10.2478/acsc-2014-0022

Archives of Control Sciences
Volume 24(LX), 2014
No. 3, *pages 375–403*

Adaptive backstepping control, synchronization and circuit simulation of a 3-D novel jerk chaotic system with two hyperbolic sinusoidal nonlinearities

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In this research work, a six-term 3-D novel jerk chaotic system with two hyperbolic sinusoidal nonlinearities has been proposed, and its qualitative properties have been detailed. The Lyapunov exponents of the novel jerk system are obtained as $L_1 = 0.07765, L_2 = 0$, and $L_3 = -0.87912$. The Kaplan-Yorke dimension of the novel jerk system is obtained as $D_{KY} = 2.08833$. Next, an adaptive backstepping controller is designed to stabilize the novel jerk chaotic system with two unknown parameters. Moreover, an adaptive backstepping controller is designed to achieve complete chaos synchronization of the identical novel jerk chaotic systems with two unknown parameters. Finally, an electronic circuit realization of the novel jerk chaotic system using Spice is presented in detail to confirm the feasibility of the theoretical model.

Key words: chaos, jerk system, novel system, adaptive control, backstepping control, chaos synchronization

1. Introduction

Chaos theory describes the qualitative study of unstable aperiodic behaviour in deterministic nonlinear dynamical systems. A chaotic system is mathematically defined as a dynamical system with at least one positive Lyapunov exponent. In simple language, a chaotic system is a dynamical system, which is very sensitive to small changes in the initial conditions. Interest in nonlinear dynamics and in particular chaotic dynamics has grown rapidly since 1963, when Lorenz published his numerical work on a simplified model of convection and discussed its implications for weather prediction [1].

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Received 18.7.2014. Revised 6.9.2014.

Nonlinear dynamics occurs widely in engineering, physics, biology and many other scientific disciplines [2]. Poincaré was the first to observe the possibility of *chaos*, in which a deterministic system exhibits aperiodic behaviour that depends on the initial conditions, thereby rendering long-term prediction impossible, since then it has received much attention [3, 4].

Chaos has developed over time. For example, Ruelle and Takens [5] proposed a theory for the onset of turbulence in fluids, based on abstract considerations about strange attractors. Later, May [6] found examples of chaos in iterated mappings arising in population biology. Feigenbaum [7] discovered that there are certain universal laws governing the transition from regular to chaotic behaviours. That is, completely different systems can go chaotic in the same way, thus, linking chaos and phase transitions.

One of the hallmarks of nonlinear dynamics is the concept of equilibrium, which helps in characterizing a system's behaviour - especially its long-term motion. There are numerous types of equilibrium behaviour that can occur in continuous dynamical systems, but such long-time behaviours are restricted by the number of degrees-of-freedom (that is, by the dimensionality) of the system. In order words, one ignores the transient behaviour of a dynamical system and only considers the limiting behaviour as $t \to \infty$.

Chaos is a kind of motion, which is erratic, but not simply quasiperiodic with large number of periods [8]. Chaotic behaviour has been observed in driven acoustic systems, resonantly forced surface water, irradiated superconducting Josephson junction, ac-driven diode circuits, driven piezoelectric resonators, periodically forced neural oscillators, Ratchets, periodically modulated Josephson junction, the rigid body, gyroscopes, etc. For the motion of a system to be chaotic, the system variables should contain nonlinear terms and it must satisfy three properties: boundedness, infinite recurrence and sensitive dependence on initial conditions.

The study of chaos in the last decades had a tremendous impact on the foundations of science and engineering and one of the most recent exciting developments in this regard is the discovery of chaos synchronization, whose possibility was first reported by Fujisaka and Yamada [9] and later by Pecora and Carroll [10].

Different types of synchronization such as complete synchronization [10], anti-synchronization [11, 12, 13], hybrid synchronization [14, 15], lag synchronization [16], phase synchronization [16, 17], anti-phase synchronization APS, generalized synchronization [19], projective synchronization [20], generalized projective synchronization [21, 22, 23], etc. have been studied in the chaos literature.

Since the discovery of chaos synchronization, different approaches have been proposed to achieve it, such as PC method [10], active control method [24, 25, 26, 27], adaptive control method [28, 29, 30, 31], backstepping control method [32, 33, 34, 35, 36, 37], sliding mode control method [38, 39, 40, 41, 42], etc.

The first famous chaotic system was accidentally discovered by Lorenz, when he was deriving a mathematical model for atmospheric convection [43]. Subsequently, Rössler discovered a chaotic system in 1976 [44], which is algebraically simpler than the Lorenz system.

Some well-known 3-D chaotic systems are Arneodo system [45], Sprott systems [46], Chen system [47], Lü-Chen system [48], Liu system [49], Cai system [50], T-system [51], etc. Many new chaotic systems have been also discovered like Li system [52], Sundarapandian system [53], Vaidyanathan systems [54, 55, 56, 57], Vaidyanathan-Madhavan system [58], Sundarapandian-Pehlivan system [59], Pehlivan-Moroz-Vaidyanathan system [60], Jafari system [61], Pham system [62], etc.

In the recent decades, there is some good interest in finding novel chaotic systems, which can be expressed by an explicit third order differential equation describing the time evolution of the single scalar variable x given by

$$\dddot{x} = j(x, \dot{x}, \ddot{x}). \tag{1}$$

The differential equation (1) is called "jerk system" because the third order time derivative in mechanical systems is called *jerk*. Thus, in order to study different aspects of chaos, the ODE (1) can be considered instead of a 3-D system. Sprott's work [46] on jerk systems inspired Gottlieb [63] to pose the question of finding the simplest jerk function that generates chaos. This question was successfully answered by Sprott [64], who proposed a jerk function containing just three terms with a quadratic nonlinearity:

$$j(x, \dot{x}, \ddot{x}) = -A\ddot{x} + \dot{x}^2 - x \quad (\text{with } A = 2.017). \tag{2}$$

Sprott showed that the jerk system with the jerk function (2) is chaotic with the Lyapunov exponents $L_1 = 0.0550, L_2 = 0$ and $L_3 = -2.0720$, and corresponding to Kaplan-Yorke dimension of $D_{KY} = 2.0265$.

In this paper, we propose a 3-D novel jerk chaotic system with two hyperbolic sinusoidal nonlinearities. First, we detail the fundamental qualitative properties of the novel jerk chaotic system. We show that the novel chaotic system is dissipative and derive the Lyapunov exponents and Kaplan-Yorke dimension of the novel jerk chaotic system.

Next, this paper derives an adaptive backstepping control law that stabilizes the novel jerk chaotic system about its unique equilibrium point at the origin, when the system parameters are unknown. The backstepping control method is a recursive procedure that links the choice of a Lyapunov function with the design of a controller and guarantees global asymptotic stability of strict feedback systems [65, 66, 67, 68].

This paper also derives an adaptive backstepping control law that achieves global chaos synchronization of the identical 3-D novel jerk chaotic systems with unknown parameters. All the main adaptive results in this paper are proved using Lyapunov stability theory. MATLAB simulations are depicted to illustrate the phase portraits of the novel jerk chaotic system, dynamics of the Lyapunov exponents, adaptive stabilization and synchronization results for the novel jerk chaotic system. Finally, an electronic circuit realization of the novel jerk chaotic system using Spice is presented to confirm the feasibility of the theoretical model.

