Theoretical and Applied Informatics
ISSN 1896-5334
Vol. 26 (2014), no. 3, 4
pp. 149 - 158

Approximation of separable numerical range using simulated annealing

PIOTR GAWRON, PRZEMYSŁAW SADOWSKI [a]

[a]Institute of Theoretical and Applied Informatics, Polish Academy of Sciences
Bałtycka 5, 44-100 Gliwice, Poland
{gawron, psadowski}@iitis.pl

Abstract: In this work we provide a method for approximating the separable numerical range of a matrix. We also recall the connection between restricted numerical range and entanglement of a quantum state. We show the possibility to establish state separability using computed restricted numerical range. In particular we present a method to obtain separability criteria for arbitrary system partition with use of the separable numerical range.

Keywords: Restricted numerical ranges, separable states, quantum entanglement

1. Introduction

The quantum information science studies the application of the quantum mechanics to processing, storing and transmitting information. The main trends of research involving the theoretical aspects of quantum computing are: the quantum information theory, the quantum computation, the quantum cryptography, the quantum communication and the quantum games. The origins of the field date back to the first decades of the twentieth century.

In the last two decades of the twentieth century, the groundwork for quantum computing systems was laid. First in the eighties, Bennett and Brassard [1] proposed quantum key distribution (QKD) protocols paving the way to a whole new field of quantum cryptography. Feynman's idea of quantum simulators [2] and Deutsch's work [3] on universal quantum computer, followed by Shor's [4] and Grover's [5] first quantum algorithms invented in the nineties of the twentieth century, created a whole new field of research on quantum computing and quantum algorithms.

One of the central concepts of quantum information theory is *quantum entanglement* [6, 7], which can be understood as non-classical correlations that exists only between quantum systems. Entanglement is the main physical phenomenon behind *i.e* the

quantum communication and cryptography, the quantum algorithms and the quantum metrology.

Investigation of the geometry of entanglement remains a subject of current scientific interests in view of applications in the theory of quantum information processing. The set Ω_N of pure quantum states belonging to a N dimensional complex Euclidean space $\mathbb{C}^N$ is known to be equivalent to the complex projective space, $\Omega_N = \mathbb{C}P^{N-1}$, of $2N-2$ real dimensions. However, as this set is embedded into the N^2-1 dimensional set $\mathcal{Q}_N$ of density matrices of dimension N by a non-linear constraint, $\rho = \rho^2$, the geometric structure of the set of mixed quantum states is rather involved, and the only simple case corresponds to the one–qubit system, $N = 2$.

2. Numerical range and derivative notions

The notion of numerical range of a given operator, also called *field of values* [8, Chapter 1], has been extensively studied during the last few decades [9, 10, 11] and its usefulness in quantum theory has been emphasized [12]. Several generalizations of numerical range are known — see *e.g.* [8, Section 1.8]. In particular, Marcus introduced the notion of *decomposable numerical range* [13, 14], the properties of which are a subject of considerable interest [15, 16].

Restricted numerical range [17, 18] and numerical shadow [19, 20, 21] are mathematical objects that have many applications in quantum information theory. For example it was shown in [18, Sec. E] that the problem of determining the product numerical range is strongly related to geometric measure of entanglement of any multipartite state $|\psi\rangle$ [22].

Other examples of restricted numerical ranges and restricted numerical shadows applications are the following:

- the block-positivity of matrices, and entanglement witnesses [18, Sec. IV. A.],
- the n-copy distillability of quantum states [18, Sec. IV. B.],
- the minimal output entropy of quantum channels [18, Sec. IV. C.],
- the local discrimination of unitary operations [18, Sec. IV. D.],
- the problems of local fidelity and geometric entanglement measures [18, Sec. E],
- the local dark spaces and quantum error correcting codes [18, Sec. IV. F],
- the local quantum control [23],
- the study of entanglement sudden death and entanglement revival [21, Sec. 6],
- quantum error correction [24].

2.1. Quantum states

Quantum states can be viewed as an extension of probability distributions. In order to set-up the vocabulary and description precision in the following section the main notions of the quantum information theory were gathered.

