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Criterions for the stability of the convex combination of polynomials**

1. Introduction

Let p_1 and p_2 be two complex and of the same degree polynomials

$$p_1(s) = \alpha_n s^n + \alpha_{n-1} s^{n-1} + \ldots + \alpha_1 s + \alpha_0 = \alpha_n (s - x_1)(s - x_2) \ldots (s - x_n),$$
$$p_2(s) = \beta_n s^n + \beta_{n-1} s^{n-1} + \ldots + \beta_1 s + \beta_0 = \beta_n (s - y_1)(s - y_2) \ldots (s - y_n), \quad (1)$$

where $\alpha_n \neq 0$, $\beta_n \neq 0$, $\alpha_k, \beta_l \in \mathbb{C}$, $k = 0, 1, \ldots, n$, $l = 0, 1, \ldots, n$ and let $C(p_1, p_2)$ be the convex combination of these polynomials

$$C(p_1, p_2) = \{\delta p_1(s) + (1 - \delta)\, p_2(s) : \delta \in [0, 1]\}.$$

We say that polynomial $p_1(s)$ is Hurwitz (resp. Schur) stable if $\mathrm{Re}(x_k) < 0$ $(k = 1, 2, \ldots, n)$ (resp. $|x_k| < 1$ $(k = 1, 2, \ldots, n)$).

We call the convex combination $C(p_1, p_2)$ Hurwitz (resp. Schur) stable if every polynomial $p(s) \in C(p_1, p_2)$ is Hurwitz (resp. Schur) stable.

According to paper [2], in 1985 a necessary and sufficient condition for Hurwitz stability of the set $C(p_1, p_2)$ was proved, for real polynomials $p_1(s)$, $p_2(s)$. In 1989 in paper [3], a necessary and sufficient condition for Hurwitz and Schur stability of the set $C(p_1, p_2)$ was given, for complex polynomials $p_1(s)$, $p_2(s)$.

In this work we introduce a new necessary and sufficient condition for Hurwitz and Schur stability of the set $C(p_1, p_2)$, where polynomials $p_1(s)$, $p_2(s)$ are complex and of the same degree. We then generalize some results from [2] and complement some of [3].

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2. A necessary and sufficient condition for the stability of convex combination of polynomials

To polynomials (1), we assign the matrix

$$\mathrm{R}(p_1, p_2) = \begin{bmatrix} \alpha_n & \alpha_{n-1} & \alpha_{n-2} & \cdots & \alpha_0 & 0 & \cdots & \cdots \\ 0 & \alpha_n & \alpha_{n-1} & \cdots & \alpha_1 & \alpha_0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \alpha_n & \cdots & \cdots & \alpha_1 & \alpha_0 \\ \cdots & \cdots & \cdots & \beta_n & \cdots & \cdots & \beta_1 & \beta_0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \beta_n & \beta_{n-1} & \cdots & \cdots & \cdots & \cdots & 0 \\ \beta_n & \beta_{n-1} & \beta_{n-2} & \cdots & \cdots & \cdots & 0 & 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} n \text{ rows} \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} n \text{ rows} \end{matrix} \tag{2}$$

It is obvious that $\mathrm{R}(p_1, p_2) \in \mathbb{C}^{(2n)\times(2n)}$ and the elements of the matrix $\mathrm{R}(p_1, p_2)$ do not depend on s. It has been proved in [1] that:

$$\det \mathrm{R}(p_1, p_2) = (-1)^{n(n-1)/2} \alpha_n^n \beta_n^n \prod_{k=1}^{n} \prod_{l=1}^{n} (x_k - y_l). \tag{3}$$

We will use the following notation:

$P_n = \{p(s\} : p(s) = \alpha_n s^n + \alpha_{n-1} s^{n-1} + \ldots + \alpha_1 s + \alpha_0,\ \alpha_n \neq 0,\ \alpha_k \in \mathbb{C}\ \forall\ k = 0, 1, \ldots, n\}$,

$$f_1(s) = \left(a_n^{(1)} + ib_n^{(1)}\right)s^n + \left(a_{n-1}^{(1)} + ib_{n-1}^{(1)}\right)s^{n-1} + \ldots + \left(a_1^{(1)} + ib_1^{(1)}\right)s + \left(a_0^{(1)} + ib_0^{(1)}\right) \tag{4}$$

$$f_2(s) = \left(a_n^{(2)} + ib_n^{(2)}\right)s^n + \left(a_{n-1}^{(2)} + ib_{n-1}^{(2)}\right)s^{n-1} + \ldots + \left(a_1^{(2)} + ib_1^{(2)}\right)s + \left(a_0^{(2)} + ib_0^{(2)}\right)$$

where:

$$a_n^{(1)} + ib_n^{(1)} \neq 0,\ a_n^{(2)} + ib_n^{(2)} \neq 0,\ a_j^{(k)} + b_j^{(k)} \in \mathbb{C}\ (k = 1, 2, \quad j = 0, 1, \ldots, n),$$

$$H_n = \{p(s) \in P_n : p(s) = \alpha_n (s - x_1)(s - x_2) \ldots (s - x_n),\ \mathrm{Re}(x_k) < 0\ (k = 1, 2, \ldots, n)\},$$

$$S_n = \{p(s) \in P_n : p(s) = \alpha_n (s - x_1)(s - x_2) \ldots (s - x_n),\ |\, x_k \,| < 1\ (k = 1, 2, \ldots, n)\},$$

$$C(f_1, f_2) = \{\delta f_1(s) + (1 - \delta) f_2(s) : \delta \in [0, 1]\}.$$

Any polynomial $f_\delta(s) \in C(f_1, f_2)$ is of the form:

$$f_\delta(s) = \sum_{k=0}^{n} (a_k(\delta) + ib_k(\delta))s^k,$$

where

$$a_k(\delta) + ib_k(\delta) = (\delta a_k^{(1)} + (1-\delta)a_k^{(2)}) + i(\delta b_k^{(1)} + (1-\delta)b_k^{(2)})$$

for $\delta \in [0, 1]$ and

$$f_\delta(s)\big|_{\delta=0} = f_2(s), \quad f_\delta(s)\big|_{\delta=1} = f_1(s).$$

Let $(a_n + ib_n)s^n + (a_{n-1} + ib_{n-1})s^{n-1} + ... + (a_1 + ib_1)s + (a_0 + ib_0) = p(s) \in P_n$.

We will use the following notation:

$$s^* = \overline{s} \quad \text{for} \quad s \in \mathbb{C},$$

$$p^*(-s^*) = \overline{p(-\overline{s})},$$

$$M(s, p) = \frac{1}{2}\left[p(s) + p^*(-s^*)\right] = a_0 + ib_1 s + a_2 s^2 + ib_3 s^3 + ...,$$

$$N(s, p) = \frac{1}{2}\left[p(s) - p^*(-s^*)\right] = ib_0 + a_1 s + ib_2 s^2 + a_3 s^3 + ...$$

There, obviously, is $p(s) = M(s, p) + N(s, p)$. The following theorem is true.

Theorem 1 ([3])

The polynomial $(a_n + ib_n)s^n + (a_{n-1} + ib_{n-1})s^{n-1} + ... + (a_1 + ib_1)s + (a_0 + ib_0) = p(s) \in P_n$ *is Hurwitz stable* ($p(s) \in H_n$) *if and only if:*

1. $a_n a_{n-1} + b_n b_{n-1} > 0$,
2. *the zeros of* $M(s, p)$ *and* $N(s, p)$ *are simple, lie on the imaginary axis of the s–plane and the zeros of* $M(s, p)$ *alternate with the zeros of* $N(s, p)$.

