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# A non integer order, state space model for one dimensional heat transfer process

KRZYSZTOF OPRZEDKIEWICZ and EDYTA GAWIN

In the paper a new, state space, non integer order model for one dimensional heat transfer process is presented. The model is based on known semigroup model. The derivative with respect to time is described by the non integer order Caputo operator, the spatial derivative is described by integer order operator. The elementary properties of the state operator are proven. The solution of state equation is calculated with the use of Laplace transform. Results of experiments show, that the proposed model is more accurate than analogical integer order model in the sense of square cost function.

**Key words:** non-integer order systems, heat transfer process, diffusion equation, infinite dimensional systems, Feller semigroups.

## 1. Introduction

Mathematical models of distributed parameter systems obtained on the basis of partial differential equations can be described in an infinite-dimensional state space, usually in a Hilbert space, but Sobolev space can also be applied. This problem has been analyzed by many authors. Fundamentals are given for example in [13]. Analysis of a hyperbolic system in Hilbert space is presented in [2]. Other references are given in the rest of the paper.

The modeling of processes and phenomena which are hard to desrcribe with the use of standard tools is one of main areas of application of non integer order calculus. Non integer models of number physical phenomena were presented by many authors, for example by [20], [6], [3], [4], [15], [22]. Analysis of anomalous diffusion problem with the use of fractional order approach and semigroup theory was presented for example by [21].

It is well known, that heat transfer processes can be also modeled with the use of non integer order approach. This problem has been investigated for example by [14], [15],

K. Oprzedkiewicz (the corresponding author) is with AGH University of Science and Technology in Krakow, Poland (Faculty of Electrical Engineering, Automatics, Computer Science and Biomedical Engineering, Dept. of Automatics and Biomedical Engineering), e-mail: kop@agh.edu.pl. E. Gawin is with High Vocational School in Tarnów, Polytechnic Institute, Poland.

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[5]. It is important to notice that all known models have a form of a transfer function or a partial differential equation. The non integer order state-space model for heat transfer process has not been presented yet. This paper presents a proposition of a new, state-space model for heat transfer process in one dimensional plant. An idea of this model bases directly on semigroup model for one-dimensional heat transfer problem.

The paper is organized as follows. At the beginning elementary ideas and definitions are recalled. Next the considered, infinite order plant and its integer order, semigroup model are presented. Then the proposition of non integer order model and its elementary properties are given. The proposed model is next verified with the use of experimental results.

## 2. Preliminaries

The set of elementary ideas and definitions recalling an idea of Gamma Euler function (see for example [11]) are given below.

**Definition 5** *The Gamma function is defined as follows*

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt. \tag{1}$$

The idea of Mittag-Leffler is now recalled. It is a non integer order generalization of exponential function $e^{\lambda t}$. It plays a crucial role in solution of fractional order (FO) state equation. The one parameter Mittag-Leffler function is defined as follows:

**Definition 6** *The one parameter Mittag-Leffler function is given by*

$$E_{\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha + 1)}. \tag{2}$$

The two parameter Mittag-Leffler function is also used.

**Definition 7** *The two parameters Mittag-Leffler function is given by*

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha + \beta)}. \tag{3}$$

Clearly, for $\beta = 1$ the two parameter function (3) turns to one parameter function (2).

The fractional order, integro-differential operator can be described by different definitions, as e.g. given by Grünvald and Letnikov, Riemann and Liouville and Caputo. All these definitions are gathered below. With respect to particular additional assumptions these definitions are equivalent.