Let $\mathbb{C}^n$ be the complex euclidean space with standard inner product. By $|\psi\rangle \in \mathbb{C}^n$ we denote normed column vector, and by $\langle\psi|$ the row vector dual to $|\psi\rangle$. With symbol † we denote hermitian conjugation. The form $\langle\psi|\phi\rangle$ denotes the vector scalar product of $|\psi\rangle$ and $|\phi\rangle$, and the form $|\psi\rangle\langle\phi|$ denotes their outer product. By $L(\mathbb{C}^n)$ we denote the set of linear operators from $\mathbb{C}^n$ to $\mathbb{C}^n$.

Let $\rho \in \Omega_n \subset L(\mathbb{C}^n)$ be a trace-one positive semi-definite linear operator *i.e.*: $\rho = \rho^\dagger$, $\rho \geq 0$ and $\mathrm{tr}\rho = 1$. Such operators are called *quantum states*. The states that are rank-one projection operators are called *pure states*. States, whose all off-diagonal elements are zero are called *classical states* and they are identified with probability distributions.

Joint state ρ of quantum systems A and B, whose corresponding spaces have dimensions n_A and n_B, $\rho \in \Omega_{n_A \times n_B} \subset L(\mathbb{C}^{n_A \times n_B})$ are called *separable* if and only if, when for some M the state can be written in the form

$$\rho = \sum_{i=1}^{M} q_i \, \rho_i^A \otimes \rho_i^B, \tag{1}$$

where $\rho_i^A \in \Omega_{n_A}$, $\rho_i^B \in \Omega_{n_B}$, $q_i > 0$ i $\sum\limits_{i=1}^{M} q_i = 1$. A state that is not separable is called *entangled state*.

One can extend these notions to case of many subsystems. The state ρ of m subsystems $A_1, \ldots, A_m$, whose corresponding spaces have dimensions n_i such that, $\rho \in \Omega_{\prod_{i=1}^m n_{A_i}} \subset L(\mathbb{C}^{\prod_{i=1}^m n_{A_i}})$ is separable with respect to a given partition $\{I_1, \ldots, I_k\}$, where I_i are disjoint subsets of the indices $I = \{1, \ldots, m\}$, $\cup_{j=1}^k I_j = I$, if and only if for some M, it can be written as

$$\rho = \sum_{i=1}^{M} p_i \rho_i^{B_1} \otimes \ldots \otimes \rho_i^{B_k}, \tag{2}$$

where B_i denotes subsystem corresponding to the partition I_i, $\rho^{B_i} \in \Omega_{n_{B_i}}$, $i = 1, \ldots, k$, $p_i > 0$ and $\sum\limits_{i=1}^{M} p_i = 1$. A state that is not separable with respect to any non-trivial partition is called *fully entangled*. If a state, for some M, can be written as

$$\rho = \sum_{i=1}^{M} p_i \rho_i^{A_1} \otimes \ldots \otimes \rho_i^{A_m}, \tag{3}$$

it is called *fully separable*.

2.2. Numerical range and restricted numerical range of a matrix

Numerical range [8] of a matrix $A \in L(\mathbb{C}^n)$ is a subset of the complex plain given by

$$\Lambda(A) = \{\langle\psi|A|\psi\rangle : |\psi\rangle \in \mathbb{C}^n, \langle\psi|\psi\rangle = 1\}. \tag{4}$$

In works [18, 17, 23] the restricted numerical range of the matrix was introduced and investigated. Restricted numerical range with respect to the set Ω^X can be defined as

$$\Lambda^X(A) = \{\mathrm{tr}\rho A : \rho \in \Omega^X\}, \tag{5}$$

where Ω^X is a subset of the set of quantum states. In the aforementioned works the applications of such a family of mathematical object in quantum information theory was presented.

As examples of subsets Ω^X for which restricted numerical range can be determined one can enumerate

- the subset of pure separable states

$$\Omega^{\text{loc}} = \{\bigotimes_{i=1}^{N} |x_i\rangle\langle x_i| : \forall_{i=1,\ldots,N}|x_i\rangle \in \mathbb{C}^{(n_i)}, \langle x_i|x_i\rangle = 1\},$$

- the subset of separable states

$$\Omega^{\text{sep}} = \{\rho : \rho = \sum_{i=1}^{N} p_i \rho_i^A \otimes \rho_i^B\}.$$

Selection of those subsets is motivated by quantum informational properties of a quantum system.

