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ORBITS OF THE KAPREKAR'S TRANSFORMATIONS – SOME INTRODUCTORY FACTS

Summary. Presented paper, above all, completes two other papers, made previously by the authors and cited in References, concerning the orbits of the Kaprekar's transformations. In the current paper many detailed facts for five initial Kaprekar's transformations (from T_2 to T_6) are described. There are introduced some new concepts and there is shown how important is the observation of numerical results giving the motivation for theoretical discussion on the Kaprekar's transformations. Moreover, the paper includes the interesting and original results concerning the fixed points and the 2-element orbits of the Kaprekar's transformations. All these results encourage to continue the discussion. The paper contains also quite large survey section devoted to the generalizations and modifications of the Kaprekar's transformations. Furthermore, some pieces of information from OEIS by N.J.A. Sloane connected with the orbits of the Kaprekar's transformations are presented here.

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PODSTAWOWE FAKTY O ORBITACH TRANSFORMACJI KAPREKARA

Streszczenie. Prezentowany artykuł uzupełnia dwie inne prace autorów, cytowane w dołączonej tu literaturze, dotyczące orbit transformacji Kaprekara. W pracy przedstawiono wiele szczegółowych faktów dla pierwszych pięciu transformacji Kaprekara (od T_2 do T_6). Wprowadzono nowe pojęcia i pokazano, jak istotna jest analiza wyników numerycznych jako motywacja do dyskusji teoretycznej transformacji Kaprekara. Ponadto w artykule zamieszczono interesujące i oryginalne wyniki dotyczące punktów stałych i 2-elementowych orbit transformacji Kaprekara. Wszystkie te wyniki wręcz zachęcają do dalszej dyskusji. Artykuł zawiera również obszerny rozdział dotyczący uogólnień i modyfikacji transformacji Kaprekara. Wreszcie mamy tu zamieszczoną garść informacji z OEIS, ufundowanej przez N.J.A. Sloane'a, związanych z orbitami transformacji Kaprekara.

1. Introduction

The current paper represents an essential supplement of two previous papers made by the authors, that is [16] and [17]. Two more papers sacrificed to the investigated subject, that is [13] and [15], are now prepared for publication by the same team of authors. The present paper is almost an independent work, excluding few selected concepts adopted from [16] and [17] (concerning especially the characteristics of the orbits of transformations, for example the Sharkovsky's order or the strong Sharkovsky's order). At the end of this paper the tables of minimal orbits of the Kaprekar's transformations T_n, for $n = 16, 17$ and 18, are included which complete the tables contained in [17]. Moreover, apart from the collection of concepts and notations (many of them are new in relation to [16] and [17]), there are discussed here "in details" the minimal orbits and the fixed points of Kaprekar's transformations T_n for $n = 2, 3, ..., 6$. Some new sequences of the fixed points and the 2-element orbits of the Kaprekar's transformations T_{an+b}, $n \in \mathbb{N}$, for the given positive integers a, b (see Chapters 3 and 4) are introduced here. In Chapter 5 devoted to the survey of many generalizations and modifications of the Kaprekar's transformations, except the broad literature presented there (even though we are familiar with a huge amount of sources devoted to the subject and the presented paper is long enough, we only managed to touch the selected threads) and the interesting factual materials, we undertook at the end of this chapter very surprising research topics. Our discoveries are related to the Ducci's transformations and some unusual recurrence sequence – we revealed some new threads connected with extending the discussion on the complex domain and we formulated few new problems. In Chapter 6 we discussed shortly the information about the "presence" of the orbits of Kaprekar's transformations in OEIS (i.e. Sloane's Online Encyclopedia of Integer Sequences).

2. Basic definitions and notions

Let us fix $n \in \mathbb{N}$, $n \geqslant 2$. Let $\alpha \in \mathbb{N}$ be any n-digit number in its decimal expansion, the digits of which are ordered in the following nondecreasing sequence

$$0 \leqslant a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n \leqslant 9.$$

Let us also assume that at least two digits, from among these ones above, are different, that is the condition $a_1 \neq a_n$ is satisfied. We take

$$T_n(\alpha) := \sum_{k=1}^{n} (a_k - a_{n-k+1}) 10^{k-1} = a_n a_{n-1} \ldots a_1 - a_1 a_2 \ldots a_n. \tag{1}$$

The map T_n is called the n-th Kaprekar's transformation. We will also use the expression: odd or even Kaprekar's transformation, depending on the parity of n.

Let us notice that in the decimal expansion of number $T_n(\alpha)$ at least two digits are then different, additionally $T_n(\alpha) < 10^n$, and finally, by completing, if necessary, the number obtained according to formula (1) with the appropriate number of zeros, we assume also that $10^{n-1} - 1 \leqslant T_n(\alpha)$. A reason of such (a little peculiar) action is the following fact (see also Theorem 2 or rather the proof of this theorem for an analytic argument):

$$T_n(a_1 a_2 \ldots a_n) = T_n(a_{\sigma(1)} a_{\sigma(2)} \ldots a_{\sigma(n)})$$

for any permutation $\sigma \in S_n$, where, as usually, S_n denotes the set of all n-element permutations (in other words, the set of elements of the symmetric n-group).

For example, we have

$$\begin{gathered} T_3(323) = 332 - 233 = 99 = 099, \\ T_3(099) = T_3(909) = T_3(990) = 891, \\ T_4(4344) = 4443 - 3444 = 999 = 0999, \\ T_4(0999) = T_4(9099) = T_4(9909) = T_4(9990) = 8991. \end{gathered} \tag{2}$$

In order to simplify the way of presentation of the decimal expansions of the discussed numbers we introduce the following notations

$$n(k\times) := \begin{cases} \underbrace{n\ldots n}_{k \text{ times}}, \text{ for } k, n \in \mathbb{N}, \\ \text{empty sequence, for } k \in \mathbb{Z}, k \leqslant 0, n \in \mathbb{N}, \end{cases}$$

$$\mathbb{N}_k^{cph} := \left\{n \in \mathbb{N}\colon 10^{k-1} - 1 \leqslant n < 10^k \wedge n \neq a(k\times), a \in \{1, 2, \ldots, 9\}\right\},$$

for $k = 2, 3, \ldots$, that is $\mathbb{N}_k^{cph}$ denotes the set composed from the number $10^{k-1} - 1 := 09((k-1)\times)$ and these k-digit natural numbers, the decimal expansion of which contains at least two different digits.