**Definition 8** *The Grünvald-Letnikov* (GL) *definition of the FO operator* ( [3], [19])*:*

$$ {}_0^{GL}D_t^{\alpha} f(t) = \lim_{h \to 0} h^{-\alpha} \sum_{j=0}^{[\frac{t}{h}]} (-1)^j \binom{\alpha}{j} f(t - jh). \tag{4}$$

*where* $\binom{\alpha}{j}$ *is a generalization of Newton symbol into real numbers:*

$$\binom{\alpha}{j} = \begin{cases} 1, & j = 0 \\ \frac{\alpha(\alpha-1)\dots(\alpha-j+1)}{j!}, & j > 0. \end{cases} \tag{5}$$

**Definition 9** *The Riemann-Liouville* (RL) *definition of the FO operator has the form*

$$ {}_0^{RL}D_t^{\alpha} f(t) = \frac{1}{\Gamma(N-\alpha)} \frac{d^N}{dt^N} \int_0^{\infty} (t-\tau)^{N-\alpha-1} f(\tau) d\tau \tag{6}$$

*where* $N-1 < \alpha < N$ *denotes the non integer order of operation and* $\Gamma$ *is the complete Gamma function expressed by* (5)*.*

The Caputo definition is as follows.

**Definition 10** *The Caputo* (C) *FO operator is defined by*

$$ {}_0^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(N-\alpha)} \int_0^{\infty} \frac{f^{(N)}(\tau)}{(t-\tau)^{\alpha+1-N}} d\tau. \tag{7}$$

For the RL or C operators the Laplace transform can be defined (see for example [9]).

**Definition 11** *The Laplace transform of Riemann-Liouville operator is given by*

$$\begin{aligned} &\mathcal{L}({}_0^{RL}D_t^{\alpha} f(t)) = s^{\alpha} F(s), \;\; \alpha < 0 \\ &\mathcal{L}({}_0^{RL}D_t^{\alpha} f(t)) = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^k {}_0D_t^{\alpha-k-1} f(0), \\ &\alpha > 0, \;\; n-1 < \alpha \leqslant n \in N. \end{aligned} \tag{8}$$

**Definition 12** *The Laplace transform for Caputo operator is defined by*

$$\begin{aligned} &\mathcal{L}({}_0^{C}D_t^{\alpha} f(t)) = s^{\alpha} F(s), \;\; \alpha < 0 \\ &\mathcal{L}({}_0^{C}D_t^{\alpha} f(t)) = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} {}_0D_t^{k} f(0), \\ &\alpha > 0, \;\; n-1 < \alpha \leqslant n \in N. \end{aligned} \tag{9}$$

Consequently, the inverse Laplace transform for non integer order function is expressed as follows (see for example [11] p.299):

$$\mathcal{L}^{-1}[s^{\alpha}F(s)] = {}_0D_t^{\alpha}f(t) + \sum_{k=0}^{n-1}\frac{t^{k-1}}{\Gamma(k-\alpha+1)}f^{(k)}(0^+)n-1<\alpha<n, \;\; n\in\mathbb{Z} \tag{10}$$

Next a linear, fractional order state equation can also be defined. It has the following form (see for example [8]), [10], [1]):

$$\begin{aligned}\frac{d^{\alpha}x(t)}{dt^{\alpha}} &= Ax(t)+Bu(t)\\ y(t) &= Cx(t)\end{aligned} \tag{11}$$

where $x(t)\in\mathbb{R}^N$ is the state vector, $u(t)\in\mathbb{R}^P$ is the control vector and $y(t)\in\mathbb{R}^R$ is the output vector, $0<\alpha<1$ is the fractional order of the equation.

If the FO operator is described with the use of Caputo definition (7), then it can be proven, that the solution of linear state equation (11) has the following form (see for example [11] p.12 ):

$$x(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau \tag{12}$$

where:

$$\Phi_0(t) = E_{\alpha}(At^{\alpha}) = \sum_{k=0}^{\infty}\frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)} \tag{13}$$

$$\Phi(t) = \sum_{k=0}^{\infty}\frac{A^k t^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)}. \tag{14}$$

## 3. The considered plant and its integer order model

Let us consider an experimental heat plant shown in Fig. 1. It has the form of a thin copper rod 30 cm long. It is heated by an electric heater of the length $\Delta x_u$ attached to the end of the rod. Resistance temperature detector of the length $\Delta x$ is placed at the other end. The input signal of the system is the standard current signal of the range 0–5 [mA]. It is amplified to the range 0–1.5 [A] to supply the heater. The signal from the resistance sensor is transformed to the standard current signal 0–5 [mA] by the transducer. The step response of the considered plant is shown in Fig. 2.