2.3. Relation with entanglement

It was shown in [18, Sec. E] that problem of determining the product numerical range is strongly related to the geometric measure of entanglement of any multipartite state $|\psi\rangle$ [22]. The measure E_g is defined as

$$E_g(|\psi\rangle) = -\log \max_{U_{\text{loc}}} |\langle\psi|U_1 \otimes U_2 \otimes \cdots \otimes U_m|0, 0, \ldots, 0\rangle|^2. \tag{6}$$

Product numerical radius $L(\rho)$, the maximum absolute value of a product numerical range $\max|\Lambda(\rho)|$, is also useful in constructing a *natural entanglement test* [25]. It has been shown, that if

$$L(\rho) < \mathrm{tr}\rho^2, \tag{7}$$

then the quantum state is entangled. Based on the above fact an experimental implementation was proposed for entanglement detection [25].

3. Algorithm

In this section we provide an algorithm for approximating numerical range of a matrix. We focus on restricted numerical range with respect to the set of separable states. In this work we will always use the notion of separable states in context of fixed system partition. In the case of having given subsystems $X_1, \ldots, X_k$ we will consider the set of separable states defined as

$$\Omega^{\mathrm{sep}}_{X_1,\ldots,X_k} = \{\rho : \rho = \sum_{i=1}^{N} p_i \rho_i^{X_1} \otimes \ldots \otimes \rho_i^{X_k}\}.$$

The algorithm will also work properly in the case of unpartitioned system giving unrestricted numerical range.

Algorithm 1 *For given matrix A, fixed k-part partition of a system, and granularity of approximations T run the following procedure for $\theta = 2\pi m/T, m = 1, \ldots, T$:*

1. $B = e^{-\mathrm{i}\theta}A$
2. $H = \frac{1}{2}(B + B^\dagger)$
3. $\mathcal{P}_1^{\max}, \ldots, \mathcal{P}_k^{\max} = \underset{\mathcal{P}_1,\ldots,\mathcal{P}_k}{\mathrm{argmax}} \langle\psi(\mathcal{P}_1)| \otimes \ldots \otimes \langle\psi(\mathcal{P}_k)|H|\psi(\mathcal{P}_1)\rangle \otimes \ldots \otimes |\psi(\mathcal{P}_k)\rangle$
4. $|\psi_{\max}\rangle = |\psi(\mathcal{P}_1^{\max})\rangle \otimes \ldots \otimes |\psi(\mathcal{P}_k^{\max})\rangle$
5. $z = \langle\psi_{\max}|A|\psi_{\max}\rangle$
6. *append* $p_m = (\Re(z), \Im(z))$ *to list of results.*

Step 3 of the algorithm can be executed using annealing algorithm. The list of elements $(p_m)_{m=1}^T$ forms the approximation of the boundary of separable numerical range.

After the the procedure is finished, additionally, the convex hull of the points in the resulting list can be computed in order to approximate the separable numerical range better.

The presented method consists of iterative procedure that allows to compute points at the border of the product numerical range of a matrix A with given system partition. As restricted separable NR is a convex hull of restricted product numerical range *i. e.* $\Lambda^{\mathrm{sep}}(A) = \mathrm{conv}(\Lambda^{\mathrm{loc}}(A))$ it is sufficient to provide the solution. The computation is done by computing points that lay on the real axis at the border after appropriate rotation of the input matrix. To get the points at the real axis we can consider the hermitian part of the matrix and compute its numerical radius. By performing the computation for different rotations distributed along the interval $[0; 2\pi]$ one obtains an approximation of the whole border of the separable numerical range.

Computing the numerical radius requires the maximization over all product states. For this purpose we use state parametrization [26] for each of the subsystems separately. State of a N-dimensional subspace can be parametrized with a sequence of $2(N-1)$ real numbers $\mathcal{P} = (\{\xi_i\}_i, \{\theta_j\}_j)$, $i = 1, 2, \ldots, N-1$, $j = 1, 2, \ldots, N-1$. The parametrization is given by:

$$|\psi(\mathcal{P})\rangle = |\psi(\{\xi_i\}_i, \{\theta_j\}_j)\rangle = \begin{bmatrix} \sqrt{p_0} \\ \sqrt{p_1} e^{\mathrm{i}\xi_1} \\ \vdots \\ \sqrt{p_{N-1}} e^{\mathrm{i}\xi_{N-1}} \end{bmatrix}, \tag{8}$$

where $p_i = \sin^2\theta_{i-1} \prod_{j=i}^{N-1} \cos^2\theta_j$ and $\theta_0 = \pi/2$. This approach enables us to compute restricted numerical range by using restricted set of parameters in step 3 of the algorithm. The state of the whole system is a product of states obtained for local parametrizations.