2. A 3-D novel jerk chaotic system

Recently, there is some interest in finding chaotic jerk functions having the special form

$$\dddot{x} + A\ddot{x} + \dot{x} = G(x), \tag{3}$$

where G is a nonlinear function having some special properties [69]. Such systems are called as *chaotic memory oscillators* in the literature. In [70], Sprott has made an exhaustive study on autonomous dissipative chaotic systems. Especially, Sprott has listed a set of 16 chaotic memory oscillators (Table 3.3, p. 74, [70]), named as $MO_0, MO_1, \ldots, MO_{15}$ with details of their Lyapunov exponents.

Sprott's system, MO_{15} is given by the differential equation

$$\dddot{x} + 0.6\ddot{x} + \dot{x} = x - 0.5\sinh(x). \tag{4}$$

It is convenient to express the Sprott ODE (4) in a system form as

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_1 - 0.5\sinh(x_1) - x_2 - 0.6x_3 \end{cases} \tag{5}$$

which is a 3-D jerk chaotic system having six terms on the R.H.S. with one hyperbolic sinusoidal nonlinearity.

We take the initial conditions for the Sprott system (5) as

$$x_1(0) = 0.8, \;\; x_2(0) = 1.2, \;\; x_3(0) = 0.5. \tag{6}$$

Then the Lyapunov exponents of the Sprott jerk system (5) are numerically obtained as

$$L_1 = 0.0601, \;\; L_2 = 0, \;\; L_3 = -0.6571. \tag{7}$$

Thus, the maximal Lyapunov exponent (MLE) of the Sprott jerk system (5) is $L_1 = 0.0601$.

Fig. 1 depicts the strange attractor of the Sprott jerk system (5) for the initial conditions (6).

In this work, we propose a new jerk system, which is given in a system form as

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_1 - a[\sinh(x_1) + \sinh(x_2)] - bx_3 \end{cases} \tag{8}$$

where a and b are positive parameters. In this paper, we shall show that the system (8) is chaotic when the parameters a and b take the values

$$a = 0.4 \;\text{ and }\; b = 0.8. \tag{9}$$

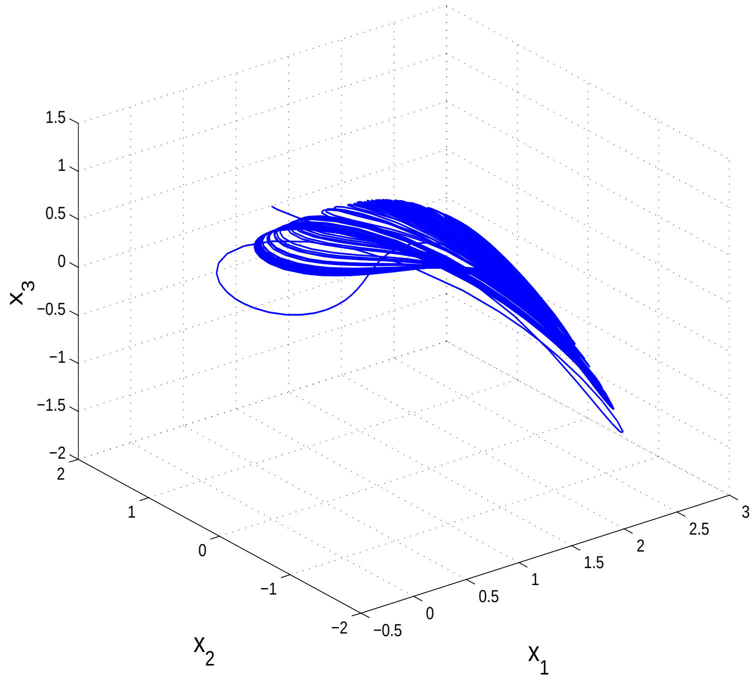

Figure 1. Strange attractor of the Sprott jerk system

We note that both the Sprott jerk system (5) and the novel jerk system (8) contain the same number of terms on the R.H.S. However, the two systems are not topologically equivalent since we have replaced the linear term $-x_2$ in the last equation of (5) with a hyperbolic sinusoidal term, *viz.* $-a\sinh(x_2)$ with $a > 0$. As a consequence, the phase portraits of the two jerk chaotic systems (5) and (8) will be different.

For the parameter values chosen in (9) and the initial conditions given in (6), the Lyapunov exponents of the novel jerk chaotic system (8) are obtained as

$$L_1 = 0.0777, \;\; L_2 = 0, \;\; L_3 = -0.8791. \tag{10}$$

The above calculations show that the maximal Lyapunov exponent (MLE) of the novel jerk chaotic system (8) is $L_1 = 0.0777$, which is greater than the MLE of the Sprott jerk system (5), viz. $L_1 = 0.0601$.

For numerical simulations of the novel jerk chaotic system (8), we use the same initial conditions (6), which were used for plotting the strange attractor of the Sprott jerk system (5).

Fig. 2 depicts the chaotic attractor of the novel jerk system (8) in 3-D view, while in Figs. 3-5, the 2-D projection of the strange chaotic attractor of the novel jerk chaotic system (8) in $(x_1,x_2),(x_2,x_3)$ and (x_3,x_1) planes, is shown, respectively.

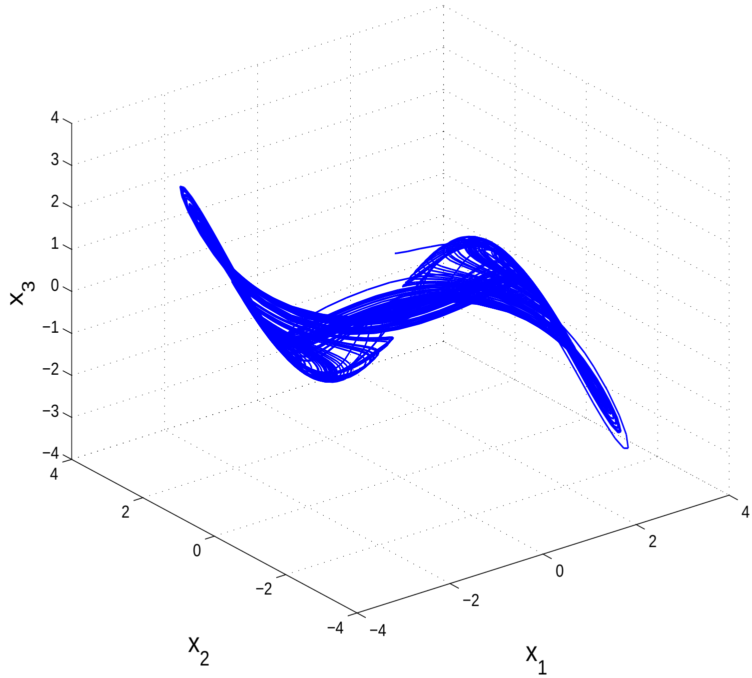

Figure 2. Strange attractor of the novel jerk system

3. Analysis of the 3-D novel jerk system

3.1. Dissipativity

In vector notation, the new jerk system (8) can be expressed as

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) = \begin{bmatrix} f_1(x_1,x_2,x_3) \\ f_2(x_1,x_2,x_3) \\ f_3(x_1,x_2,x_3) \end{bmatrix}, \tag{11}$$

where

$$\begin{cases} f_1(x_1,x_2,x_3) &= x_2 \\ f_2(x_1,x_2,x_3) &= x_3 \\ f_3(x_1,x_2,x_3) &= x_1 - a(\sinh(x_1) + \sinh(x_2)) - bx_3. \end{cases} \tag{12}$$

Let Ω be any region in $\Re^3$ with a smooth boundary and also, $\Omega(t) = \Phi_t(\Omega)$, where Φ_t is the flow of f. Furthermore, let $V(t)$ denote the volume of $\Omega(t)$. By Liouville's theorem, we know that

$$\dot{V}(t) = \int\limits_{\Omega(t)} (\nabla \cdot f)\, dx_1\, dx_2\, dx_3. \tag{13}$$