The fundamental mathematical model describing the heat conduction in the plant is the partial differential equation of the parabolic type with the homogeneous Neumann boundary conditions at the ends, the homogeneous initial condition, the heat exchange

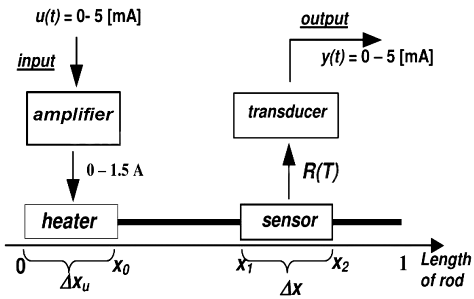


Figure 1. An experimental heat plant

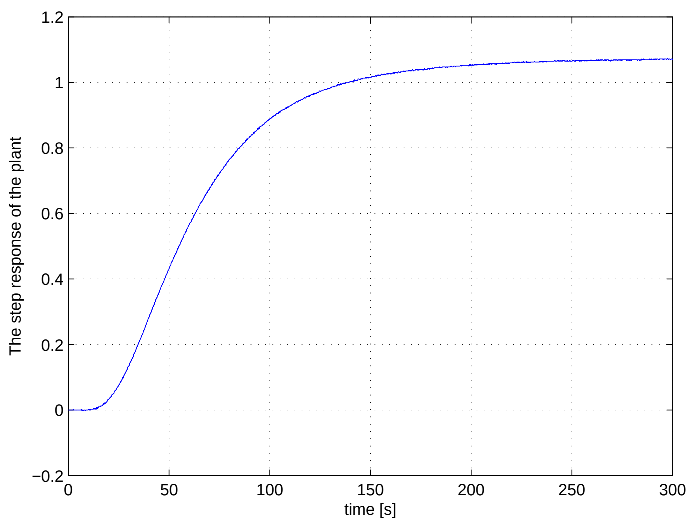


Figure 2. The step response of the experimental heat plant

along the length of rod and distributed control and observation. This equation has the following form (see [16] [18]):

$$\begin{cases} \dfrac{\partial Q(x,t)}{dt} = a\dfrac{\partial^2 Q(x,t)}{\partial x^2} - R_a Q(x,t) + b(x)u(t), \\ \dfrac{\partial Q(0,t)}{dx} = 0, t \geqslant 0 \\ \dfrac{\partial Q(1,t)}{dx} = 0, t \geqslant 0 \\ Q(x,0) = 0, 0 \leqslant x \leqslant 1 \\ y(t) = y_0 \int_0^1 Q(x,t)c(x)dx \end{cases} \tag{15}$$

where $Q(x,t)$ denotes temperature in time $t$ and point $x$, $R_a$, $a$ denotes uncertain coefficients of heat conduction and heat exchange, $b(x)$ denotes the control function, $c(x)$ is an observation function and $y_0$ denotes the steady state gain of the system.

The heat equation (15) can be shown as an equivalent abstract initial problem in Hilbert space $X = L^2(0,1)$ with standard scalar product. This issue was discussed for example in [13]. In our case the abstract form of the heat equation (15) is as follows:

$$\begin{cases} \dot{Q}(t) = AQ(t) + Bu(t) \\ Q(0) = 0 \\ y(t) = y_0 CQ(t) \end{cases} \tag{16}$$

where:
$AQ = aQ'' - R_a Q$,
$D(A) = \left\{Q \in H^2(0,1) : Q'(0) = 0, Q'(1) = 0\right\}$,
$a, R_a > 0$,
$H^2(0,1) = \left\{u \in L^2(0,1) : u', u'' \in L^2(0,1)\right\}$,
$CQ(t) = \langle c, Q(t)\rangle, Bu(t) = bu(t)$,
$\langle u, v\rangle = \int_0^1 u(x)v(x)dx$ (denotes the standard scalar product).