4. Results

In principle by finding an appropriate matrix and calculating its restricted numerical range one can provide a proof of the non-separability of a quantum state.

Various matrices can provide different separability criteria. Obviously, for a given state, finding a matrix that allows to determine that the state is not separable according to a given partition of the system and thus it is entangled, is a difficult task.

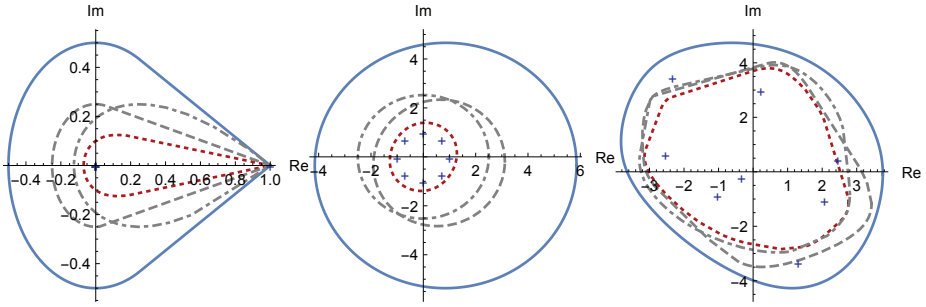

Fig. 1: Numerical range (blue solid line), and separable numerical range with respect to partition $2 \times 2 \times 2$ (red dotted line), 4×2 (dashed line), 2×4 (dot-dashed line) of (a) matrix A_1, (b) matrix A_2, (c) a generic (drawn from Ginibre ensemble) non-normal matrix of dimension 8. Crosses indicate spectrum of the matrix.

Below we provide three examples. Consider three matrices

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{9}$$

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 10 \\ 0 & e^{\frac{i\pi}{4}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{\frac{3i\pi}{4}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-\frac{3i\pi}{4}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{i\pi}{4}} \end{pmatrix}, \tag{10}$$

and A_3 — a generic matrix drawn from Ginibre ensemble [27]. Illustrations of the way Algorithm 1 approximates separable numerical ranges for the above matrices are provided in Fig. 1.

The figure depicts the numerical range and the separable numerical ranges of various 8×8 matrices. The external border contains images $\mathrm{tr}A\rho$ of all quantum states. The states whose images lay outside the grey dashed-dotted region are not 2×4 separable, similarly those states whose images lay outside of the dashed region are not 4×2 separable. Those states whose images lay outside the red dotted region are genuinely entangled.

Acknowledgments

PS was supported by the Polish Ministry of Science and Higher Education within "Diamond Grant" Programme under the project number 0064/DIA/2013/42.

5. Streszczenie

W pracy przedstawiamy metodę przybliżania separowalnego obrazu numerycznego macierzy. Przybliżamy również związek pomiędzy ograniczonymi obrazami numerycznymi macierzy oraz splątaniem stanów kwantowych. Wskazujemy możliwość zbadania separowalności stanu używając wyliczonego ograniczonego obrazu numerycznego. W szczególności przedstawiamy metodę uzyskania kryterium separowalności, dla dowolnego podziału układów kwantowych, używając separowalnego obrazu numerycznego.

References

[1] C. H. Bennett and G. Brassard. Quantum cryptography: Public key distribution and coin tossing. In *Proceedings of IEEE International Conference on Computers, Systems and Signal Processing*, volume 175. Bangalore, India, 1984.

[2] R. P. Feynman. Simulating physics with computers. *International journal of theoretical physics*, 21(6):467–488, 1982.

[3] D. Deutsch. Quantum theory, the Church-Turing principle and the universal quantum computer. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, 400(1818):97–117, 1985.

[4] P. W. Shor. Algorithms for quantum computation: discrete logarithms and factoring. In *Foundations of Computer Science, 1994 Proceedings., 35th Annual Symposium on*, pages 124–134. IEEE, 1994.

[5] L. K. Grover. A fast quantum mechanical algorithm for database search. In *Proceedings of the twenty-eighth annual ACM symposium on Theory of computing*, pages 212–219. ACM, 1996.