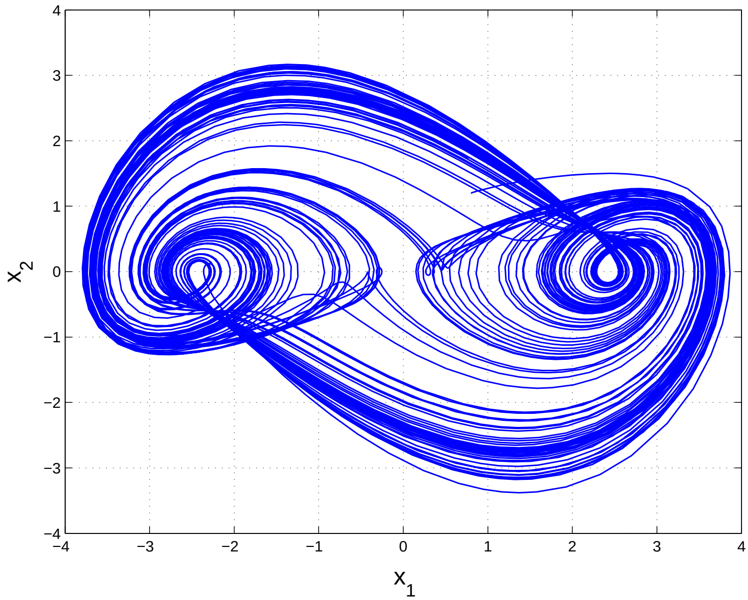

Figure 3. 2-D projection of the novel jerk system on (x_1, x_2)-plane

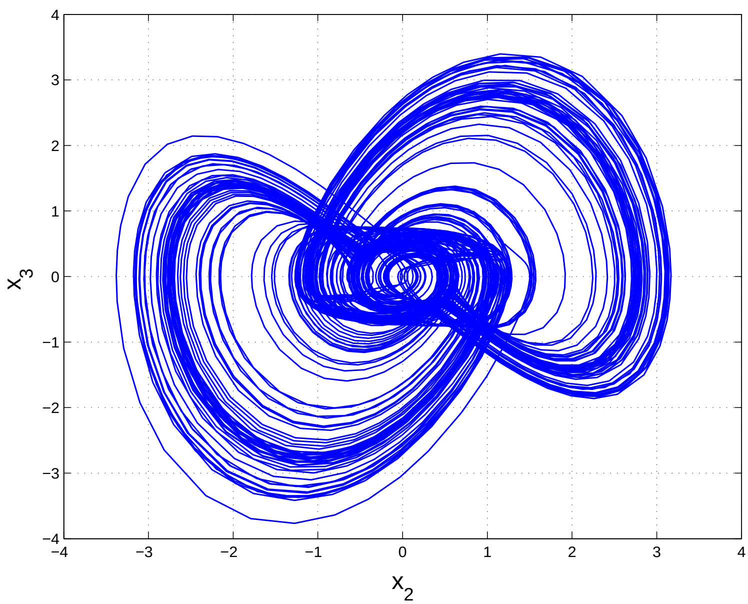

Figure 4. 2-D projection of the novel jerk system on (x_2, x_3)-plane

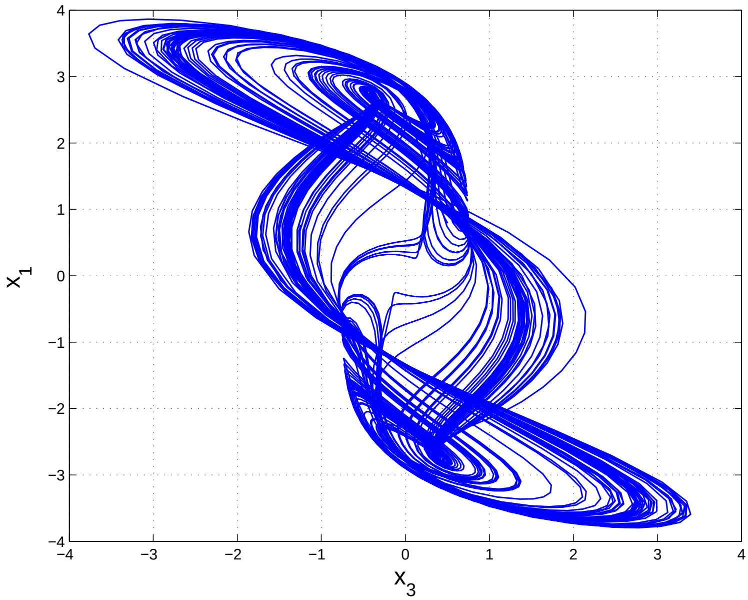

Figure 5. 2-D projection of the novel jerk system on (x_3, x_1)-plane

The divergence of the novel jerk system (11) is found as:

$$\nabla \cdot f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} = -b < 0 \tag{14}$$

since b is a positive parameter. Inserting the value of $\nabla \cdot f$ from (14) into (13), we get

$$\dot{V}(t) = \int_{\Omega(t)} (-b)\, dx_1\, dx_2\, dx_3 = -bV(t). \tag{15}$$

Integrating the first order linear differential equation (15), we get

$$V(t) = \exp(-bt)V(0). \tag{16}$$

Since $b > 0$, it follows from Eq. (16) that $V(t) \to 0$ exponentially as $t \to \infty$. This shows that the novel 3-D jerk chaotic system (8) is dissipative. Hence, the system limit sets are ultimately confined into a specific limit set of zero volume, and the asymptotic motion of the novel jerk chaotic system (8) settles onto a strange attractor of the system.

3.2. Equilibrium points

The equilibrium points of the 3-D novel jerk chaotic system (8) are obtained by solving the equations

$$\left.\begin{array}{lcccc} f_1(x_1,x_2,x_3) & = & x_2 & = & 0 \\ f_2(x_1,x_2,x_3) & = & x_3 & = & 0 \\ f_3(x_1,x_2,x_3) & = & x_1 - a\left[\sinh(x_1)+\sinh(x_2)\right] - bx_3 & = & 0 \end{array}\right\}. \tag{17}$$

We take the parameter values as in the chaotic case, *viz.* $a = 0.4$ and $b = 0.8$. Thus, the equilibrium points of the system (8) are characterized by the equations

$$x_1 - 0.4\sinh(x_1) = 0, \;\; x_2 = 0, \;\; x_3 = 0. \tag{18}$$

Solving the system (18), we get the equilibrium points of the system (8) as

$$E_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \; E_+ = \begin{bmatrix} 2.5527 \\ 0 \\ 0 \end{bmatrix} \text{ and } E_- = \begin{bmatrix} -2.5527 \\ 0 \\ 0 \end{bmatrix}. \tag{19}$$

To test the stability type of the equilibrium points E_0, E_+ and E_-, we calculate the Jacobian matrix of the novel jerk chaotic system (8) at any point $\boldsymbol{x} = \boldsymbol{x}^*$:

$$J(\boldsymbol{x}^*) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 - 0.4\cosh(x_1) & -0.4\cosh(x_2) & -0.8 \end{bmatrix}. \tag{20}$$

We note that

$$J_0 \triangleq J(E_0) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.6 & -0.4 & -0.8 \end{bmatrix} \tag{21}$$

which has the eigenvalues

$$\lambda_1 = 0.5368, \;\; \lambda_{2,3} = -0.6684 \pm 0.8191i. \tag{22}$$

This shows that the equilibrium point E_0 is a saddle-focus point. Next, we note that

$$J_+ \triangleq J(E_+) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1.5839 & -0.4 & -0.8 \end{bmatrix} \tag{23}$$

which has the eigenvalues

$$\lambda_{1,2} = 0.2805 \pm 1.0416i, \;\; \lambda_3 = -1.3611. \tag{24}$$

This shows that the equilibrium point E_+ is also a saddle-focus point. Since $J(E_-) = J(E_+)$, it is immediate that E_- is also a saddle-focus point. Hence, the novel jerk chaotic system (8) has three equilibrium points E_0, E_+, E_- defined by (19), which are saddle-foci.