Remark 1. Identifying the notation $09(k\times)$ with the $(k+1)$-element sequence of the respective numbers, similarly as the numbers from set $\mathbb{N}_{k+1}^{cph}$ for every $k = 1, 2, 3, \ldots$, we can write that

$$\mathbb{N}_k^{cph} \cap \mathbb{N}_{k+1}^{cph} = \emptyset,\ k \in \mathbb{N}.$$

On the other hand, treating only formally this identification procedure, we get

$$\mathbb{N}_k^{cph} \cap \mathbb{N}_{k+1}^{cph} = \{9(k\times)\}.$$

By using the above introduced notations we can additionally formulate the following theorem referring to examples (2):

Theorem 2. *If $n \in \mathbb{N}_k^{cph}$, then $T_k(n) \geqslant 9((k-1)\times)$ for every $k = 2, 3, \ldots$*

Proof. Let us assume that the digits of the decimal expansion of number $n \in \mathbb{N}_k^{cph}$ are ordered in the nondecreasing sequence $0 \leqslant a_1 \leqslant \ldots \leqslant a_k \leqslant 9$, $a_1 < a_k$. Then we have

$$T_k(n) = a_k a_{k-1} \ldots a_1 - a_1 a_2 \ldots a_k.$$

In consequence, if $a_k \geqslant 2 + a_1$, then $T_k(n) > 10^{k-1}$, whereas if $a_k = 1 + a_1$, then

$$T_k(n) \geqslant 9((k-1)\times),$$

however the equality holds here only if $a_1 = a_2 = \ldots = a_{k-1}$ and $a_k = 1 + a_1$, which determines, with accuracy to the location of digits, the following nine numbers

$$01((k-1)\times),\ 12((k-1)\times), \ldots, 89((k-1)\times).$$

□

Remark 3. (Concerning the proof of Theorem 2.) Examples (2), taking into account an additional zero, are typical for discussing the operators T_3 and T_4.

Corollary 4. *We have*

$$T_k\colon \mathbb{N}_k^{cph} \to \mathbb{N}_k^{cph}$$

for every $k = 2, 3, \ldots$ *Moreover we get*

$$T_k\left(09(k\times)\right) = 89((k-1)\times)1,$$

$$T_k\left(a(k\times)(a-1)\right) = 09(k\times)$$

for every $a = 2, 3, \ldots, 9$ *and* $k = 2, 3, \ldots$

Remark 5. A reason for introducing the set $\mathbb{N}_k^{cph}$ in this paper was to eliminate from the discussion the trivial fixed point (i.e. the zero number) of transformations T_k. Let us notice that in papers [16, 17] the trivial fixed point of transformations T_k is allowed. In this case we had the more general definition of the Kaprekar's transformation

$$T_k\colon \{0\} \cup \mathbb{N}_k^{cph} \to \{0\} \cup \mathbb{N}_k^{cph},$$

where $k \geqslant 2$. Certainly $T_k(0) = 0$. Let us also notice that we have then

$$T_k\left(\left\{\alpha \in \mathbb{N}\colon 10^{k-1} \leqslant \alpha < 10^k\right\}\right) = \{0\} \cup \left\{\alpha \in \mathbb{N}\colon 10^{k-1} - 1 \leqslant \alpha > 10^k\right\}.$$

3. Iterations of operators T_2, T_3, T_4 and T_5

One can easily verify (calculations by hand are sufficient to this aim) that

$$T_2(\mathbb{N}_2^{cph}) = \{A(9-A)\colon A = 0, 1, \ldots, 8\},$$

$$T_2^n(\mathbb{N}_2^{cph}) = \{A(9-A)\colon A = 0, 1, \ldots, 4\} \text{ for } n \geqslant\ 2,$$

$$T_3(\mathbb{N}_3^{cph}) = \{A9(9-A)\colon A = 0, 1, \ldots, 8\},$$

$$T_3^k(\mathbb{N}_3^{cph}) = \{A9(9-A)\colon A = 4, 5, \ldots, 10-k\} \text{ for } k = 2, 3, \ldots, 6,$$

$$T_3^n(\mathbb{N}_3^{cph}) = \{495\} \text{ for } n \geqslant\ 6.$$

Next, the set $T_4(N_4^{cph})$ is represented by the following numbers [16]:

$$9 \times A(A+B)A, \text{ when } A + B \leqslant 9,$$

$$9 \times (A+1)(A+B-10)A, \text{ when } A+B>9,$$

where $A = a - d$, $B = b - c$ and a, b, c, d are the digits of the 4-digit number n and such that $0 \leqslant d \leqslant c \leqslant b \leqslant a \leqslant 9$, $d < a$.

One of the numbers described by these formulae is the number $686 \times 9 = 6174$. This is the only fixed point of operator T_4 and, simultaneously, this is the unique possible orbit of this operator obtained, which is especially interesting, after at most seven iterations of operator T_4. More precisely, we have

$$T_4^6(N_4^{cph}) = \{6174, 4176, 8352, 8532\} \text{ and } T_4^7(N_4^{cph}) = \{6174\}. \tag{3}$$

Number 6174 will be called the classical Kaprekar's constant. The given below decomposition of this constant is interesting

$$6174 = \sqrt{2} \times 3 \times 343 \times 3 \times \sqrt{2} = 2^1 \times 3^2 \times 7^3.$$

Furthermore, in the orbits of operator T_5 there appear only the numbers of form: $ABA \times 99$, where $0 \leqslant B \leqslant A \leqslant 9$ – these numbers create the image of operator T_5 (see [16]).

One can obtain in this way 54 numbers generating one 2-element orbit and two 4-element orbits of operator T_5. There are certainly all the orbits of this operator, therefore this operator, unlike operator T_4, does not possess any fixed point!

We noticed additionally the following interesting relations

$$T_5(99 \times AAA) = T_5(99 \times (1110 - AAA)) = T_5(99 \times BAB) = T_5(99 \times BBB),$$

for $A = 1, 2, 3, 4$, where $B := 10 - A$.

Number $T_5(99 \times 444) = T_5^3(99 \times 333) = 61974$ belongs to one of the 4-element orbits (number 61974 is the extension of the classical Kaprekar's constant with digit 9), whereas the number $T_5^3(99 \times 111) = T_5(99 \times 222) = 85932$ after the next superposition with operator T_5 falls into the second 4-element orbit.

The 2-element orbit has the form

$$\Big(T_5(505 \times 99) = 545 \times 99,\ 606 \times 99\Big).$$

Set $T_5^n(\mathbb{N}_5^{cph})$ stabilizes starting from $n = 4$.

3.1. Historical remarks

The Kaprekar's transformations T_n, $n \in \mathbb{N}$, $n \geqslant 2$, were defined by Hindu mathematician Dattathreya Ramachandra Kaprekar in paper [18] where the Kaprekar's constant 6174 was only announced. Only in his next paper [19], published after six years, he proved that $T_4^{\mathbf{7}}(\mathbb{N}_4^{cph}) = \{6174\}$ and the constant **7** is the lowest number as possible (see the first equality in relations (3)).

Properties of operators T_3, T_5 were investigated as well by some other mathematicians (among others, by Ch. Trigg – the known American popularizer of mathematics – in paper [28], by K.E. Eldridge and S. Sagong in paper [10]) and also in various number systems.