The polynomial $f_\delta(s) \in C(f_1, f_2)$, that is:

$$f_\delta(s) = M(s, f_\delta) + N(s, f_\delta) \tag{5}$$

where:

$$M(s, f_\delta) = \frac{1}{2}\left[f_\delta(s) + f_\delta^*(-s^*)\right] = \delta M(s, f_1) + (1-\delta)M(s, f_2)$$

$$N(s, f_\delta) = \frac{1}{2}\left[f_\delta(s) - f_\delta^*(-s^*)\right] = \delta N(s, f_1) + (1-\delta)N(s, f_2) \tag{6}$$

$$\forall \delta \in [0, 1].$$

Matrix (2) formed with the coefficients of $M(s, f_\delta)$, $N(s, f_\delta)$ will be denoted by $\mathrm{R}(f_1, f_2, \delta)$. It is obvious that $\mathrm{R}(f_1, f_2, \delta) \in \mathbb{C}^{(2n)\times(2n)}$ and every element of this matrix is a linear combination of coefficients of polynomials f_1, f_2. Let us denote:

$$\mathrm{R}(f_1) = \mathrm{R}(f_1, f_2, \delta)\big|_{\delta=1}, \quad \mathrm{R}(f_2) = \mathrm{R}(f_1, f_2, \delta)\big|_{\delta=0},$$

and:

$$\Delta(\delta) = \det \mathrm{R}(f_1, f_2, \delta), \quad \Delta(1) = \det \mathrm{R}(f_1), \quad \Delta(0) = \det \mathrm{R}(f_2).$$

The following theorem was proved for the family $C(f_1, f_2)$.

Theorem 2 ([3])

If $f_1(s), f_2(s) \in H_n$ *and*

$$a_n(\delta)a_{n-1}(\delta) + b_n(\delta)b_{n-1}(\delta) > 0 \quad \forall \delta \in (0,1) \tag{7}$$

then the convex combination $C(f_1, f_2)$ *is Hurwitz stable* ($C(f_1, f_2) \subset H_n$) *if and only if* $\Delta(\delta) \neq 0$ *for all* $\delta \in [0, 1]$.

Now we shall prove a theorem which is complementary to Theorem 2.

Theorem 3

If $f_1(s), f_2(s) \in H_n$ *and coefficients of any polynomial* $f_\delta(s) \in C(f_1, f_2)$ *satisfy inequality (7), then the following conditions are equivalent:*

(i) $C(f_1, f_2) \subset H_n$,

(ii) $\Delta(\delta) \neq 0 \quad \forall \delta \in (0,1)$,

(iii) $\lambda_k(\mathrm{R}(f_1)\mathrm{R}^{-1}(f_2)) \notin (-\infty, 0) \quad \forall k = 1, 2, ..., 2n$,

where $\lambda_k(\mathrm{R}(f_1)\mathrm{R}^{-1}(f_2))$ $k = 1, 2, ..., 2n$ *are eigenvalues of the matrix* $\mathrm{R}(f_1)\mathrm{R}^{-1}(f_2)$.

Proof

In this proof the reasoning developed in paper [2] will be applied.

(i) $\Rightarrow$ (ii) It follows from Theorem 2.

(ii) $\Rightarrow$ (iii) Let $\Delta(\delta) \neq 0 \quad \forall \delta \in (0, 1)$.

From (5) i (6) it follows that:

$$\mathrm{R}(f_1, f_2, \delta) = \delta \mathrm{R}(f_1) + (1-\delta)\mathrm{R}(f_2), \; \delta \in (0, 1). \tag{8}$$

The assumption $\Delta(\delta) \neq 0 \quad \forall \delta \in (0, 1)$ and condition (8) imply:

$$\Delta(\delta) = \det(\delta \mathrm{R}(f_1) + (1-\delta)\mathrm{R}(f_2)) \neq 0, \; \forall \delta \in (0, 1).$$