3.3. Point reflection symmetry

We define a new set of coordinates as

$$\begin{aligned} \xi_1 &= -x_1 \\ \xi_2 &= -x_2 \\ \xi_3 &= -x_3. \end{aligned} \tag{25}$$

We find that

$$\begin{aligned} \dot{\xi}_1 &= -x_2 = \xi_2 \\ \dot{\xi}_2 &= -x_3 = \xi_3 \\ \dot{\xi}_3 &= -x_1 + a\sinh(x_1) + a\sinh(x_2) + bx_3 = \xi_1 - a\sinh(\xi_1) - a\sinh(\xi_2) - b\xi_3. \end{aligned} \tag{26}$$

This calculation shows that the 3-D novel jerk chaotic system (8) is invariant under the transformation of coordinates

$$(x_1, x_2, x_3) \mapsto (-x_1, -x_2, -x_3). \tag{27}$$

In mathematics, reflection through the origin refers to the point reflection of $\mathfrak{R}^n$ across the origin of $\mathfrak{R}^n$. Reflection through the origin is an orthogonal transformation corresponding to scalar multiplication by -1, and can also be written as $-I$, where I is the identity matrix. In $\mathfrak{R}^3$, the point reflection symmetry is characterized by $(x_1, x_2, x_3) \mapsto (-x_1, -x_2, -x_3)$. Thus, the novel jerk chaotic system (8) has point reflection symmetry about the origin. Hence, it follows that any non-trivial trajectory of the novel jerk chaotic system (8) must have a twin trajectory.

3.4. Lyapunov exponents and Kaplan-Yorke dimension

For the parameter values $a = 0.4$ and $b = 0.8$, the Lyapunov exponents are numerically obtained using MATLAB as

$$L_1 = 0.0771, \;\; L_2 = 0, \;\; L_3 = -0.8791. \tag{28}$$

Thus, the maximal Lyapunov exponent (MLE) of the novel jerk system (8) is positive, which means that the system has a chaotic behavior. Since $L_1 + L_2 + L_3 = -0.082 < 0$, it follows that the novel jerk chaotic system (8) is dissipative. Also, the Kaplan-Yorke dimension of the novel jerk chaotic system (8) is obtained as

$$D_{KY} = 2 + \frac{L_1 + L_2}{|L_3|} = 2.0877, \tag{29}$$

which is fractional.

4. Adaptive control of the 3-D novel jerk chaotic system with unknown parameters

In this section, we use backstepping control method to derive an adaptive feedback control law for globally stabilizing the 3-D novel jerk chaotic system with unknown parameters. Thus, we consider the 3-D novel jerk chaotic system given by

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = x_1 - a\left[\sinh(x_1) + \sinh(x_2)\right] - bx_3 + u \end{cases} \tag{30}$$

where a and b are unknown constant parameters, and u is a backstepping control law to be determined using estimates $\hat{a}(t)$ and $\hat{b}(t)$ for a and b, respectively.

The parameter estimation errors are defined as:

$$\begin{cases} e_a(t) = a - \hat{a}(t) \\ e_b(t) = b - \hat{b}(t). \end{cases} \tag{31}$$

Differentiating (31) with respect to t, we obtain the following equations:

$$\begin{cases} \dot{e}_a(t) = -\dot{\hat{a}}(t) \\ \dot{e}_b(t) = -\dot{\hat{b}}(t). \end{cases} \tag{32}$$

Next, we shall state and prove the main result of this section.

Theorem 10 *The 3-D novel jerk chaotic system (30), with unknown parameters a and b, is globally and exponentially stabilized by the adaptive feedback control law,*

$$u(t) = -4x_1 - 5x_2 - (3 - \hat{b}(t))x_3 + \hat{a}(t)\left[\sinh(x_1) + \sinh(x_2)\right] - kz_3 \tag{33}$$

where $k > 0$ is a gain constant,

$$z_3 = 2x_1 + 2x_2 + x_3, \tag{34}$$

and the update law for the parameter estimates $\hat{a}(t), \hat{b}(t)$ is given by

$$\begin{cases} \dot{\hat{a}}(t) = -\left[\sinh(x_1) + \sinh(x_2)\right]z_3 \\ \dot{\hat{b}}(t) = -x_3 z_3. \end{cases} \tag{35}$$

Proof We prove this result via backstepping control method and Lyapunov stability theory. First, we define a quadratic Lyapunov function

$$V_1(z_1) = \frac{1}{2}z_1^2 \tag{36}$$

where $z_1 = x_1$. Differentiating V_1 along the dynamics (30), we get

$$\dot{V}_1 = z_1\dot{z}_1 = x_1x_2 = -z_1^2 + z_1(x_1 + x_2). \tag{37}$$

Now, we define

$$z_2 = x_1 + x_2. \tag{38}$$

Using (38), we can simplify the equation (37) as

$$\dot{V}_1 = -z_1^2 + z_1z_2. \tag{39}$$

Secondly, we define a quadratic Lyapunov function

$$V_2(z_1, z_2) = V_1(z_1) + \frac{1}{2}z_2^2 = \frac{1}{2}\left(z_1^2 + z_2^2\right). \tag{40}$$

Differentiating V_2 along the dynamics (30), we get

$$\dot{V}_2 = -z_1^2 - z_2^2 + z_2(2x_1 + 2x_2 + x_3). \tag{41}$$

Now, we define

$$z_3 = 2x_1 + 2x_2 + x_3. \tag{42}$$

Using (42), we can simplify the equation (41) as

$$\dot{V}_2 = -z_1^2 - z_2^2 + z_2z_3. \tag{43}$$

Finally, we define a quadratic Lyapunov function

$$V(z_1, z_2, z_3, e_a, e_b) = V_2(z_1, z_2) + \frac{1}{2}z_3^2 + \frac{1}{2}e_a^2 + \frac{1}{2}e_b^2 \tag{44}$$

which is a positive definite function on $\mathfrak{R}^5$. Differentiating V along the dynamics (30), we get

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 + z_3(z_3 + z_2 + \dot{z}_3) - e_a\dot{\hat{a}} - e_b\dot{\hat{b}}. \tag{45}$$

Eq. (45) can be written compactly as

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 + z_3S - e_a\dot{\hat{a}} - e_b\dot{\hat{b}} \tag{46}$$

where

$$S = z_3 + z_2 + \dot{z}_3 = z_3 + z_2 + 2\dot{x}_1 + 2\dot{x}_2 + \dot{x}_3. \tag{47}$$

A simple calculation gives

$$S = 4x_1 + 5x_2 + (3-b)x_3 - a[\sinh(x_1) + \sinh(x_2)] + u. \tag{48}$$

Substituting the adaptive control law (33) into (48), we obtain

$$S = -(a-\hat{a})[\sinh(x_1) + \sinh(x_2)] - (b-\hat{b})x_3 - kz_3. \tag{49}$$

Using the definitions (32), we can simplify (49) as

$$S = -e_a[\sinh(x_1) + \sinh(x_2)] - e_b x_3 - k z_3. \tag{50}$$

Substituting the value of S from (50) into (46), we obtain

$$\dot{V} = -z_1 - z_2 - (1+k)z_3^2 + e_a\left[-[\sinh(x_1) + \sinh(x_2)]z_3 - \dot{\hat{a}}\right] + e_b\left[-x_3 z_3 - \dot{\hat{b}}\right]. \tag{51}$$