Moreover, let us notice that the discussion on operators T_n for greater values of n was omitted for many years, which in fact was the main cause of our interest in this subject-matter. The reason of this discussion missing was firstly the weakly developed computer science and afterwards, probably, the, not attractive enough, substantial character of this problem – very unfair judgement according to us, the authors of the current paper and the papers [13, 15–17], is it not?

4. Iterations of operator T_6

Set $T_6(\mathbb{N}_6^{cph})$ is created by the numbers described by means of one of the given below seven formulae (see [16]):

$$9 \times A(A+B)(A+B+C)(A+B)A,$$

where $0 \leqslant C \leqslant B \leqslant A \leqslant A+B+C \leqslant 9$,

$$9 \times A(A+B+1)(A+B+C-10)9A,$$

where $0 \leqslant C \leqslant B \leqslant A \leqslant A+B \leqslant 8$ and $10 \leqslant A+B+C < 20$,

$$9 \times (A+1)0(A+B+C-10)9A,$$

where $1 \leqslant C \leqslant B \leqslant A \leqslant 9$ and $A+B=9$,

$$9 \times (A+1)(A+B-9)(A+B+C-9)(A+B-10)A,$$

where $0 \leqslant C \leqslant B \leqslant A \leqslant 9$ and $10 \leqslant A+B \leqslant A+B+C \leqslant 18$,

$$9 \times (A+1)(A+B-8)(A+B+C-19)(A+B-10)A,$$

where $0 \leqslant C \leqslant B \leqslant A \leqslant 9$, $A+B+C \geqslant 19$ and $A+B \leqslant 17$,

$$9 \times 110(C-1)89,$$

where $C \geqslant 1$,

$$9 \times 109989.$$

The image of operator T_6 consists of 219 numbers. Set $T_6^n(\mathbb{N}_6^{cph})$ stabilizes not before $n = 13$, the respective so called maxinvariant set is composed of two fixed points and the elements of the unique 7-element orbit of operator T_6. Set $T_6^{12-n}(\mathbb{N}_6^{cph})$ consists of $10 + 2n$ elements for $n = 0, 1, \ldots, 4$ and, respectively, of 21, 25, 31, 40, 53, 82 elements for $n = 5, 6, \ldots, 10$.

For each $A \in \{5, 6, \ldots, 9\}$ we have

$$T_6^{13}(99A(A-5)00) = 420876.$$

This number belongs to the 7-element orbit of operator T_6, moreover, the numbers $T_6^n(99A(A-5)00)$, $n = 1, \ldots, 13$ are all different. Let us notice that

$$T_6^{12}(99A(A-5)00) = 652644,$$

which can be also described by formula

$$T_6^{12}(\mathbb{N}_6^{cph}) \backslash T_6^{13}(\mathbb{N}_6^{cph}) = \{652644\}.$$

Moreover we have

$$T_6^{11}(\mathbb{N}_6^{cph}) \backslash T_6^{12}(\mathbb{N}_6^{cph}) = \{620874, 749943\},$$
$$T_6^{10}(\mathbb{N}_6^{cph}) \backslash T_6^{11}(\mathbb{N}_6^{cph}) = \{651744, 629964\},$$
$$T_6^{9}(\mathbb{N}_6^{cph}) \backslash T_6^{10}(\mathbb{N}_6^{cph}) = \{873522, 864432\},$$
$$T_6^{8}(\mathbb{N}_6^{cph}) \backslash T_6^{9}(\mathbb{N}_6^{cph}) = \{310887, 820872\}.$$

Remark 6. If $abcdef \in T_6(\mathbb{N}_6^{cph})$ and $a_1b_1c_1d_1e_1f_1 \in T_6(\mathbb{N}_6^{cph})$, and these numbers are different, then $abc \neq a_1b_1c_1$.

Number 549945 is the fixed point of operator T_6 and it will be called the singular fixed point of operator T_6 with respect to the following property:

if $k \in \mathbb{N}_6^{cph}$ and $n \in \mathbb{N}$ and

$$T_6^n(k) = 549945 \neq k,$$

then $n = 1$.

Let us notice also that if $k \in \mathbb{N}_6^{cph}$ and $T_6(k) = 549945$, then the decimal expansion of number k is a "permutation" of digits of the number

$$(5+e)(5+e)ddee,$$

where $0 \leqslant e \leqslant 4$, $e \leqslant d \leqslant 5+e$. Number **631764** is the second fixed point of operator T_6 and it will be called the Kaprekar's type fixed point of operator T_6 (with the bold typing the classical Kaprekar's constant is marked). It possesses the following property.

If $k \in \mathbb{N}_6^{cph}$, $n \in \mathbb{N}$ and $T_6^n(k) = 631764 \neq k$, then $n \leqslant 4$ and moreover:

— if $n = 1$, then the decimal expansion of number k is a "permutation" of digits of the number
$$(6+f)(3+e)(2+d)def,$$
where $0 \leqslant f \leqslant 3$, $f \leqslant e \leqslant d \leqslant 1+e \leqslant 4+f$;

— if $n = 2$, then the decimal expansion of number k is a "permutation" of digits of one of two following numbers
$$(4+f)(3+e)(2+d)def, 0 \leqslant f \leqslant 5, f \leqslant e \leqslant d \leqslant 1+e \leqslant 2+f$$
or
$$(6+e)(6+e)(2+d)dee, 0 \leqslant e \leqslant 3, e \leqslant d \leqslant 4+e;$$

— if $n = 3$, then the decimal expansion of number k is a "permutation" of digits of one of four following numbers
$$(7+f)(6+e)(4+d)def, 0 \leqslant f \leqslant 2, f \leqslant e \leqslant d \leqslant 2+e \leqslant 3+f,$$
$$(7+e)(6+d)(6+d)dde, 0 \leqslant e \leqslant 2, e \leqslant d \leqslant 1+e,$$
$$(7+f)(4+e)ddef, 0 \leqslant f \leqslant 2, f \leqslant e \leqslant d \leqslant 4+e \leqslant 7+f$$
and
$$(7+f)(6+e)ddef, 0 \leqslant f \leqslant 2, f \leqslant e \leqslant d \leqslant 6+e \leqslant 7+f;$$

— and finally if $n = 4$, then the decimal expansion of number k is a "permutation" of digits of one of three following numbers
$$(8+e)(8+e)(3+d)dee, 0 \leqslant e \leqslant 1, e \leqslant d \leqslant 5+e,$$
$$(8+e)(8+e)(5+d)dee, 0 \leqslant e \leqslant 1, e \leqslant d \leqslant 3+e,$$
$$(8+e)(8+e)(7+d)dee, 0 \leqslant e \leqslant 1, e \leqslant d \leqslant 1+e.$$

4.1. More about the fixed points of the Kaprekar's transformations

In reference to the singular fixed point of operator T_6 let us note yet that the number given below

$$5((n-1)\times)49(n\times)4((n-1)\times)5$$

is the fixed point of operator T_{3n} for every $n \in \mathbb{N}$.

