quite obvious. The goal is to find the set of coefficients of matrix G with minimal square values. Minimization with respect to equality constraints (15) can be interpreted as searching for minimal gains, providing that eigenvalues of $(A–GC)$ are all equal –4 (multiplicity 3). In other words, there are desired dynamical properties of the observer, and optimization of (16) is an attempt to find matrix gain G which can assure those properties having minimal coefficients.

Theoretically minimization of (16) is possible analytically *via* Lagrange multipliers method (see [4]), however it requires to analytically solve 9 nonlinear algebraic equations, which is rather difficult task. On the other hand the Optimization toolbox of Matlab has an implementation of sequential quadratic programming (SQP) method for constrained optimization which can be used for solving this problem.

The main obstacle for numerical optimization of multi variable problems is that there are no guarantees that the obtained minimizer is a global one. Moreover difficulties in choosing initial conditions, which should fulfill the constraints (15), don't allow multiple experiments needed to increase the probability of finding the global minimizer. In spite of these inconveniences optimization for the considered system was performed, and the best results were achieved for initial conditions being solutions of single output design tasks (11) and (13). What is interesting, probably due to symmetric nature of matrix A, the solution was not unique. Two symmetrical sets of gains were obtained with the same value of performance index. These solutions were

$$G^{*}=\begin{bmatrix} 3.0860 & 0.6775 \\ 1.7139 & 2.5086 \\ -1.1585 & 2.9140 \end{bmatrix} \tag{17}$$

and

$$G^{**}=\begin{bmatrix} 2.9140 & -1.1585 \\ 2.5086 & 1.7139 \\ 0.6775 & 3.0860 \end{bmatrix} \tag{18}$$

For both of them the performance index value was

$$J_1(G^{*})=J_1(G^{**})=29.0466.$$

Also there should be mentioned that the SQP based method wasn't the only one tried (experiments were performed also among the others with genetic algorithms) but was the one that provided the best results.

6. Minimal dynamical error

Different optimization approach regarded the problem of dynamical error. One of the possible approach to minimize it is by minimization of the following functional

$$J(G)=\int_0^{\infty} e^2(t)dt \tag{19}$$

Which is the integral of the square of the estimation error. Because of the specifics of the system, most important is to minimize the estimation error of the second state variable x_2. Performance index (19) is one of the ways to measure the properties of dynamical systems, and actually is a norm in some important function spaces. However, there is a question of computing such performance index. One can apply the following theorem.

Theorem 2 (see [7])

The system of the form

$$\dot{x}=Fx \tag{20}$$

is considered. Let $\lambda(F)$ be in left open half-plane. Then there exist matrix $H = H^{\mathrm{T}} \geq 0$ that is the solution of Lyapunov equation

$$F^T H + HF = -D \tag{21}$$

and

$$x(0)^T Hx(0) = \int_0^{\infty} x(t)^T Dx(t)dt \tag{22}$$

where $D = D^{\mathrm{T}} \geq 0$.

Setting

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{23}$$

and $F = (A–GC)$ one can compute dynamical error of estimation of second state variable using equality (22).

The main obstacle in this approach, and in general in performance index (19), is that it is dependent on the initial error of estimation which is usually unavailable. However, it is known from the theory of quadratic forms (see [7]) that

$$x(0)^T Hx(0) \leq \| x(0) \|^2 \lambda_{\max}(H) \tag{24}$$

Using this bound on the dynamical error, one can formulate the following performance index to be minimized with respect to equality constraints (15)

$$J_2(G) = \lambda_{\max}(H) \tag{25}$$

Where H is a solution of Lyapunov equation (21) with $F = (A–GC)$ and D given by equality (23). Value of this performance index can be computed by a suitable function of Matlab Control System toolbox. That function may be applied to solve Lyapunov equation in every optimization step. That approach in computing (25) along with the implementation of SQP method from Optimization toolbox was utilized to solve the optimization problem. The following solution was obtained

$$G^{***} = \begin{bmatrix} 2.8976 & 0.5863 \\ 2.7712 & 2.7531 \\ 0.8762 & 3.1024 \end{bmatrix} \tag{26}$$

with performance index value

$$J_2(G^{***}) = 0.2255.$$

Again, there are no guarantees that the solution (26) is global or unique, however, after many tests it was the one with the best value of performance index (25).