[6] I. Bengtsson and K. Życzkowski. *Geometry of quantum states: an introduction to quantum entanglement*. Cambridge University Press, 2006.

[7] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki. Quantum entanglement. *Reviews of Modern Physics*, 81(2):865, 2009.

[8] R. A. Horn and C. R. Johnson. *Topics in Matrix Analysis*. Cambridge University Press, 1994.

[9] K. E. Gustafson and D. K. M. Rao. *Numerical Range: The Field of Values of Linear Operators and Matrices*. Springer-Verlag, New York, 1997.

[10] T. Ando and C. K. Li. Special issue: The numerical range and numerical radius. *Linear and Multilinear Algebra*, 37(1–3), 1994. Ando, T. and Li, C. K., editors.

[11] C. K. Li. *C*-numerical ranges and *c*-numerical radii. *Linear and Multilinear Algebra*, 37(1–3):51–82, 1994.

[12] D. W. Kribs, A. Pasieka, M. Laforest, C. Ryan, and M. P. Silva. Research problems on numerical ranges in quantum computing. *Linear and Multilinear Algebra*, 57:491–502, 2009.

[13] M. Marcus. *Finite Dimensional Multilinear Algebra. Part I.* Marcel Dekker, New York, U.S.A., 1973.

[14] M. Marcus and B. Wang. Some variations on the numerical range. *Linear and Multilinear Algebra*, 9:111–120, 1980.

[15] N. Bebiano, C. K. Li, and J. da Providencia. The numerical range and decomposable numerical range of matrices. *Linear and Multilinear Algebra*, 29:195–205, 1991.

[16] C. K. Li and A. Zaharia. Induced operators on symmetry classes of tensors. *Trans. Am. Math. Soc.*, pages 807–836, 2001.

[17] Z. Puchała, P. Gawron, J.A. Miszczak, Ł. Skowronek, M .D. Choi, and K. Życzkowski. Product numerical range in a space with tensor product structure. *Linear Algebra and its Applications*, 434(1):327–342, 2011.

[18] P. Gawron, Z. Puchała, J.A. Miszczak, Ł. Skowronek, and K. Życzkowski. Restricted numerical range: A versatile tool in the theory of quantum information. *Journal of Mathematical Physics*, 51:102204, 2010.

[19] C. F. Dunkl, P. Gawron, J. A. Holbrook, J. A. Miszczak, Z. Puchała, and K. Życzkowski. Numerical shadow and geometry of quantum states. *Journal of Physics A: Mathematical and Theoretical*, 44:335301, 2011.

[20] C.F. Dunkl, P. Gawron, J.A. Holbrook, Z. Puchala, and K. Życzkowski. Numerical shadows: measures and densities on the numerical range. *Linear Algebra and its Applications*, 2011.

[21] Z. Puchała, J. A. Miszczak, P. Gawron, C. F. Dunkl, J. A. Holbrook, and K. Życzkowski. Restricted numerical shadow and geometry of quantum entanglement. *Journal of Physics A: Mathematical and Theoretical*, 45:415309, 2012.

[22] T.-C. Wei and P. M. Goldbart. Geometric measure of entanglement and applications to bipartite and multipartite quantum states. *Physical Review A*, 68(4):042307, 2003.

[23] T. Schulte-Herbrüggen, G. Dirr, U. Helmke, and S. J. Glaser. The significance of the C-numerical range and the local c-numerical range in quantum control and quantum information. *Linear and Multilinear Algebra*, 56(1-2):3–26, 2008.

[24] C.-K. Li and Y.-T. Poon. Quantum error correction and generalized numerical ranges. *arXiv preprint arXiv:0812.4772*, 2008.

[25] Ł. Rudnicki, Z. Puchała, P. Horodecki, and K. Życzkowski. Constructive entanglement test from triangle inequality. *Journal of Physics A: Mathematical and Theoretical*, 47(42):424035, 2014.

[26] T. Radtke and S. Fritzsche. Simulation of n-qubit quantum systems. iv. parametrizations of quantum states, matrices and probability distributions. *Computer Physics Communications*, 179(9):647–664, Nov 2008.

[27] B. Collins, P. Gawron, A. E. Litvak, and K. Życzkowski. Numerical range for random matrices. *Journal of Mathematical Analysis and Applications*, 418(1):516–533, 2014.