Proof. We have (minuend and subtrahend are written in a way giving the possibility to verify directly the obtained difference – we will continue this fashion for all verified differences in the rest of this paper):

$$\begin{array}{rccccc}
 & 9(\quad n\times \quad) & & 5(n\times) & 4(\quad n\times \quad) & \\
- & 4(\quad n\times \quad) & & 5(n\times) & 9(\quad n\times \quad) & \\
\hline
 & 5((n-1)\times) & 4 & 9(n\times) & 4((n-1)\times) & 5
\end{array}$$

$\square$

Next, the Kaprekar's type fixed point of operator T_6 is the second element of the following sequence of the fixed points

$$63(n\times)176(n\times)4,\ n = 0, 1, 2, \ldots,$$

of the Kaprekar's transformation T_{2n+4}, respectively, for $n = 0, 1, 2 \ldots$

Rather as a curiosity, we introduce two more disjoint sequences of the fixed points of the Kaprekar's transformations.[1] At first for transformations T_{2n+15}, $n \geqslant 1$, we have respectively

$$9(n\times)87654209875432 10((n-1)\times)1, \quad n \geqslant 1.$$

Proof. Let us notice that

$$\begin{array}{rcccccccccccccccc}
 & 9((n+1)\times) & 8 & 8 & 7 & 7 & 6 & 5 & 5 & 4 & 4 & 3 & 2 & 2 & 1 & 1 & 0(\quad n\times \quad) \\
- & 0(\ n\times\ )\,1 & 1 & 2 & 2 & 3 & 4 & 4 & 5 & 5 & 6 & 7 & 7 & 8 & 8 & 9 & (\ (n+1)\times\ ) \\
\hline
 & 9(\ n\times\ )\,8 & 7 & 6 & 5 & 4 & 2 & 0 & 9 & 8 & 7 & 5 & 4 & 3 & 2 & 1 & 0((n-1)\times)\,1
\end{array}$$

$\square$

And now for transformations T_{8n+3}, $n \geqslant 2$, we get the sequence of the following fixed points

$$\big\{9\mathbf{8}7((n-1)\times)\mathbf{6}5((n-1)\times)\mathbf{43}((n-1)\times)2\mathbf{1}((n-2)\times)0\mathbf{9}8((n-1)\times)$$
$$\mathbf{76}((n-1)\times)\mathbf{5}4((n-1)\times)\mathbf{32}((n-1)\times)11;\quad n \geqslant 2\big\}.$$

[1] Other sequences of the fixed points of the Kaprekar's transformations are presented in paper [17].

Let us observe that we start here from the fixed point of the transformation T_{19}! With the bold typing we identified the unique fixed point of transformation T_{11}.

Proof. Let us notice that

$$\begin{array}{rl} & 9\,9\,8(n\times)7(n\times)6(n\times)5(n\times)4(n\times)3(n\times)2(n\times)1(n\times)\ 0 \\ - & 0\,1(n\times)2(n\times)3(n\times)4(n\times)5(n\times)6(n\times)7(n\times)8(n\times)\ 9\,9 \\ \hline & 987(n-1\times)65(n-1\times)43(n-1\times)21(n-2)\times 098(n-1\times)\ 76(n-1\times)54(n-1\times)32(n-1\times)11 \end{array}$$

□

5. Announcement of our new paper [13]

By observing the obtained numerical results describing the orbits of transformations T_n for $n \leqslant 35$ we manage to prove the following theorem.

Theorem 7. *Each Kaprekar's transformation T_{2k-1}, $k \in \mathbb{N}$, $k \geqslant 7$ possesses the following 2-element orbit*

$$(873(n\times)209876(n\times)22,$$

$$966543((n-2)\times)296((n-1)\times)54331),$$

where $n := k-5$, that is n takes the values starting from 2, 3, ...

Proof. Let us notice that

$$\begin{array}{rcccccccccccc} & 9 & 9 & 6 & ((n+1)\times) & 5 & 5 & 4 & 4 & 3(\quad n\times\quad) & 2 & 1 \\ - & 1 & 2 & 3 & (\quad n\times\quad)4 & 4 & 5 & 5 & 6 & ((n+1)\times) & 9 & 9 \\ \hline & 8 & 7 & 3 & (\quad n\times\quad)2 & 0 & 9 & 8 & 7 & 6(\quad n\times\quad) & 2 & 2 \end{array}$$

and

$$\begin{array}{rcccccccccccc} & 9 & 8 & 8 & 7 & 7 & 6(\quad n\times\quad) & 3(\quad n\times\quad) & 2 & 2 & 2 & 0 \\ - & 0 & 2 & 2 & 2 & 3 & (\quad n\times\quad)6 & (\quad n\times\quad)7 & 7 & 8 & 8 & 9 \\ \hline & 9 & 6 & 6 & 5 & 4 & 3((n-2)\times)29 & 6((n-1)\times\quad)5 & 4 & 3 & 3 & 1 \end{array}$$

for every $n = 2, 3, \ldots$ □

Remark 8. We observed also that only for $k = 12$ and $k = 13$ the transformation T_{2k-1} possesses one more 2-element orbit (only one in each case!). So, for $k = 12$ this is the orbit

$$(87765543319997665443222,\ 87764442219997775553222),$$

whereas for $k = 13$ this is the orbit

$$(887665443219997765543321 2,\ 8876644422199977755533212).$$

This remark does not have to be surprising if we put our attention to the fact that our numerical observation includes only the orbits of transformations T_n with indices $n \leqslant 35$.

6. Generalizations and modifications of the Kaprekar's transformations

Certainly, it seems completely natural to ask for the possibility of modifying the operators T_n, $n \in \mathbb{N}$ and the consequences resulting from such modifications. For instance, in paper [11] the operator T_4 is replaced by operator

$$k(\mathbb{a}) := a_4a_3a_2a_1 - a_2a_1a_4a_3,$$

where $\mathbb{a} \in \mathbb{N}_4^{cph}$ and a_1, a_2, a_3, a_4 forms the sequence of digits of $\mathbb{a}$ in an ascending order.

This operator does not possess the fixed point but it has the 2-element orbit $\{\mathbf{2}17\mathbf{8}, 6\mathbf{53}4\}$ and, what is surprising, $T_4(\mathbf{2538}) = 6174$. In the mentioned paper much more is proven, that is, if we would consider this problem with respect to the equivalents of 4-digit numbers in any base $b \in \mathbb{N}$, $b \geqslant 2$, it means with respect to the numbers of form

$$\mathbb{x} = (b_1b_2b_3b_4)_b = b_1b^3 + b_2b^2 + b_3b + b_4,$$

where $b_i \in \{0, 1, \dots b-1\}$, $i = 1, 2, 3, 4$, and

$$a_1 = b_{\sigma(1)} \geqslant a_2 = b_{\sigma(2)} \geqslant a_3 = b_{\sigma(3)} \geqslant a_4 = b_{\sigma(4)},$$

for the respective ordering permutation σ (with the same constraint as for decimal expansion that $a_1 > a_4$ which means that we eliminate from discussion b numbers of the form $(\beta\beta\beta\beta)_b$, $\beta \in \{0, 1, \dots b-1\}$), then operator k would possess the fixed point only if $b = 2$ or $b = 3 \cdot 2^n$.