7. Simulations

Now, to illustrate the effects of presented designs, simulational results will be shown. Simulations were performed in following conditions:

- Differential equation (1) was being solved by Runge–Kutta 4th order method with discretization step 0.01 s.
- The disturbances were modelled with uniformly distributed random numbers with amplitude 0.1 which were used as noise of output measurements.
- Observer had zero initial conditions and the system initial conditions were chosen relatively large to make results more visible.
- Applied control signal $u(t)$ is depicted on Figure 2.

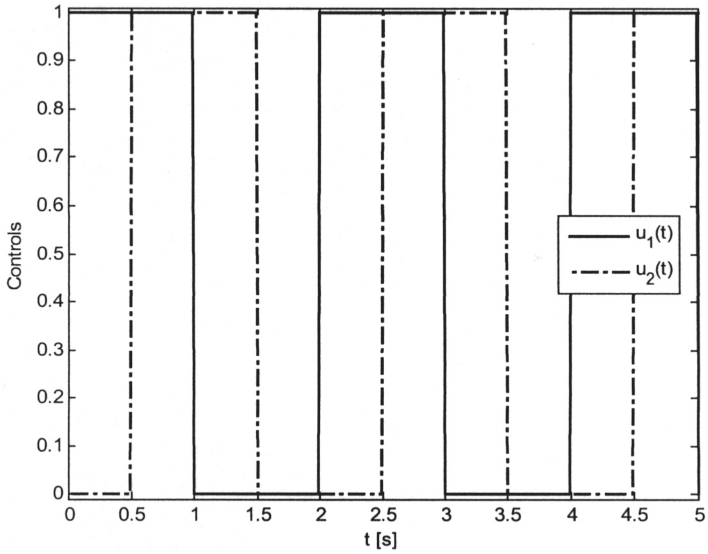

Fig. 2. Applied control signal

7.1. Single output designs

Both of single output designs have produced very similar results. Difference between them originates from differences in disturbances of output measurements. Plots of x_2 and its estimate from either observer are presented in Figures 3 and 5, while corresponding to them error plots are respectably in Figures 4 and 6. One can see that both of these observers are producing estimate, that has rather large overshoot. Also the influence of measurement noise is visible during steady state estimation (after third second of simulation time).

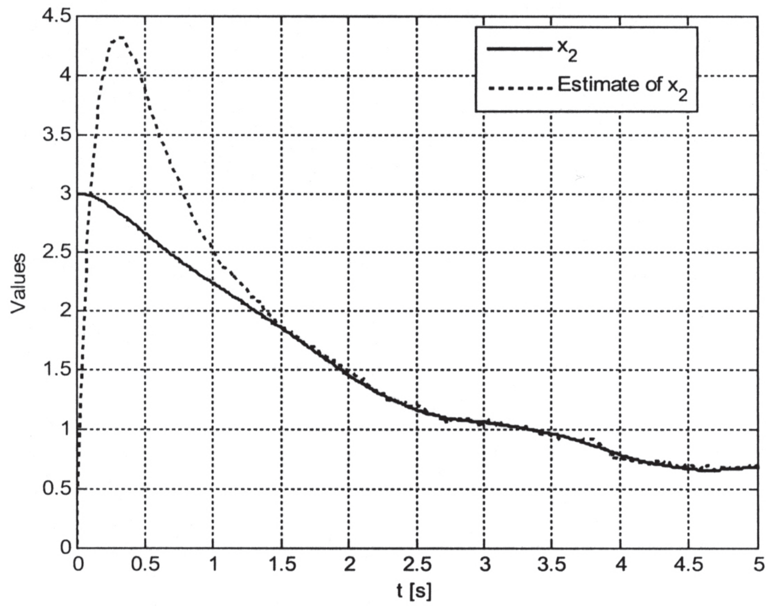

Fig. 3. Second state variable and its estimate produced by observer using output x_1