Substituting the update law (35) into (51), we get

$$\dot{V} = -z_1^2 - z_2^2 - (1+k)z_3^2, \tag{52}$$

which is a negative semi-definite function on $\Re^5$. From (52), it follows that the vector $\boldsymbol{z}(t) = (z_1(t), z_2(t), z_3(t))$ and the parameter estimation error $(e_a(t), e_b(t))$ are globally bounded, i.e.

$$\begin{bmatrix} z_1(t) & z_2(t) & z_3(t) & e_a(t) & e_b(t) \end{bmatrix} \in \mathrm{L}_\infty. \tag{53}$$

Also, it follows from (52) that

$$\dot{V} \leqslant -z_1^2 - z_2^2 - z_3^2 = -\|\mathbf{z}\|^2. \tag{54}$$

That is,

$$\|\mathbf{z}\|^2 \leqslant -\dot{V}. \tag{55}$$

Integrating the inequality (55) from 0 to t, we get

$$\int_0^t |\boldsymbol{z}(\tau)|^2\, d\tau \leqslant V(0) - V(t). \tag{56}$$

From (56), it follows that $\boldsymbol{z}(t) \in \mathrm{L}_2$. From Eq. (30), it can be deduced that $\dot{\boldsymbol{z}}(t) \in \mathrm{L}_\infty$. Thus, using Barbalat's lemma, we conclude that $\boldsymbol{z}(t) \to \mathbf{0}$ exponentially as $t \to \infty$ for all initial conditions $\boldsymbol{z}(0) \in \Re^3$. Hence, it is immediate that $\boldsymbol{x}(t) \to \mathbf{0}$ exponentially as $t \to \infty$ for all initial conditions $\boldsymbol{x}(0) \in \Re^3$. This completes the proof. □

For the numerical simulations, the classical fourth-order Runge-Kutta method with step size $h = 10^{-8}$ is used to solve the system of differential equations (30) and (35), when the adaptive control law (33) is applied.

The parameter values of the novel jerk chaotic system (30) are taken as $a = 0.4$ and $b = 0.8$, and the positive gain constant as $k = 8$. Furthermore, as initial conditions of the novel jerk chaotic system (30), we take $x_1(0) = 6.2$, $x_2(0) = -12.5$ and $x_3(0) = 4.1$. Also, as initial conditions of the parameter estimates $\hat{a}(t)$ and $\hat{b}(t)$, we take $\hat{a}(0) = 18.7$ and $\hat{b}(0) = 11.5$. In Fig. 6, the exponential convergence of the controlled states $x_1(t), x_2(t), x_3(t)$ is depicted, when the adaptive control law (33) and (35) are implemented.

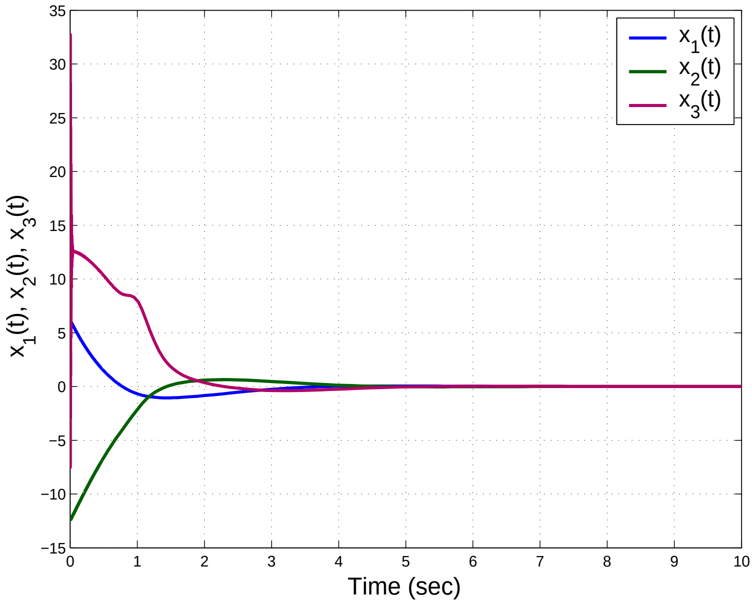

Figure 6. Time-history of the controlled states $x_1(t), x_2(t), x_3(t)$

5. Adaptive synchronization of the identical 3-D novel jerk chaotic systems with unknown parameters

In this section, we use backstepping control method to derive an adaptive control law for globally and exponentially synchronizing the identical 3-D novel jerk chaotic systems with unknown parameters. As the master system, we consider the 3-D novel jerk chaotic system given by

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = x_1 - a[\sinh(x_1) + \sinh(x_2)] - bx_3 \end{cases} \tag{57}$$

where x_1, x_2, x_3 are the states of the system, and a and b are unknown constant parameters. As the slave system, we consider the 3-D novel jerk chaotic system given by

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = y_3 \\ \dot{y}_3 = y_1 - a[\sinh(y_1) + \sinh(y_2)] - by_3 + u \end{cases} \tag{58}$$

where y_1, y_2, y_3 are the states of the system, and u is a backstepping control to be determined using estimates $\hat{a}(t)$ and $\hat{b}(t)$ for a and b, respectively. We define the synchronization errors between the states of the master system (57) and the slave system (58) as

$$\begin{cases} e_1 &= y_1 - x_1 \\ e_2 &= y_2 - x_2 \\ e_3 &= y_3 - x_3. \end{cases} \tag{59}$$

Then the error dynamics is easily obtained as

$$\begin{cases} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= e_3 \\ \dot{e}_3 &= e_1 - be_3 - a[\sinh(y_1) + \sinh(y_2)] \\ &+ a[\sinh(x_1) + \sinh(x_2)] + u. \end{cases} \tag{60}$$

The parameter estimation errors are defined as:

$$\begin{cases} e_a(t) &= a - \hat{a}(t) \\ e_b(t) &= b - \hat{b}(t). \end{cases} \tag{61}$$

Differentiating (61) with respect to t, we obtain the following equations:

$$\begin{cases} \dot{e}_a(t) &= -\dot{\hat{a}}(t) \\ \dot{e}_b(t) &= -\dot{\hat{b}}(t). \end{cases} \tag{62}$$

Next, we shall state and prove the main result of this section.