In this last case one can prove that

$$k^i(x) = \left(2^n, 2^n - 1, 2^{n+1} - 1, 2^{n+1}\right)_b \text{ dla } i \geqslant 2n + 3.$$

Let us also notice that operator T_5 in bases $b < 13$ was investigated by the, mentioned before, C. Trigg in paper [28].

Authors of the current paper, apart from the discussed here Kaprekar's transformations, have also defined and investigated few generalizations of these maps (paying special attention to the description of minimal orbits). Exceptionally in this section we will also discuss not only the numbers from set $\mathbb{N}_n^{cph}$ but all the natural n-digit numbers, for every $n \in \mathbb{N}$, $n \geqslant 2$ (see [15, 16]). We distinguish the following generalizations of the Kaprekar's transformations:

— the symmetric Kaprekar's transformations

Let $a_1 a_2 \ldots a_n$ be the decimals representation of natural number a, $10^{n-1} \leqslant a < 10^n$. Then the n-th symmetric Kaprekar's transformation M is defined by the formula

$$M(a_1 a_2 \ldots a_n) = \sum_{k=1}^{n} |c_k - b_k| 10^{k-1},$$

where $(b_1, b_2, \ldots, b_n)$ and $(c_1, c_2, \ldots, c_n)$ are the sequences, nondecreasing and nonincreasing, respectively, composed of the digits $a_1, a_2, ..., a_n$. We include to the set of n-digit numbers also the number zero. The orbits of operators M for the odd values $n \leqslant 19$, although "quite easy" to calculate even by hand, surprise yet with their final form. We will present here only few quantitative pieces of information.

So, if $n = 2k + 1$, $1 \leqslant k \leqslant 5$, then M possesses only the fixed points and k-element orbits, for $n = 13$ operator M possesses two fixed points, 0 and 65432101...6, four 2-element cycles, eleven 3-element cycles and 827 cycles of length 6 (sic). For $n = 15$ the operator M possesses 44 fixed points, 342 different 2-elements orbits and 2678 different 4-elements orbits. For $n = 17$ the operator M possesses only 6 fixed points, 32 different 2-element orbits and 6060 different 4-element orbits. Finally, for $n = 2^k$ the operator M possesses only trivial orbit $= \{0\}$ for every $k \in \mathbb{N}$ (it follows from Theorem 9 given below).

— the nonoptimal Kaprekar's transformations

One of the examples of this transformation, called by us the Q-Kaprekar's transformation, is defined as

$$Q_n(A) := (a_n - a_2)10^{n-1} + (a_{n-1} - a_1)10^{n-2} + \sum_{k=1}^{n-2} (a_k - a_{n-k+1})10^{k-1},$$

where $0 \leqslant a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n \leqslant 9$ are the all digits of decimal expansion of number A. We note that, in contrast to the Kaprekar's transformation T_4, the transformation Q_4 possesses two 2-element orbits: $\{2187, 6543\}$ and $\{3285, 5274\}$ and the trivial fixed point. Next, Q_5 possesses the trivial fixed point and the

2-element orbit $\{52974, 54963\}$ (to the contrast, transformation T_5 has four different orbits). Transformations Q_6 and T_6 have both three fixed points and, respectively, the 8-element orbit and the 7-element orbit. Transformations Q_7 and T_7 possess both the trivial fixed point and one 8-element orbit (but of the different orbit types – see [16] for the respective definition).

— general Kaprekar's transformations

We take that the natural number A, $10^{n-1} \leqslant A < 10^n$, possesses the following decimal expansion $A = d_1 d_2 \ldots d_n$. Let $a_1 := \max\{d_1, d_2, \ldots, d_n\}$, $a_2 := \max\{d_2, d_3, \ldots, d_n\}$ and in general $a_k := \max\{d_k, d_{k+1}, \ldots, d_n\}$, for $k = 1, \ldots, n$. The general Kaprekar's transformations are defined by relations

$$\begin{aligned} d_{\sigma,\pi}(A) &:= \sum_{k=1}^{n} |d_{\sigma(k)} - d_{\pi(k)}| 10^{n-k}, \\ d_{\sigma,\pi}^{weak}(A) &:= \left| \sum_{k=1}^{n} (d_{\sigma(k)} - d_{\pi(k)}) 10^{n-k} \right|, \end{aligned}$$

and

$$\begin{aligned} D_{f,g}(A) &:= \sum_{k=1}^{n} |d_{f(k)} - d_{g(k)}| 10^{n-k}, \\ D_{f,g}^{weak}(A) &:= \left| \sum_{k=1}^{n} (d_{f(k)} - d_{g(k)}) 10^{n-k} \right|, \\ R_f(A) &:= \sum_{k=1}^{n} |a_k - a_{f(k)}| 10^{n-k}, \\ R_f^{weak}(A) &:= \left| \sum_{k=1}^{n} (a_k - a_{f(k)}) 10^{n-k} \right|, \end{aligned}$$

for any permutations σ, π on set $\{1, \ldots, n\}$ and for any functions $f, g\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$.

At the end we should recall the oldest known modification of the Kaprekar's transformation, that is the Ducci's transformation. Italian mathematician Enrico Ducci (1864–1940) at the end of XIX century made many remarks concerning the following map $D_n\colon \mathbb{Z}^n \to \mathbb{Z}^n$, $n \geqslant 3$, defined by formula

$$D_n(\mathbb{x}) := \big[|x_1 - x_2|, |x_2 - x_3|, \ldots, |x_n - x_1| \big],$$

where $\mathbb{x} := [x_1, x_2, \ldots, x_n]$. This map is called today the nth Ducci's map (for every $n \geqslant 3$). Some property of the iteration of this map became a ground of a problem[2] which often appears in various task books – including the books of recreational-cognitive character (see the books by Kordemskii [20] and Kurlyandchik [21]) as well as the scientific articles (especially concerning the number theory, see the papers by Calkin [6], Beardon [3], Ludington Furno [24] and the works cited by these authors). We formulate the mentioned property of Ducci's map in the form of the following theorem:

Theorem 9. *For each $\mathbb{x} \in \mathbb{Z}^n$ there exists $m = m(\mathbb{x}) \in \mathbb{N}$ such that*

$$D_n^m(\mathbb{x}) = [\underbrace{0, \ldots, 0}_{n\text{-}times}]$$

if and only if n is the power of number two (that is $n = 2^k$ for some $k \in \mathbb{N}$).

Proof of this theorem was published for the first time by C. Ciamberlini and A. Marengoni in paper [8] (see also Beardon's paper [3]).