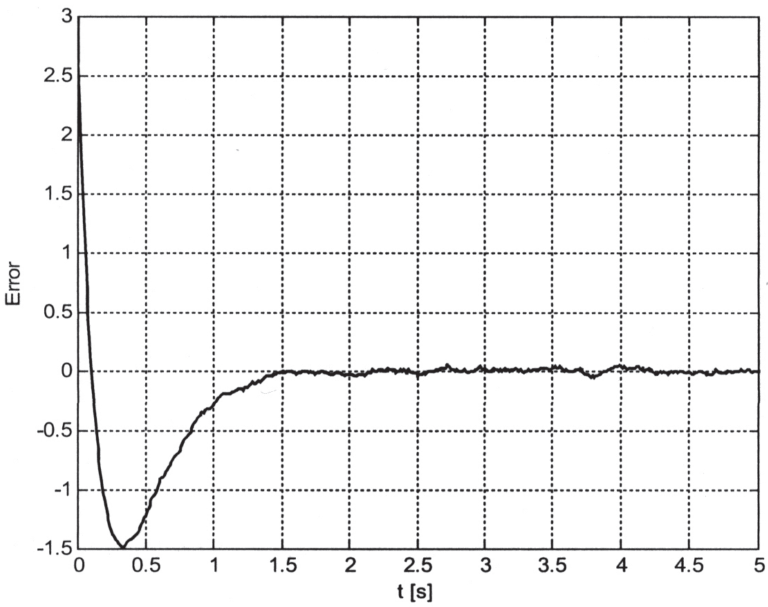

Fig. 4. Estimation error of second state variable of observer using output x_1

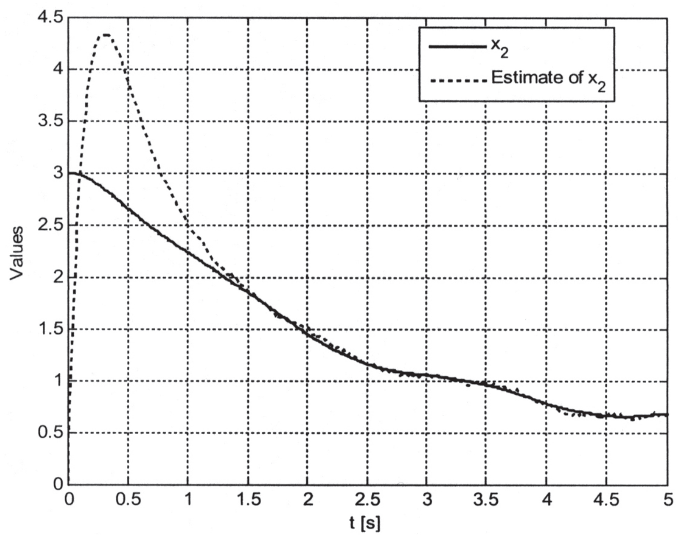

Fig. 5. Second state variable and its estimate produced by observer using output x_3