Theorem 11 *The identical 3-D novel jerk chaotic systems (57) and (58) with unknown parameters a and b are globally and exponentially synchronized by the adaptive control law*

$$\begin{cases} u(t) &= -4e_1 - 5e_2 - (3 - \hat{b}(t))e_3 - \hat{a}(t)[\sinh(x_1) + \sinh(x_2)] \\ & +\hat{a}(t)[\sinh(y_1) + \sinh(y_2)] - kz_3 \end{cases} \tag{63}$$

where $k > 0$ is a gain constant,

$$z_3 = 2e_1 + 2e_2 + e_3, \tag{64}$$

and the update law for the parameter estimates $\hat{a}(t), \hat{b}(t)$ is given by

$$\begin{cases} \dot{\hat{a}}(t) &= [\sinh(x_1) + \sinh(x_2) - \sinh(y_1) - \sinh(y_2)]z_3 \\ \dot{\hat{b}}(t) &= -e_3 z_3. \end{cases} \tag{65}$$

Proof We prove this result via backstepping control method and Lyapunov stability theory. First, we define a quadratic Lyapunov function

$$V_1(z_1) = \frac{1}{2} z_1^2 \tag{66}$$

where

$$z_1 = e_1. \tag{67}$$

Differentiating V_1 along the error dynamics (60), we get

$$\dot{V}_1 = z_1 \dot{z}_1 = e_1 e_2 = -z_1^2 + z_1(e_1 + e_2). \tag{68}$$

Now, we define

$$z_2 = e_1 + e_2. \tag{69}$$

Using (69), we can simplify the equation (68) as

$$\dot{V}_1 = -z_1^2 + z_1 z_2. \tag{70}$$

Secondly, we define a quadratic Lyapunov function

$$V_2(z_1, z_2) = V_1(z_1) + \frac{1}{2} z_2^2 = \frac{1}{2}\left(z_1^2 + z_2^2\right). \tag{71}$$

Differentiating V_2 along the error dynamics (60), we get

$$\dot{V}_2 = -z_1^2 - z_2^2 + z_2(2e_1 + 2e_2 + e_3). \tag{72}$$

Now, we define

$$z_3 = 2e_1 + 2e_2 + e_3. \tag{73}$$

Using (73), we can simplify the equation (72) as

$$\dot{V}_2 = -z_1^2 - z_2^2 + z_2 z_3. \tag{74}$$

Finally, we define a quadratic Lyapunov function

$$V(z_1, z_2, z_3, e_a, e_b) = V_2(z_1, z_2) + \frac{1}{2} z_3^2 + \frac{1}{2} e_a^2 + \frac{1}{2} e_b^2 \tag{75}$$

which is a positive definite function on $\mathfrak{R}^5$. Differentiating V along the error dynamics (60), we get

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 + z_3(z_3 + z_2 + \dot{z}_3) - e_a \dot{\hat{a}} - e_b \dot{\hat{b}}. \tag{76}$$

Eq. (76) can be written compactly as

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 + z_3 S - e_a \dot{\hat{a}} - e_b \dot{\hat{b}} \tag{77}$$

where

$$S = z_3 + z_2 + \dot{z}_3 = z_3 + z_2 + 2\dot{e}_1 + 2\dot{e}_2 + \dot{e}_3. \tag{78}$$

A simple calculation gives

$$\begin{cases} S = 4e_1 + 5e_2 + (3-b)e_3 - a[\sinh(y_1) + \sinh(y_2)] \\ \quad\quad + a[\sinh(x_1) + \sinh(x_2)] + u. \end{cases} \tag{79}$$

Substituting the adaptive control law (63) into (48), we obtain

$$S = (a - \hat{a})[\sinh(x_1) + \sinh(x_2) - \sinh(y_1) - \sinh(y_2)] - (b - \hat{b})e_3 - kz_3. \tag{80}$$

Using the definitions (62), we can simplify (80) as

$$S = e_a[\sinh(x_1) + \sinh(x_2) - \sinh(y_1) - \sinh(y_2)] - e_b e_3 - kz_3. \tag{81}$$

Substituting the value of S from (81) into (77), we obtain

$$\begin{cases} \dot{V} = -z_1 - z_2 - (1+k)z_3^2 + e_b\left[-e_3 z_3 - \dot{\hat{b}}\right] \\ \quad\quad + e_a\left[[\sinh(x_1) + \sinh(x_2) - \sinh(y_1) - \sinh(y_2)]z_3 - \dot{\hat{a}}\right]. \end{cases} \tag{82}$$

Substituting the update law (65) into (82), we get

$$\dot{V} = -z_1^2 - z_2^2 - (1+k)z_3^2, \tag{83}$$

which is a negative semi-definite function on $\mathfrak{R}^5$. From (83), it follows that the vector $\boldsymbol{z}(t) = (z_1(t), z_2(t), z_3(t))$ and the parameter estimation error $(e_a(t), e_b(t))$ are globally bounded, i.e.

$$\begin{bmatrix} z_1(t) & z_2(t) & z_3(t) & e_a(t) & e_b(t) \end{bmatrix} \in \mathrm{L}_\infty. \tag{84}$$

Also, it follows from (83) that

$$\dot{V} \leqslant -z_1^2 - z_2^2 - z_3^2 = -\|\mathbf{z}\|^2. \tag{85}$$

That is,

$$\|\mathbf{z}\|^2 \leqslant -\dot{V}. \tag{86}$$

Integrating the inequality (86) from 0 to t, we get

$$\int_0^t |\boldsymbol{z}(\tau)|^2 \, d\tau \leqslant V(0) - V(t). \tag{87}$$

From (87), it follows that $\boldsymbol{z}(t) \in \mathrm{L}_2$. From Eq. (60), it can be deduced that $\dot{\boldsymbol{z}}(t) \in \mathrm{L}_\infty$. Thus, using Barbalat's lemma, we conclude that $\boldsymbol{z}(t) \to \mathbf{0}$ exponentially as $t \to \infty$ for all initial conditions $\boldsymbol{z}(0) \in \mathfrak{R}^3$. Hence, it is immediate that $\boldsymbol{e}(t) \to \mathbf{0}$ exponentially as

$t \to \infty$ for all initial conditions $\boldsymbol{e}(0) \in \Re^3$. This completes the proof. □

For the numerical simulations, the classical fourth-order Runge-Kutta method with step size $h = 10^{-8}$ is used to solve the system of differential equations (57) and (58).

The parameter values of the novel jerk chaotic systems are taken as $a = 0.4$ and $b = 0.8$, and the positive gain constant as $k = 8$. Furthermore, as initial conditions of the master chaotic system (57), we take $x_1(0) = 2.3$ $x_2(0) = 3.5$ and $x_3(0) = -0.7$. As initial conditions of the slave chaotic system (58), we take $y_1(0) = 3.5$, $y_2(0) = -4.2$ and $y_3(0) = 1.5$. Also, as initial conditions of the parameter estimates $\hat{a}(t)$ and $\hat{b}(t)$, we take $\hat{a}(0) = 8.2$ and $\hat{b}(0) = 12.5$.

In Figs. 7-10, the complete synchronization of the identical 3-D jerk chaotic systems (57) and (58) is shown, when the adaptive control law and the parameter update law are impelemented.

Also, in Fig. 10, the time-history of the synchronization errors $e_1(t), e_2(t), e_3(t)$, is shown.