Next, M. Burmester, R. Forcade and E. Jacobs in paper [5] proved that if number n, in the above theorem, is not the integer power of number two, then for every $\mathbb{x} \in \mathbb{Z}^n$ there exists $m = m(\mathbb{x}) \in \mathbb{N}$ such that the vector $D_n^m(\mathbb{x})$ belongs to one of the orbits of operator D_n. Each orbit of operator D_n is composed from the vectors of the form

$$k[\underbrace{x_1, x_2, \ldots, x_n}_{\text{binary vector}}],$$

where $k \in \mathbb{N}$, $x_i \in \{0, 1\}$ for every $i = 1, 2, \ldots, n$.

The length of orbits of operator D_n was discussed for the first time by A. Ehrlich in paper [9], and next by N.J. Calkin (with colleagues) in [5].

6.1. Ducci's transformations on the real vectors

Already in 1949 Mosche Lotan in paper [22] (see also [29]) proved that for every $\mathbb{x} \in \mathbb{R}^4$, either after the finite number of operator D_4 iterations, vector $\mathbb{x}$ transforms into the zero vector, or

$$D_4(\mathbb{x}) = \alpha[1, q, q^2, q^3],$$

[2]Unfortunately this problem does not have a unified name. It is known as the "Ducci Sequences", the "N-number Game", the "Four Number Game", the "Difference boxes" and the playing "Diffy" (see [1, 27]).

where $\alpha \in \mathbb{R}$ and q is the unique real root of equation

$$q^3 - q^2 - q - 1 = 0.$$

We have $q = \frac{1}{3}\left(1 + \sqrt[3]{19 - 3\sqrt{33}} + \sqrt[3]{19 + 3\sqrt{33}}\right) \approx 1.83929$. Moreover, we get then

$$D_4^n(\mathbb{x}) = \alpha|1-q|^{n-1}[1, q, q^2, q^3],$$

for every $n = 1, 2, 3, \ldots$ Also the following "corollary" (observed by us) resulting from the Lothan Theorem holds:

Theorem 10. *Let $\mathbb{x} \in \mathbb{R}^4$ and $\mathbb{x} = [x_1, x_2, x_3, x_4]$. If one of the following nine conditions is satisfied then there exists $m = m(\mathbb{x}) \in \mathbb{N}$ such that $D_4^m(\mathbb{x}) = \mathbb{O}$:*

1° $D_4(\mathbb{x}) = [a, b, c, d]$ *and* $b + d \neq b(a + c)$;

2° $\max\{x_1, x_3\} \leqslant \min\{x_2, x_4\}$;

3° $\min\{x_1, x_3\} \geqslant \max\{x_2, x_4\}$;

4° $\min\{x_1, x_3\} \leqslant \min\{x_2, x_4\} \leqslant \frac{x_1+x_3}{2} \leqslant \max\{x_2, x_4\} \leqslant \max\{x_1, x_3\}$;

5° $\min\{x_2, x_4\} \leqslant \min\{x_1, x_3\} \leqslant \frac{x_2+x_4}{2} \leqslant \max\{x_1, x_3\} \leqslant \max\{x_2, x_4\}$;

6° $\min\{x_1, x_3\} \leqslant \min\{x_2, x_4\}$ *and* $\max\{x_2, x_4\} \leqslant \frac{x_1+x_3}{2}$;

7° $\min\{x_2, x_4\} \leqslant \min\{x_1, x_3\}$ *and* $\max\{x_1, x_3\} \leqslant \frac{x_2+x_4}{2}$;

8° $\frac{x_1+x_3}{2} \leqslant \min\{x_2, x_4\}$ *and* $\max\{x_2, x_4\} \leqslant \max\{x_1, x_3\}$;

9° $\frac{x_2+x_4}{2} \leqslant \min\{x_1, x_3\}$ *and* $\max\{x_1, x_3\} \leqslant \max\{x_2, x_4\}$.

We observe that if the condition 2° or 3° is satisfied then $m = 4$, whereas, if one of the conditions $4°, 5°, \ldots, 9°$ is satisfied then $m = 7$. All the needed verifying calculations were made by hand – we used almost four sheets of paper!

Let us notice that, similarly as for case $k = 4$, for the any $k \in \mathbb{N}$, $k \geqslant 3$, we have

$$D_k^n(\mathbb{q}_k) = |1 - q_{k-1}|^{n-1}\mathbb{q}_k,$$

where $\mathbb{q}_k := [1, q_{k-1}, q_{k-1}^2, \ldots, q_{k-1}^{k-1}]$ and q_{k-1} is a real root of the equation

$$q^{k-1} - q^{k-2} - \ldots - q - 1 = 0.$$

Remark 11. Let us set

$$f_k(q) := \begin{cases} q^k - \frac{q^k-1}{q-1}, \text{ if } q \neq 1, \\ 1-k, \text{ if } q = 1. \end{cases}$$

Then we have

$$q\, f_{k-1}(q) = f_k(q) + 1 \tag{4}$$

and on the basis of the above equation and the above definition we can prove by induction that

— if k is odd then $f_k(q)$ possesses only one real root (of multiplicity one): $0 < q_k < 2$,

— if k is even then $f_k(q)$ possesses two real roots (both of multiplicity one): $q_k^{(1)} < 0 < q_k^{(2)}$,

— and additionally

$$1 = q_1 \leqslant q_{2k-1} < q_{2k}^{(2)} < q_{2k+1}, \quad k \in \mathbb{N}. \tag{5}$$

Moreover, we can prove that if $f_k(q) = 0$, then $f_{k-1}(q) = \frac{1}{q}$ and $q = 2 - \frac{1}{q^k}$. Therefore, if we set

$$\gamma_k := \begin{cases} q_k, \text{ if } k \text{ is odd}, \\ q_k^{(2)}, \text{ if } k \text{ is even}, \end{cases}$$

then

$$\lim_{k\to\infty} \gamma_k = 2,$$

$$\lim_{k\to\infty} f_{k-1}(\gamma_k) = \lim_{k\to\infty} \frac{1}{\gamma_k} = \frac{1}{2}.$$

Hare, Prodinger, and Shallit proved in [14, Theorem 1.1] that

$$\gamma_k = 2 - 2\sum_{n=1}^{\infty} \frac{1}{n}\binom{n(k+1)-2}{n-1}\frac{1}{2^{n(k+1)}}, \tag{6}$$

$$\frac{1}{2-\gamma_k} = 2^k - \frac{k}{2} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}\binom{n(k+1)}{n+1}\frac{1}{2^{n(k+1)}}, \tag{7}$$

$$\frac{1}{\gamma_k} = \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}\binom{n(k+1)}{n-1}\frac{1}{2^{n(k+1)}}. \tag{8}$$

The formula (6) was discovered in 1998 by Wolfram (see [30, Theorem 3.9]).