Since the polynomial $f_2(s)$ is stable, $R^{-1}(f_2)$ exists because of (3) and Theorem 1, which says that $\Delta(0) \neq 0$. Using some of properties of matrix determinants, we conclude that:

$$\det(\delta R(f_1)R^{-1}(f_2) + (1-\delta)I) \neq 0,$$

and the last condition implies

$$\det\left(\frac{\delta-1}{\delta}I - R(f_1)R^{-1}(f_2)\right) \neq 0, \quad \forall\delta \in (0,1).$$

From this relation we infer that the eigenvalues $\lambda_k(R(f_1)R^{-1}(f_2)) \notin (-\infty, 0)$ for $k = 1, 2, ..., 2n$, which finishes the proof of the implication (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i) We assume that $\lambda_k(R(f_1)R^{-1}(f_2)) \notin (-\infty, 0)$ $k = (1, 2, ..., 2n)$. Hence

$$\det\left(\frac{\delta-1}{\delta}I - R(f_1)R^{-1}(f_2)\right) \neq 0, \quad \forall\delta \in (0,1),$$

and then

$$\Delta(\delta) = \det(\delta R(f_1) + (1-\delta)R(f_2)) \neq 0, \quad \forall\delta \in (0,1) \tag{9}$$

Conditions (7) i (9) imply the stability of the family $C(f_1, f_2)$ via Theorem 2. □

Similar results can be achieved in the case of a convex combination of two complex Schur stable polynomials.

Let $(a_n + ib_n)s^n + (a_{n-1} + ib_{n-1})s^{n-1} + ... + (a_1 + ib_1)s + (a_0 + ib_0) = p(s) \in P_n$.

We accept further notation:

$$p^*((s^*)^{-1}) = \overline{p((\overline{s})^{-1})},$$

$$D_1(s,p) = \frac{1}{2}[p(s) + s^n p^*((s^*)^{-1})] = \frac{1}{2}[(a_0 + a_n + i(b_0 - b_n)) +$$

$$+(a_1 + a_{n-1} + i(b_1 - b_{n-1}))s + ... + (a_n + a_0 + i(b_n - b_0))s^2]^n,$$

$$D_2(s,p) = \frac{1}{2}[p(s) - s^n p^*((s^*)^{-1})] = \frac{1}{2}[(a_0 - a_n + i(b_0 + b_n)) +$$

$$+(a_1 - a_{n-1} + i(b_1 + b_{n-1}))s + ... + (a_n - a_0 + i(b_n + b_0))s^2]^n,$$

for $s \neq 0$. Obviously, $p(s) = D_1(s,p) + D_2(s,p)$. For the polynomial $p(s) \in P_n$ the Theorem 4 is true.

Theorem 4 ([3])

The polynomial $(a_n + ib_n)s^n + (a_{n-1} + ib_{n-1})s^{n-1} + \ldots + (a_1 + ib_1)s + (a_0 + ib_0) = p(s) \in P_n$ *is Schur stable* ($p(s) \in S_n$) *if and only if:*

1. $|(a_0 + ib_0)/(a_n + ib_n)| < 1$,
2. *the zeros of* $D_1(s, p)$ *and* $D_2(s, p)$ *are simple, lie on the circus* $|s| = 1$ *and the zeros of* $D_1(s, p)$ *alternate with the zeros of* $D_2(s, p)$.

Let $f_1(s)$, $f_2(s)$ be polynomials defined in (4) and $f_\delta(s) \in C(f_1, f_2)$. We can write the polynomial $f_\delta(s)$ in the form:

$$f_\delta(s) = D_1(s, f_\delta) + D_2(s, f_\delta),$$

where:

$$D_1(s, f_\delta) = \frac{1}{2}[f_\delta(s) + s^n f_\delta^*((s^*)^{-1})] = \delta D_1(s, f_1) + (1-\delta)D_1(s, f_2),$$

$$D_2(s, f_\delta) = \frac{1}{2}[f_\delta(s) - s^n f_\delta^*((s^*)^{-1})] = \delta D_2(s, f_1) + (1-\delta)D_2(s, f_2),$$