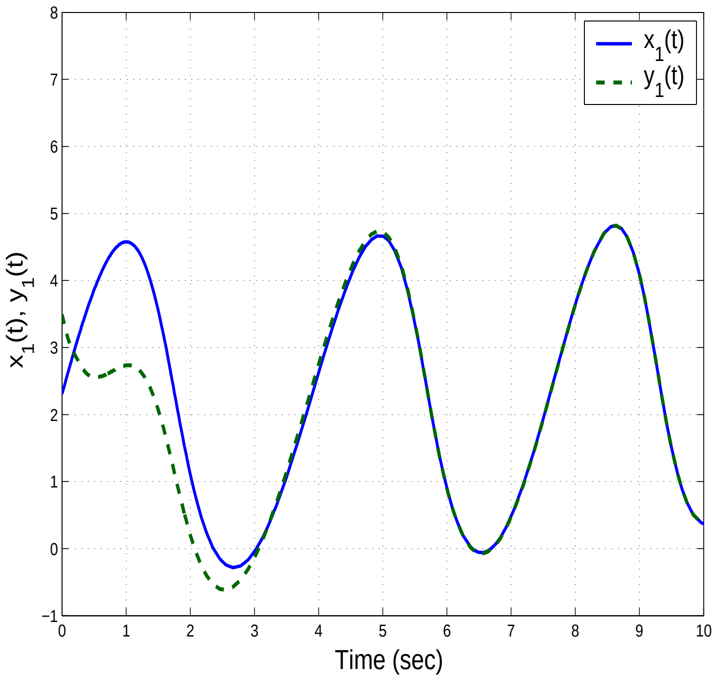

Figure 7. Synchronization of the states $x_1(t)$ and $y_1(t)$

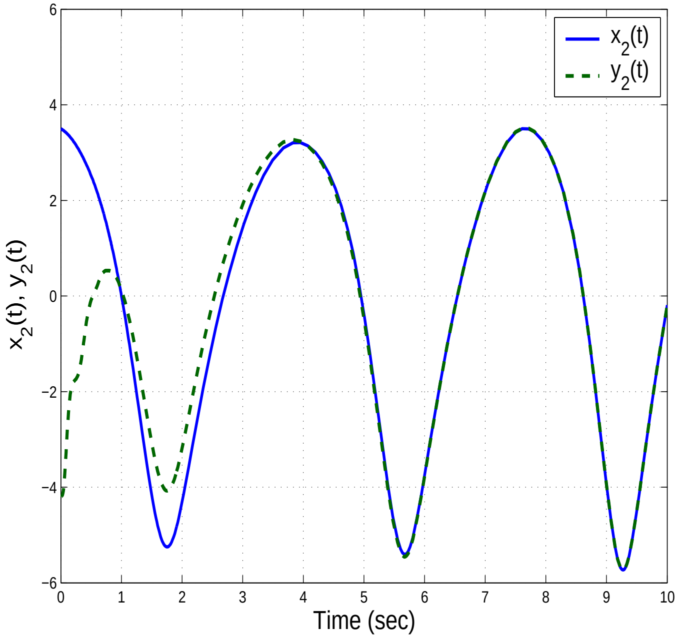

Figure 8. Synchronization of the states $x_2(t)$ and $y_2(t)$

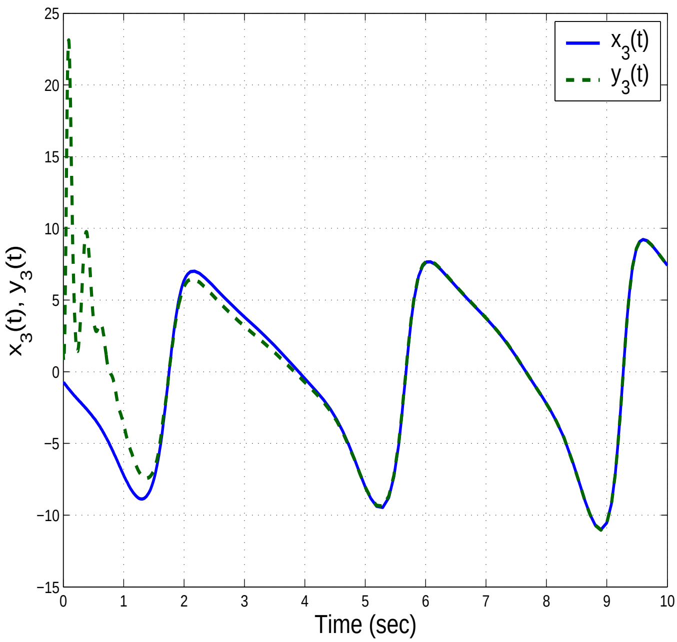

Figure 9. Synchronization of the states $x_3(t)$ and $y_3(t)$

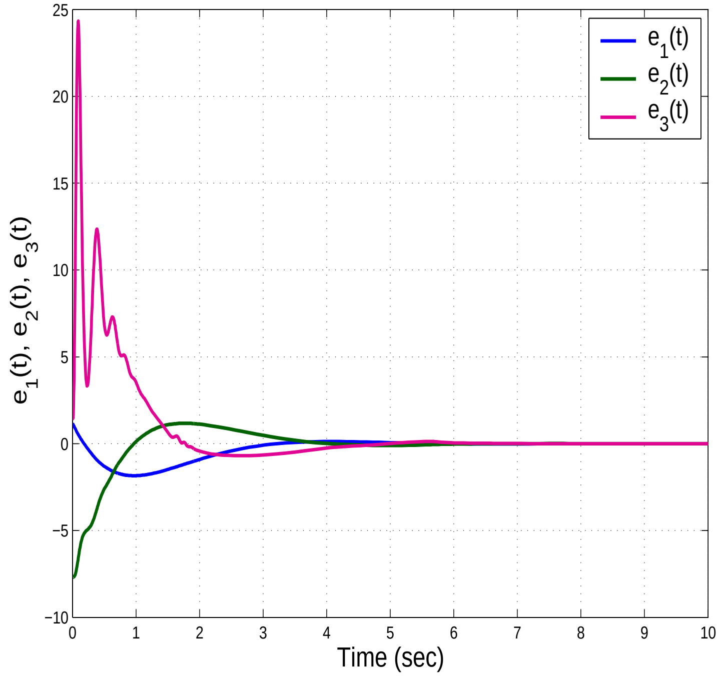

Figure 10. Time-history of the synchronization errors $e_1(t), e_2(t), e_3(t)$

6. Circuit realization of the novel jerk system

In this section, we design an electronic circuit modeling new jerk system (8). The circuit in Fig. 11 has been designed following an approach based on operational amplifiers [59, 60, 70] where the state variables x_1, x_2, and x_3 of the system (8) are associated with the voltages across the capacitors C_1, C_2, and C_3, respectively. In Fig. 11, there are three operational amplifiers, which are connected as integrators (U_1, U_2, and U_3). The nonlinear equations for the electronic circuit are derived as follows:

$$\begin{cases} \dot{x}_1 = \frac{1}{R_1C_1}x_2 \\ \dot{x}_2 = \frac{1}{R_2C_2}x_3 \\ \dot{x}_3 = \frac{1}{R_3C_3}x_1 - \frac{1}{R_4C_3}\sinh(x_1) - \frac{1}{R_5C_3}\sinh(x_2) - \frac{1}{R_6C_3}x_3. \end{cases} \tag{88}$$

where the values of components are chosen as: $R_1 = R_2 = R_3 = R_7 = R_8 = R_9 = R_{10} = R_{11} = R_{12} = 10k\Omega$, $R_4 = R_5 = 0.25k\Omega$, $R_6 = 12.5k\Omega$, and $C_1 = C_2 = C_3 = 10nF$. The power supplies of all active devices are $\pm 15V_{DC}$. As shown in Fig. 11, nonlinearities of the circuit are two inverting hyperbolic sinusoidal blocks, each block can be realized approximately by common electronic components as presented in Fig. 12.