For γ_3 we obtain the relation

$$(\gamma_3 - 1)^3 = 2 - 2(\gamma_3 - 1)^2, \tag{9}$$

which implies the following nested radicals expansion:

$$\gamma_3 - 1 = \left(2(1 - t_n)\right)^{1/3}, \tag{10}$$

where

$$\begin{aligned} t_{k+1} &= \left(2|1 - t_k|\right)^{2/3}, \quad k = 0, 1, \ldots, n-1, \\ t_0 &= (\gamma_3 - 1)^2. \end{aligned}$$

But no other t_0 in sufficient small neighborhood of $(\gamma_3 - 1)^2$ there exists such that these procedure is convergent. To the contrast if we write (9) in the form

$$\gamma_3 - 1 = \sqrt{\frac{2}{2 + (\gamma_3 - 1)}},$$

then the sequence

$$t_{k+1} = \sqrt{\frac{2}{2 + t_k}}, \quad k = 0, 1, \ldots,$$

is convergent to $\gamma_3 - 1$ for every $t_0 > 2^{-1/3} - 2 \approx -1.2063$, since $\left(\sqrt{\frac{2}{2+x}}\right)' = \frac{-1}{\sqrt{2(2+x)^3}}$. More other informations on real and complex zeros of polynomials $f_k(q)$ (and simultaneously their derivatives and integrals!) could be found in Zhu and Grossman paper [31].

At the end of this subsection we note that M. Misiurewicz and A. Schinzel in [25] proved, among others, that for every $\mathbb{x} \in \mathbb{R}^n$ and a limit point $\mathbb{p}$ of the sequence $\{D_n^k(\mathbb{x})\}_{k=1}^{\infty}$ we have $\mathbb{p} = c\,\mathbb{v}$, where $\mathbb{v} \in \{0,1\}^n$, and

$$c := \lim_{k\to\infty} ||D_n^k(\mathbb{x})||_{max} \quad \left(||\mathbb{b}||_{max} := \max_{1\leqslant l\leqslant n} \{|b_l|\} \right).$$

Additionally, let us note some, unsolved yet, problem referring to the subject matter of the infinite Ducci's transformation. Let $\{p_n\}_{n=1}^{\infty}$ be the increasing sequence of all primes and let us set $d_0(n) = p_n$, $n \geqslant 1$, and

$$d_{k+1}(n) := |d_k(n) - d_k(n+1)|, \quad k \geqslant 0, \quad n \geqslant 1,$$

then a well known Gilbreath's conjecture claims that

$$d_k(1) = 1 \quad \text{for all } \ k \geqslant 1$$

– see the paper by Andrew M. Odlyzko [26] for a background of this problem.

7. Geometric and periodic properties of some recurrence sequence

In [4] it is proven that the sequence $\{a_n\}_{n=0}^{\infty}$ of real numbers, satisfying relation $a_{n+1} = |a_n| - a_{n-1}$, is always periodic with period 9. We observe additionally that the respective sequence of complex elements does not possess this property, for example:

— if $a_0 = 1$, $a_1 = i$ then $a_2 = 0$, $a_3 = -i$, $a_4 = 1$, $a_5 = 1 + i$, $a_6 = \sqrt{2} - 1$, $a_7 = \sqrt{2} - 2 - i$, $a_8 = 1 - \sqrt{2} + \sqrt{7 - 4\sqrt{2}}$, $a_9 = 3 - 2\sqrt{2} + \sqrt{7 - 4\sqrt{2}} + i$. From numerical calculations (with working precision 600 in Mathematica v.8 software) it follows that the sequence $\{\Im(a_n)\}_{n=0}^{\infty}$ is periodic with period 4 (the repeated sequence of the form 0, 1, 0, -1). This property holds in all complex (not real) cases which can be proven easily by observing the basic recurrence formula for a_n. In consequence, the sequence $\{a_n\}$ may have only the periods $4k$ for some $k \in \mathbb{N}$. By symbolic calculations in Mathematica software we deduced that $k = 1$ is impossible[3];

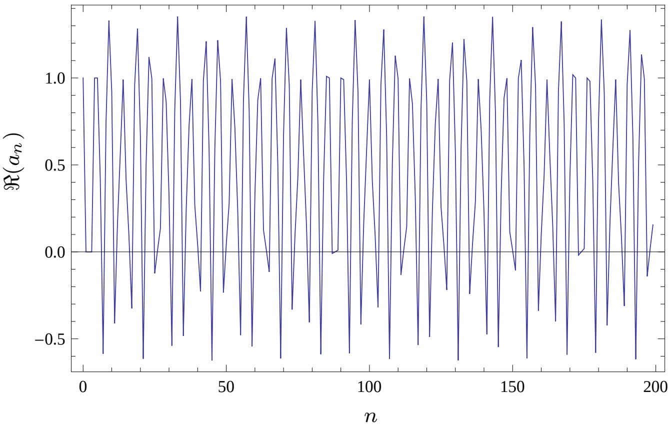

Fig. 1. Plot of function $n \mapsto \Re(a_n)$, where $a_0 = 0$, $a_1 = i$; the successive points are connected with lines

Rys. 1. Wykres funkcji $n \mapsto \Re(a_n)$, gdzie $a_0 = 0$, $a_1 = i$; kolejne punkty połączone są odcinkami

[3] How did we test the "nonperiodicity" of $\{a_n\}$? For example, at first we found the sequence $\{|\Re\,(a_n - a_{160})|\}_{n=0}^{600}$ and next we sorted this one in nondecreasing order. In all discussed here cases of $\{a_n\}$ we got the respective sorted sequences with values $\geqslant 0.1$ for $n \geqslant 3$.

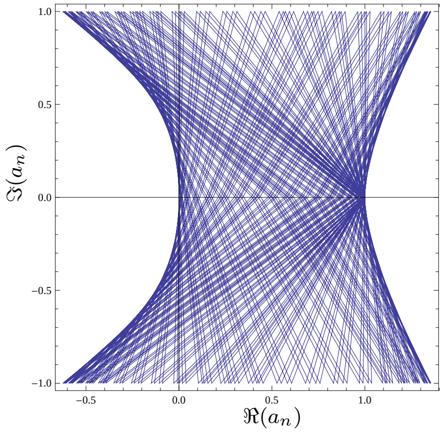

Fig. 2. Plot in the Gauss plane where elements of sequence $\{a_n\}$, where $a_0 = 0$, $a_1 = i$, are connected with lines

Rys. 2. Wykres na płaszczyźnie Gaussa, na którym elementy ciągu $\{a_n\}$, gdzie $a_0 = 0$, $a_1 = i$, połączone są odcinkami

— if $a_0 = 5 + 3i$, $a_1 = 3 - 4i$ then $a_2 = -3i$, $a_4 = 4i$, $a_5 = 4 + 3i$, $a_6 = 5 - 4i$, $a_7 = \sqrt{41} - 4 - 3i, \ldots$, also in this case the sequence $\{\Im(a_n)\}_{n=0}^{\infty}$ is periodic with period 4 (the repeated sequence of the form 3, −4, −3, 4);