The following set of the eigenvectors for the state operator $A$ creates the orthonormal basis of the state space:

$$h_i = \begin{cases} 0, i = 0 \\ \sqrt{2}cos(i\pi x), i = 1, 2, ... \end{cases} \tag{17}$$

The discrete spectrum of the state operator for integer order model $A$ is a set of the single, real eigenvalues, which are expressed as follows:

$$\lambda_i = -a\pi^2 i^2 - R_a, i = 0, 1, 2, .... \tag{18}$$

In the state space basis defined by set of eigenvectors (17) the operators $A$, $B$ and $C$ have the following matrix representation:

$$A = diag\{\lambda_0, \lambda_1, \lambda_2, ..\} \tag{19}$$

$$B = [b_0, b_1, b_2, ...]^T \tag{20}$$

where $b_i = \langle b, h_i\rangle$, $b(x)$ denotes the control function:

$$b(x) = \begin{cases} 1, x \in [0, x_0] \\ 0, x \notin [0, x_0] \end{cases} \tag{21}$$

$$C = [c_0, c_1, c_2, ...] \tag{22}$$

and where $c_i = \langle c, h_i \rangle$, $c(x)$ denotes the output sensor function:

$$c(x) = \begin{cases} 1, x \in [x_1, x_2] \\ 0, x \notin [x_1, x_2]. \end{cases} \tag{23}$$

From (21) and (23) it turns out, that the control function $b(x)$ and output function $c(x)$ are the interval constant functions. Assume, that the control function $u(t) = 1(t)$.Then the solution of (16) has the following form [16]:

$$y_{IO}(t) = y_0 \sum_{i=1}^{\infty} \left( \frac{e^{\lambda_i t} - 1}{\lambda_i} \right) \langle b, h_i \rangle \langle c, h_i \rangle. \tag{24}$$

The basic features of the discussed parabolic IO system have been analyzed. It can be proved, that the state operator $A$ for considered system is: negative, self-adjoint and it has the compact inverse operator. For known coefficients $a$ and $R_a$ equations (16)-(23) give a good description of considered real experimental heat object. By the "cutting" the further elements of infinite-dimensional operators $A$, $B$ and $C$ we obtain its finite-dimensional approximation, which is a useful tool for numeric modelling of discussed plant. In this case the operators $A$, $B$ and $C$ can be interpreted as matrices. If these parameters are not exactly known, then an interval model can be applied (see [16], [17], [18]).

It is important to notice that the approach using the semigroup theory presented above can be extended to non integer order systems. This problem is discussed for example in [21], [23].

## 4. The proposed non integer order model of the system

The proposed, time non integer order model is obtained after replacing the 1'st order differential with respect to time by suitable non integer order differential. This is motivated by the fact that the dynamics of the heat exchange between the heater and the rod and between the rod and the sensor is not exactly described by equation (15). The non integer order differentiation is expected to provide better description of these processes.

Assume that non integer order operator is described by the Caputo definition (7). Then the heat transfer equation turns to the following form:

$$\begin{cases} {}^C D_t^{\alpha} Q(x,t) = a \dfrac{\partial^2 Q(x,t)}{\partial x^2} - R_a Q(x,t) + b(x) u(t) \\ \dfrac{\partial Q(0,t)}{dx} = 0, t \geqslant 0 \\ \dfrac{\partial Q(1,t)}{dx} = 0, t \geqslant 0 \\ Q(x,0) = 0, 0 \leqslant x \leqslant 1 \\ y(t) = y_0 \int_0^1 Q(x,t) c(x) dx \end{cases} \tag{25}$$

where $0 < \alpha < 2$ denotes the non integer order of the system. Other parameters are the same as in IO model (16). Now we need to express (25) as the state equation in the Hilbert space analogically to (16). The proposed state equation can be written as:

$$\begin{cases} {}^{C}D_t^{\alpha}Q(t) = AQ(t) + Bu(t) \\ Q(0) = 0 \\ y(t) = y_0 CQ(t) \end{cases} \tag{26}$$

where:

$$\begin{cases} AQ = aQ'' - R_a Q, \\ D(A) = \left\{ Q \in H^2(0,1) : Q'(0) = 0, Q'(1) = 0 \right\}, \\ a, R_a > 0, \\ H^2(0,1) = \left\{ u \in L^2(0,1) : u', u'' \in L^2(0,1) \right\}, \\ CQ(t) = \langle c, Q(t) \rangle, Bu(t) = bu(t) \end{cases} \tag{27}$$

Here we can apply Feller semigroups, which are presented in [7], [12] and [21]. The semigroup definition is described as follows.