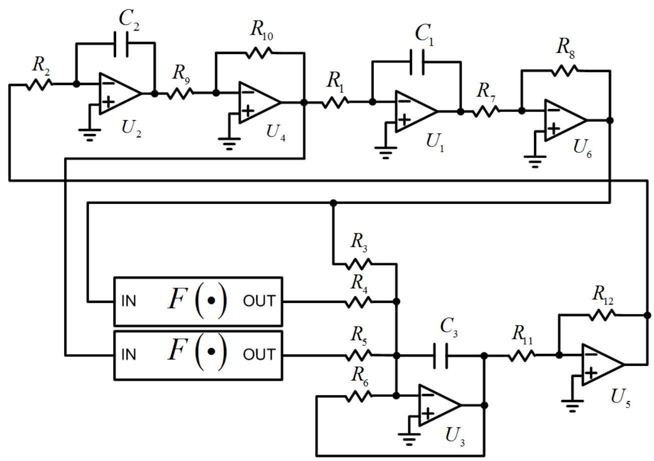

Figure 11. Circuit diagram for realizing the novel jerk system

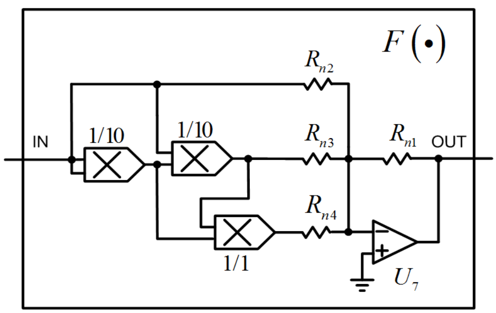

Figure 12. Designed electronic circuit modeling of the inverting hyperbolic sinusoidal block. Here the values of components are selected as: $R_{n1} = 10k\Omega$, $R_{n2} = 1M\Omega$, $R_{n3} = 60k\Omega$, and $R_{n4} = 120k\Omega$

It is well know that it is possible to express the hyperbolic sinusoidal function as Taylor series [71]:

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}. \tag{89}$$

Hence, the corresponding circuital equation of each block is given as

$$F(x_i) = -\frac{R_{n1}}{R_{n2}} x_i - \frac{R_{n1}}{R_{n3}} x_i^3 - \frac{R_{n1}}{R_{n4}} x_i^5 \approx \delta \sinh(x_i) \tag{90}$$

where $\delta = -\frac{1}{10^2}$ is the scaling factor.

The designed circuit is implemented by using the electronic simulation package NI Multisim. The obtained results are presented in Figs. 13-15. Here Figs. 13, 14 and 15 display the (x_1, x_2), (x_2, x_3) and (x_3, x_1) phase portraits respectively. Good qualitative agreement between the numerical results and the electronic simulation results shows the correction and feasibility of novel chaotic jerk system (8).

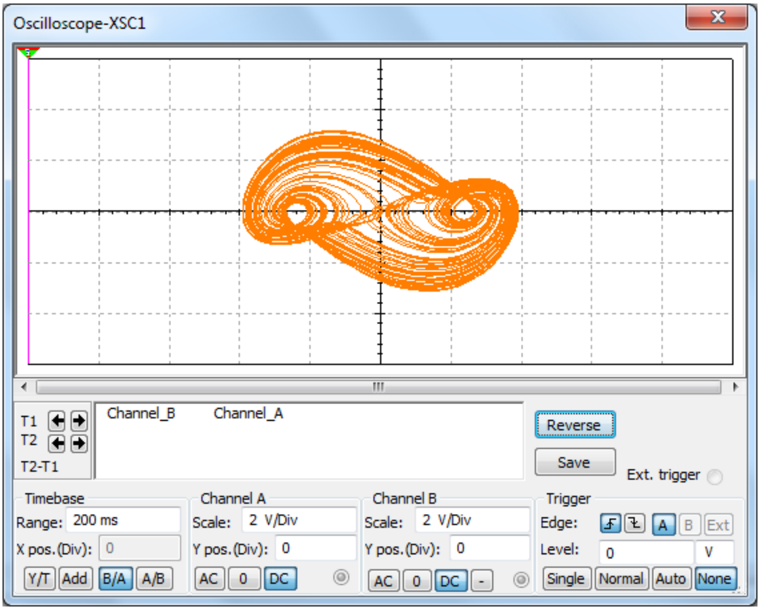

Figure 13. 2-D projection of the designed electronic circuit on (x_1, x_2)-plane obtained from NI Multisim.

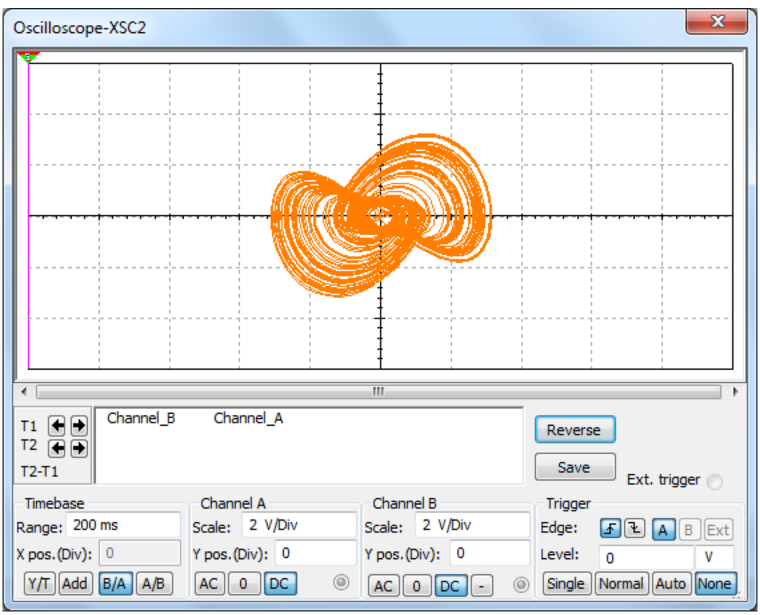

Figure 14. 2-D projection of the designed electronic circuit on (x_2, x_3)-plane obtained from NI Multisim.

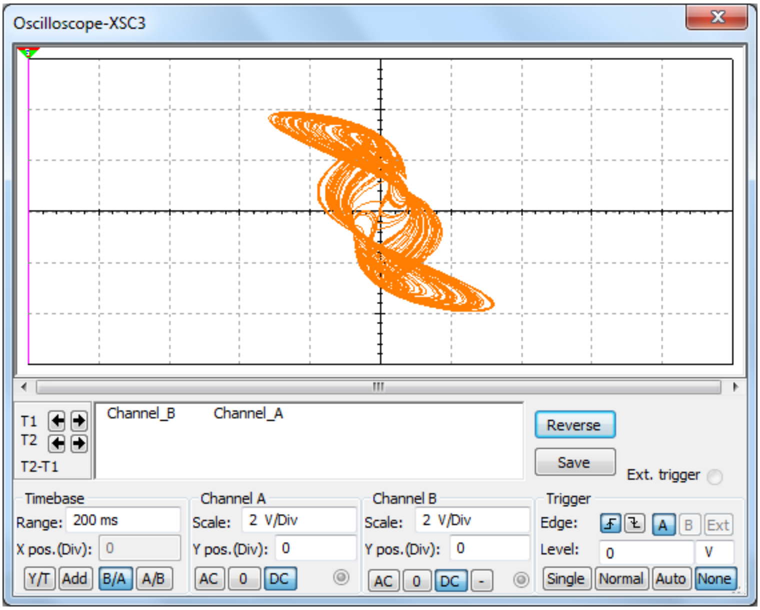

Figure 15. 2-D projection of the designed electronic circuit on (x_3, x_1)-plane obtained from NI Multisim.

7. Conclusion

In this paper, we proposed a novel six-term jerk chaotic system with two hyperbolic sinusoidal nonlinearities. Dynamic characteristics of new system has been discovered. It is worth noting that the possibilities of control and synchronization of such system with unknown parameters are verified by constructing an adaptive backstepping controller. The main results were established using adaptive control theory and Lyapunov stability theory. Moreover, the correction and feasibility of novel theoretical system are confirmed through Spice results which are obtained from the designed electronic circuit. It is possible to use the new jerk system in potential chaos-based applications such as secure communications, random generation, or path planning for autonomous mobile robots. It is believed that the unknown dynamical behaviors of such strange chaotic jerk systems should be further investigated in the future researches.

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