$$\forall \delta \in [0, 1].$$

Let us build matrix (2) with the coefficients of polynomials $D_1(s, f_\delta)$, $D_2(s, f_\delta)$, which we denote by $\mathrm{R}_S(f_1, f_2, \delta)$. As it was in Hurwitz's stability case, this matrix is of order $2n$ and its elements depend on $\delta \in [0, 1]$. Let us denote:

$$\mathrm{R}_S(f_1) = \mathrm{R}_S(f_1, f_2, \delta)\big|_{\delta=1}, \quad \mathrm{R}_S(f_2) = \mathrm{R}_S(f_1, f_2, \delta)\big|_{\delta=0},$$

and:

$$\Delta_S(\delta) = \det \mathrm{R}_S(f_1, f_2, \delta), \quad \delta \in [0, 1], \quad \Delta_S(1) = \det \mathrm{R}_S(f_1), \quad \Delta_S(0) = \det \mathrm{R}_S(f_2).$$

The Theorem 5 is true.

Theorem 5 ([3])

If polynomials $f_1(s)$ *and* $f_2(s)$ *are in* S_n *and*

$$|(a_0(\delta) + ib_0(\delta))/(a_n(\delta) + ib_n(\delta)| < 1, \ \forall \delta \in (0, 1) \tag{10}$$

then the family $C(f_1, f_2)$ *is Schur stable* ($C(f_1, f_2) \subset S_n$) *if and only if the determinant* $\Delta_S(\delta) \neq 0$ *for all* $\delta \in [0, 1]$.

Now we shall prove a theorem which is complementary to Theorem 5.

Theorem 6

If $f_1(s), f_2(s) \in S_n$ *and coefficients of any polynomial* $f_\delta(s) \in C(f_1, f_2)$ *satisfy inequality (10) then the following conditions are equivalent:*

(i) $C(f_1, f_2) \subset S_n$,

(ii) $\Delta_S(\delta) \neq 0 \quad \forall \delta \in (0, 1)$,

(iii) $\lambda_k(\mathrm{R_S}(f_1)\mathrm{R_S}^{-1}(f_2)) \notin (-\infty, 0) \quad \forall k = 1, 2, ..., 2n$,

where $\lambda_k(\mathrm{R_S}(f_1)\mathrm{R_S}^{-1}(f_2))$ $k = 1, 2, ..., 2n$ *are eigenvalues of the matrix* $\mathrm{R_S}(f_1)\mathrm{R_S}^{-1}(f_2)$.

Proof of this Theorem is analogous to proof of Theorem 3. It is enough to notice that

$$\mathrm{R}_S(f_1, f_2, \delta) = \delta \mathrm{R}_S(f_1) + (1 - \delta)\mathrm{R}_S(f_2),\ \delta \in (0, 1)$$

and to use Theorems 4 and 5.

Example 1

Let us consider two complex Hurwitz polynomials:

$$f_1(s) = (1 - i)s^2 + (2 - 2i)s + (2 - 2i) = (1 - i)(s + 1 - i)(s + 1 + i),$$

$$f_2(s) = (1 + i)s^2 + (3 + 3i)s + (2 + 2i) = (1 + i)(s + 2)(s + 1),$$

and the convex combination of these polynomials:

$$C(f_1, f_2) = \{\delta f_1(s) + (1 - \delta)f_2(s) : \delta \in [0, 1]\} =$$

$$= \{f(s) = [1 + i(1 - 2\delta)]s^2 +$$

$$+ [3 - \delta + i(3 - 5\delta)]s + [2 + i(2 - 4\delta)]\}.$$

The matrices $\mathrm{R}(f_1)$, $\mathrm{R}(f_2)$ *are of the form:*

$$\mathrm{R}(f_1) = \begin{bmatrix} 1 & -2i & 2 & 0 \\ 0 & 1 & -2i & 2 \\ 0 & -i & 2 & -2i \\ -i & 2 & -2i & 0 \end{bmatrix}, \quad \mathrm{R}(f_2) = \begin{bmatrix} 1 & 3i & 2 & 0 \\ 0 & 1 & 3i & 2 \\ 0 & i & 3 & 2i \\ i & 3 & 2i & 0 \end{bmatrix}$$