**Definition 13** *The semigroup is a set S coupled with a binary operation* $T\,(T : S \times S \to S)$ *which is associative. That is,* $\forall x, y, z \in S, \quad T(T(x,y),z) = T(x, T(y,z))$.

The uniqueness of the solution gives reveals the semigroup property, which is given by:

$$T(t+s) = T(t)T(s), t, s > 0 \tag{28}$$

The semigroup property (28) of the family of functions, $\{T(t) : t \geqslant 0\}$, is a composition. Notice that $T(0)$ is the identity operator.

A strongly continuous positive contraction semigroup on $C_\infty(S)$ is called a Feller semigroup on $S$. We have differential representation of the operators which has a form of Feller semigroup.

According to [21] we conclude, that the non integer order system can be expressed as

$${}^{C}D_t^{\alpha}Q(t) = AQ(t) + Bu(t), Q(0) = 0$$

where $0 < \alpha < 1, t \geqslant 0$ and $A$ is the generator of bounded continuous Feller semigroup $T(t)_{t \geqslant 0}$ in the Hilbert space $H^2(0,1)$.

Elementary properties of the parabolic IO system have been analyzed and it can be proved, that the state operator $A$ for integer order system described by (19) is: negative, self-adjoint and has the compact inverse operator.

Now can formulate an analogous properties for non integer system.

**Proposition 1** Elementary properties of state operator $A$
*The state operator A for non integer order system described by* (26) *is: negative, self-adjoint and has the compact inverse operator.*

**Proof** Let $P \in H^2(0,1)$ then by (27)

$$\begin{aligned}\langle P, AQ\rangle = \int_0^1 P(x)[aQ''(x) - R_a Q(x)]dx = a\int_0^1 P(x)Q''(x)dx - R_a\langle P,Q\rangle = \\ a(P(1)Q'(1) - P(0)Q'(0)) - a\int_0^1 P'(x)Q'(x)dx - R_a\langle P,Q\rangle = \\ -aQ(1)P'(1) + \int_0^1 [aP''(x) - R_a P(x)]Q(x)dx \supset Q \in D(a).\end{aligned} \tag{29}$$

By the theorem of Riesz functional linear and limited in Hilbert space

$$H^2(0,1) \supset D(A) \ni Q \to \int_0^1 [aP''(x) - R_a P(x)Q(x)]dx \in H^2(0,1) \tag{30}$$

extends to functional linear and limited on $H^2(0,1)$.

Other expressions represent functional linear densely defined on $H^2(0,1)$, extended to functional linear and limited defined on the whole $H$, only if $p'(1) = 0$. Therefore, in accordance with the definition of the operator conjugated we have

$$D(A^*) = \left\{P \in H^2(0,1) : P'(0) = 0, P'(1) = 0\right\} = D(A) \tag{31}$$

and

$$A^*P = aP'' - R_a P = AP \tag{32}$$

which means that the operator A is self-adjoint. Furthermore

$$\begin{aligned}\langle Q, AQ\rangle = \int_0^1 Q(x)[aQ''(x) - R_a Q(x)]dx = \\ aQ(1)Q'(1) - aQ(0)Q'(0) - a\int_0^1 [Q'(x)]^2 dx - R_a||Q^2|| \leqslant -R_a||Q^2|| \supset Q \in D(A)\end{aligned} \tag{33}$$

therefore the operator $-A + R_a I$ is non-negative.