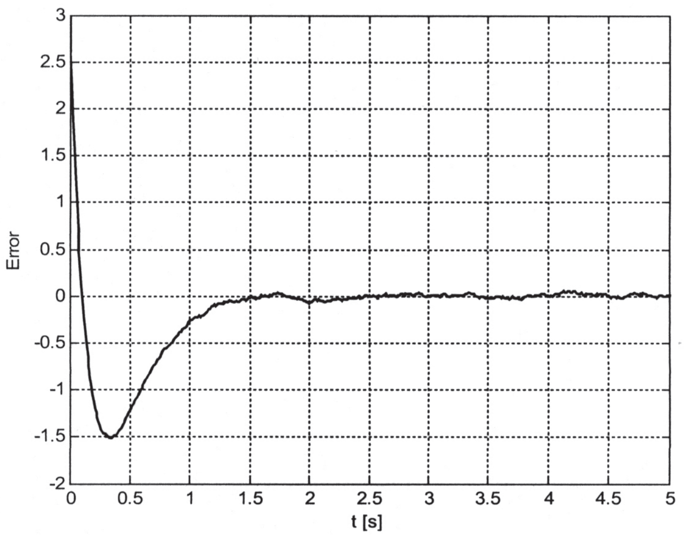

Fig. 6. Estimation error of second state variable of observer using output x_3

7.2. Minimal disturbance augmentation design

This design in experiments has produced the best results. In Figure 7, there is a plot of x_2 and its estimate one can see that, there are no overshoot and also the disturbances are not visible in the steady state.

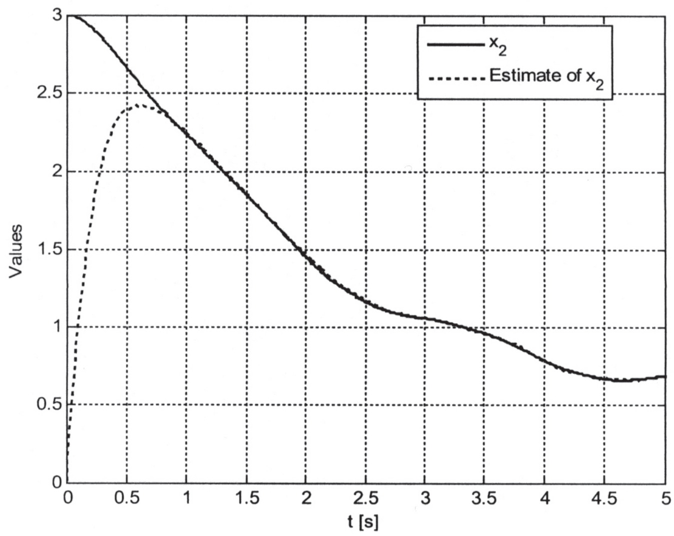

Fig. 7. Second state variable and its estimate produced by observer using both outputs with gain given by (17)

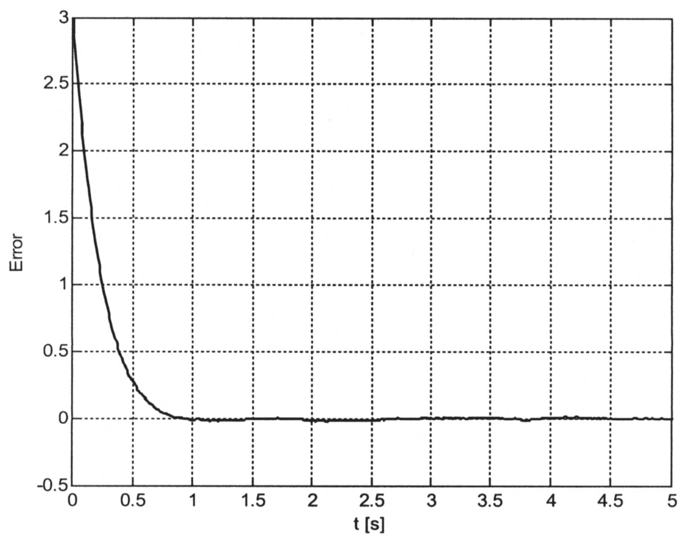

Fig. 8. Estimation error of second state variable of observer using both outputs with gain given by (17)

Moreover error plot in Figure 8 supports this observation, showing convergence of error to zero. Also it is worthwhile to note that this convergence is faster than for single output design. Only one of the symmetrical solutions of minimal disturbance augmentation problems is presented, however its counterpart produces very similar outcome.