and the matrix

$$\mathrm{R}(f_1)\mathrm{R}^{-1}(f_2) = -\frac{1}{6}\begin{bmatrix} -1 & 0 & 0 & 5i \\ 0 & -1 & 5i & 0 \\ 0 & 5i & 1 & 0 \\ 5i & 0 & 0 & 1 \end{bmatrix}.$$

Since the characteristic polynomial of the matrix $R(f_1)\,R^{-1}(f_2)$ *is of the form*

$$\det(\lambda I - R(f_1)R^{-1}(f_2)) = \left(\lambda^2 + \frac{2}{3}\right)^2,$$

then $\lambda_k(R(f_1)R^{-1}(f_2)) \notin (-\infty, 0)$ $(k = 1, 2, 3, 4)$. *Hence the family* $C(f_1, f_2)$ is Hurwitz stable for all $\delta \in [0, 1]$.

Example 2

Consider the following two complex Schur stable polynomials:

$$f_1(s) = -8s^2 - \frac{4}{3}is + 2\left(1 + \frac{1}{3}i\right),$$

$$f_2(s) = 3s^2 - i$$

and the convex combination of these polynomials:

$$C(f_1, f_2) = \left\{ f(s) = (-11\delta + 3)s^2 - \frac{4}{3}\delta is + 2\delta + i\left(\frac{5}{3}\delta - 1\right), \quad \delta \in [0, 1] \right\}$$

The matrices $R_S(f_1)$, $R_S(f_2)$ *are of the form:*

$$R_S(f_1) = \begin{bmatrix} -3 - \frac{1}{3}i & 0 & -3 + \frac{1}{3}i & 0 \\ 0 & -3 - \frac{1}{3}i & 0 & -3 + \frac{1}{3}i \\ 0 & -5 + \frac{1}{3}i & -\frac{4}{3}i & 5 + \frac{1}{3}i \\ -5 + \frac{1}{3}i & -\frac{4}{3}i & 5 + \frac{1}{3}i & 0 \end{bmatrix},$$

$$R_S(f_2) = \frac{1}{2}\begin{bmatrix} 3 + i & 0 & 3 - i & 0 \\ 0 & 3 + i & 0 & 3 - i \\ 0 & 3 - i & 0 & -3 - i \\ 3 - i & 0 & -3 - i & 0 \end{bmatrix}$$

and the matrix

$$\mathrm{R}_S(f_1)\mathrm{R}_S^{-1}(f_2) = -\frac{1}{6}\begin{bmatrix} 13 & 0 & 0 & -3i \\ 0 & 13 & -3i & 0 \\ i(3-i) & 6i & 22 & -i(3+i) \\ 6i & i(3+i) & i(3-i) & 22 \end{bmatrix}.$$

For $\lambda = -2$

$$\det(\lambda \mathrm{I} - \mathrm{R}_S(f_1)\mathrm{R}_S^{-1}(f_2)) = 0,$$

therefore the family $C(f_1, f_2)$ is not Schur stable. For $\delta = \frac{1}{3}$ *polynomial*

$$-\frac{2}{9}\left[3s^2 + 2is + (-3+2i)\right] = f_{\frac{1}{3}}(s) \in C(f_1, f_2)$$

is not Schur stable.

3. Conclusion

In this work necessary and sufficient conditions for Hurwitz (Schur) stability of convex combination of two complex polynomials of the same degree were proved and examples were given to show the use of those theorems. Theorems 3 and 6, proved in this paper, give answers for the question of stability of a convex combination of complex polynomials of the same degree, faster than theorems known from paper [3].

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