To prove the compactness $A^{-1}$ we consider the auxiliary operator $G$:

$$GQ = Q'', D(G) = D(A).$$

We need to solve the following system of equations:

$$\begin{cases} Q''(x) = f(x), 0 \leqslant x \leqslant 1 \\ Q'(1) = 0 \\ Q'(0) = 0. \end{cases} \tag{34}$$

The system has exactly one solution, namely

$$Q(x) = -\int_0^1 y f(y) dy + \int_0^x (x-y) f(y) dy$$

thus, the operator exists and has the form:

$$G^{-1} = \langle g, f \rangle h + Kf,$$

where

$$\begin{aligned} g, h \in H^2(0,1), \\ g(x) = x, \\ h(x) = -1, 0 \leqslant x \leqslant 1, \end{aligned} \tag{35}$$

and $K$ is an integral operator

$$(Kf)(x) = \int_0^x k(x-y) f(y) dy$$

with kernel

$$k(x,y) = \begin{cases} x-y, 0 \leqslant y \leqslant x \\ 0, x \leqslant y \leqslant 1. \end{cases} \tag{36}$$

$K$ is a Hilbert-Schmidt operator, so the operator $G^{-1}$ is compact. We conclude then, that $G$ has a compact resolvent $(\lambda I - G)^{-1}$, and

$$A^{-1} = \frac{1}{a}(-\frac{R_a}{a} I + G)^{-1}.$$

It means finally that the operator $A^{-1}$ is compact. □

The solution of state equation (26) can be calculated with the use of Laplace transform for C operator defined by (10) with assumptions that initial condition is equal zero: $Q(x,0) = 0, 0 \leqslant x \leqslant 1$ and state and control operators are described by (19)-(21). Assuming the control signal of the Heaviside function form $u(t) = 1(t)$ and applying (10), we obtain the solution as follows:

$$y_{FO}(t) = y_0 \sum_{n=1}^{\infty} \frac{(E_\alpha(\lambda_n t^\alpha) - 1(t))}{\lambda_n} \langle b, h_n \rangle \langle c, h_n \rangle. \tag{37}$$

Note, that the same solution can be calculated with the use of (12):

$$\begin{gathered}Q(t)=\Phi_0(t)x_0+\int_0^t \Phi(t-\tau)Bd\tau=\\ \sum_{k=0}^{\infty}\frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}x_0+\int_0^t\sum_{k=0}^{\infty}\frac{A^k(t-\tau)^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)}Bd\tau=\\ \sum_{k=0}^{\infty}\frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}x_0+B\sum_{k=0}^{\infty}\frac{A^k t^{(k+1)\alpha}}{\Gamma((k+1)\alpha)(k+1)\alpha}.\end{gathered}\tag{38}$$

## 5. Experimental results

Experiments were done using experimental system shown in Fig. 1. The step response of the system was tested in time range from 0 to $T_f = 300[s]$. Parameters of the proposed new model were calculated with the use of least square method and typical cost function:

$$I_1=\int_0^{T_f}(y-y_e)^2dt.\tag{39}$$

Optimal parameters in the sense of the above cost function are given in table 3. Diagrams of the absolute error of approximation for model size $N = 15$ and $N = 25$ are given in Figs. 3 and 4, respectively. The comparison is also illustrated by the results for integer order model given in the table 3 and shown in Fig. 5.

Table 3. Parameters of IO and FO models and cost function (39).

| Model | Integer Order | | Non Integer Order | |
|---|---|---|---|---|
| order $N$ | 15 | 25 | 15 | 25 |
| $\alpha$ | 1 | 1 | 0.9752 | 0.9706 |
| $a_w$ | 0.0009 | 0.0009 | 0.0008 | 0.0009 |
| $R_a$ | 0.0275 | 0.0274 | 0.0385 | 0.0391 |
| Cost function(39) | 0.0072 | 0.0056 | 0.0036 | 7.7362e-04 |

From Tab. 3 and figures 3, 4, 5 we can conclude immediately, that the proposed non integer order model describes the considered infinite-dimensional plant better than integer order model presented in the first part of this paper.