7.3. Minimal dynamical error design

Gains chosen by minimization of performance index (25) have resulted in plots presented on Figures 9 and 10. One can see that in this case, the convergence is fastest of all considered observers.

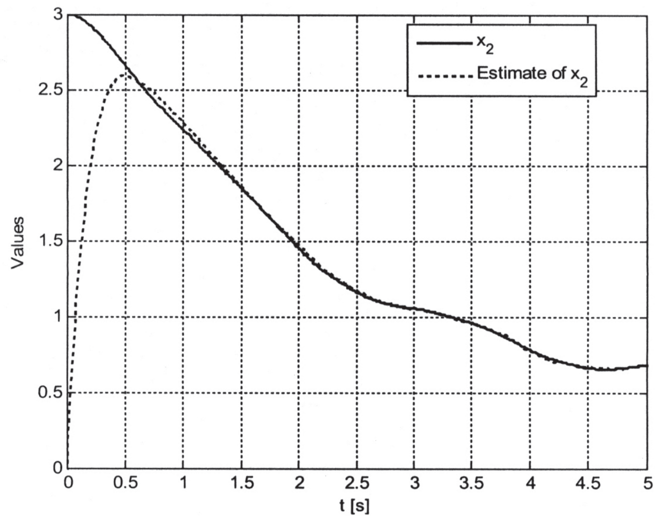

Fig. 9. Second state variable and its estimate produced by observer using both outputs with gain given by (26)

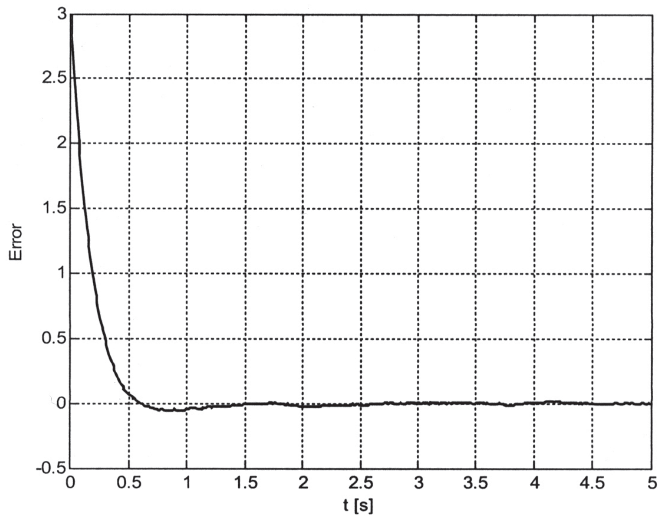

Fig. 10. Estimation error of second state variable of observer using both outputs with gain given by (26)

Overshoot exists but is much smaller than in the case of single output designs. Similarly impact of disturbances in steady state is also smaller. Also it should be noted that this observer produced the smallest value of integral of square of estimation error of second state variable (right side of (22)).

8. Concluding remarks

The focus of this paper was to present the benefits of multi output design in comparison to classical single output. It was shown, on the example of RC ladder network, that systems observable from more than one output give possibility of joining pole placement designs with optimality. Proposed in this paper performance indexes (16) and (25) are not the only ones worth considering. For example one should consider possibilities of minimizing $\boldsymbol{H}_2$ norm of the system or increasing robustness of the pole placement.

In author's opinion current state of optimization methods implemented in numerical packages (not only Matlab but also for example Octave or Mathematica) make constrained optimization of parameters worth considering as a design tool. Moreover, complicated to compute analytically performance indexes such as (25) can be easily used, because power of currently available personal computers allows very fast computation of them. For example computation of performance index takes about 0.001 s. and the whole optimization needs 0.1 second to be performed.

In conclusion, one can see, that if it is possible, and has low economical repercussions, observer designs for linear systems, based on multiple outputs, allow much more design possibilities than single output cases.

Results presented in this paper however lead to interesting questions regarding applying them in nonlinear systems. For example, performance index (25) might be used for observers designed via output injection where error dynamics is also linear (see [2]).

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