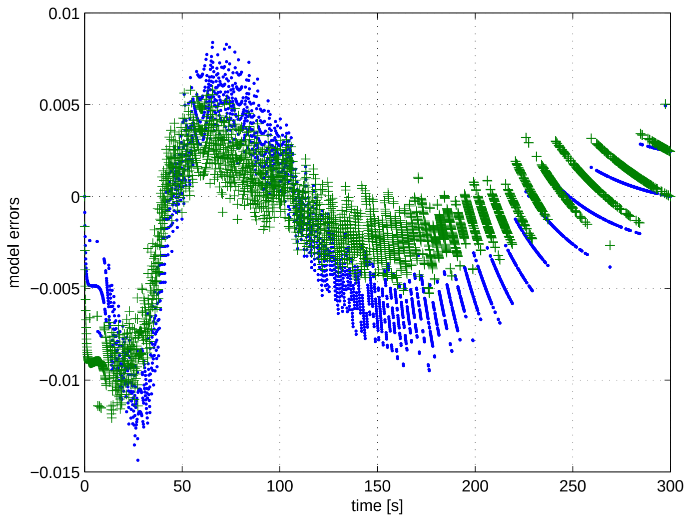


Figure 3. The absolute error for integer order (.) and non integer order (+) models, order $N = 15$.

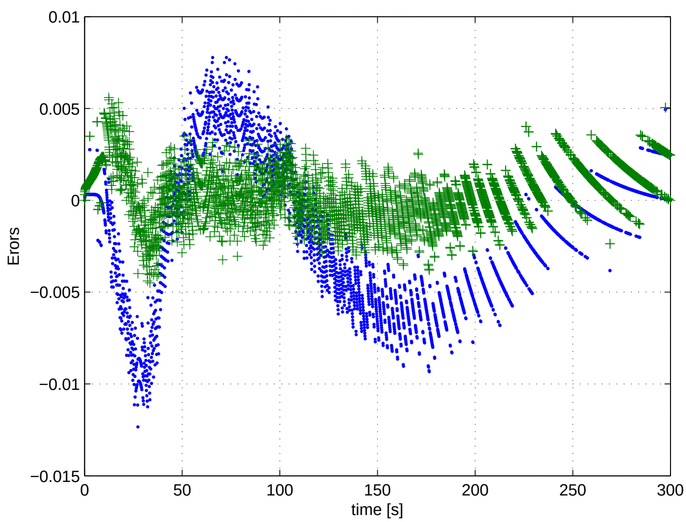


Figure 4. The absolute error for integer order (.) and non integer order (+) models, order $N = 25$.

## 6. Final conclusions

We can conclude that the proposed fractional order, state space model for one dimensional heat plant can be built via generalization of integer order, abstract model. Furthermore the accuracy of the proposed model in the sense of square cost function is better than analogical integer order model. Parameters of the proposed model can be numerically calculated.

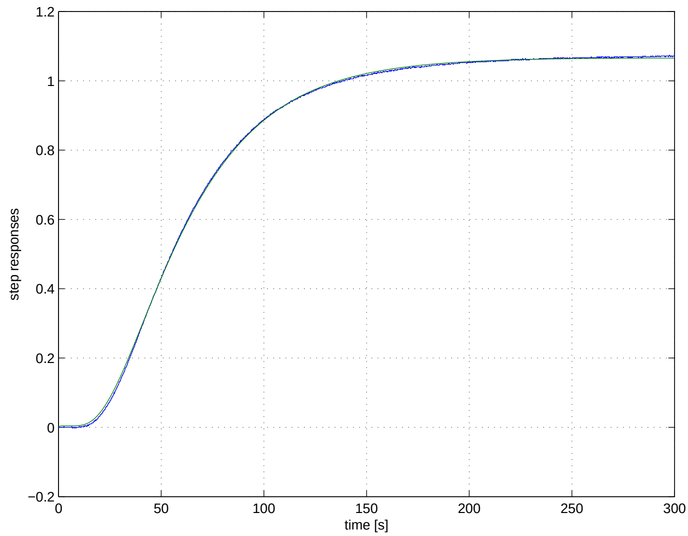


Figure 5. The step responses of real plant and non integer order model for order $N = 25$.

Further analysis of the presented problem will cover proposition and investigation of the state space, fractional order model with non integer order space derivative, described by Riesz operator and the next model with both derivatives expressed by non integer order operators.

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