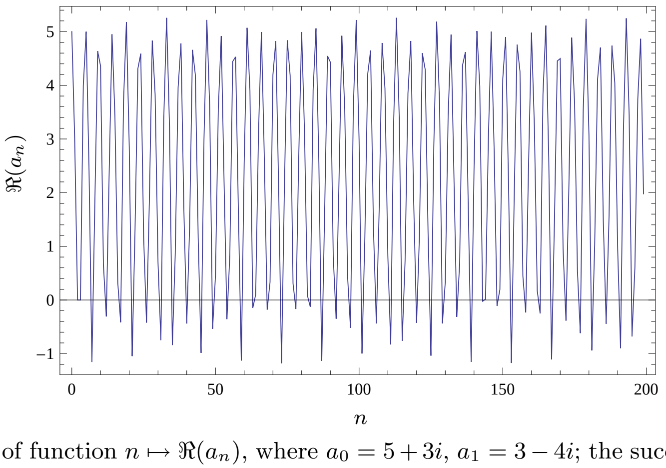

Fig. 3. Plot of function $n \mapsto \Re(a_n)$, where $a_0 = 5+3i$, $a_1 = 3-4i$; the successive points are connected with lines

Rys. 3. Wykres funkcji $n \mapsto \Re(a_n)$, gdzie $a_0 = 5 + 3i$, $a_1 = 3 - 4i$; kolejne punkty połączone są odcinkami

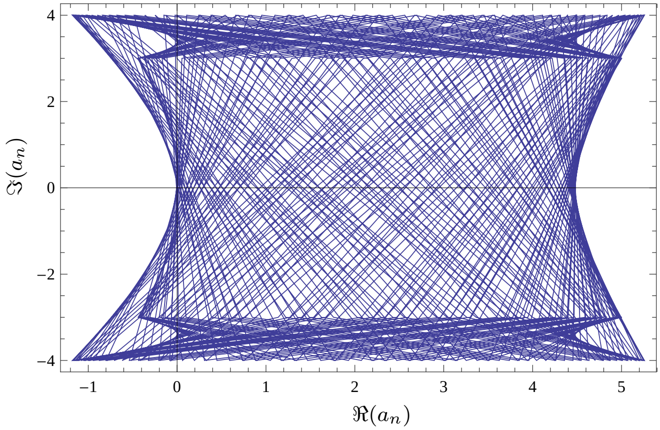

Fig. 4. Plot in the Gauss plane where elements of sequence $\{a_n\}$, where $a_0 = 5 + 3i$, $a_1 = 3 - 4i$, are connected with lines

Rys. 4. Wykres na płaszczyźnie Gaussa, na którym elementy ciągu $\{a_n\}$, gdzie $a_0 = 5 + 3i$, $a_1 = 3 - 4i$, połączone są odcinkami

— if $a_0 = 1+i$, $a_1 = 1-i$ then $a_2 = \sqrt{2}-1-i \approx 0.414214-i$, $a_3 = \sqrt{4-2\sqrt{2}}-1+i \approx 0.0823922+i$, $a_4 = 1-\sqrt{2}+\sqrt{1+\left(\sqrt{4-2\sqrt{2}}-1\right)^2}+i \approx 0.589175+i$, $a_5 \approx 1.07827-i$, $a_6 \approx 0.881423-i$, $a_7 \approx 0.25474+i$, similarly as in previous cases the sequence $\{\Im(a_n)\}_{n=0}^{\infty}$ is periodic with period 4 (the repeated sequence of the form 1, −1, −1, 1).

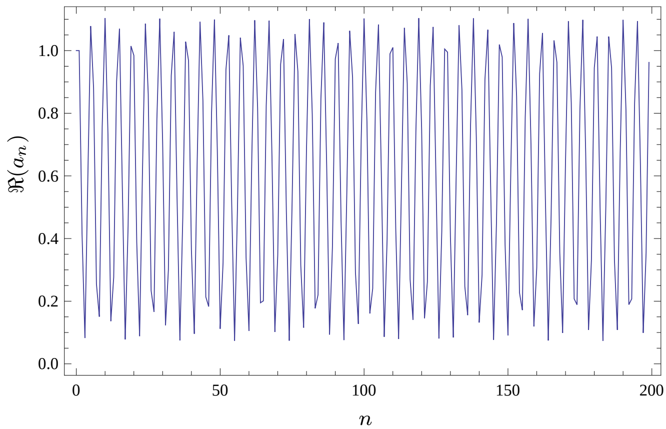

Fig. 5. Plot of function $n \mapsto \Re(a_n)$, where $a_0 = 1 + i$, $a_1 = 1 - i$; the successive points are connected with lines

Rys. 5. Wykres funkcji $n \mapsto \Re(a_n)$, gdzie $a_0 = 1+i$, $a_1 = 1-i$; kolejne punkty połączone są odcinkami

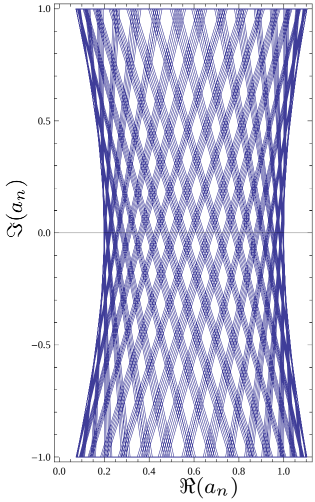

Fig. 6. Plot in the Gauss plane where elements of sequence $\{a_n\}$, where $a_0 = 1 + i$, $a_1 = 1 - i$ are connected with lines

Rys. 6. Wykres na płaszczyźnie Gaussa, na którym elementy ciągu $\{a_n\}$, gdzie $a_0 = 1+i$, $a_1 = 1 - i$, połączone są odcinkami

We are surprised by the phenomenon of the "almost periodicity" of $\{\Re(a_n)\}_{n=0}^{\infty}$ as well as by the geometry of patterns illustrating the location of elements of sequence $\{a_n\}$.

However we do not know if the respective complex sequence is periodic and especially under which initial conditions it may happen. We are quite inspired by a problem of this type, so probably we will put our attention on it in the separate paper.

8. On the orbits of Kaprekar's transformations in OEIS

In the commonly known Sloane's online encyclopaedia of the sequences of integer numbers OEIS one can find, among others, the following sequences connected with the orbits of Kaprekar's transformations:

— A164731(n) – the number of cycles of n-digit numbers (including fixed points) under the n-th Kaprekar transformation; in other bases: A004526 (base 2), A165006 (base 3), A165025 (base 4), A165045 (base 5), A165064 (base 6), A165084 (base 7), A165103 (base 8), A165123 (base 9);

— A164732(n) – the number of n-digit numbers in a cycle (including fixed points) under the n-th Kaprekar transformation;

— A151949(n) $= T_n(n)$, $n = 0, 1, 2, \ldots$ where $T_0(0) := 0$.

The number of 3-element minimal orbits of Kaprekar's transformation T_{2n+8} is equal to $A140226(n) = \frac{n(11+n^2)}{3}$ for $n = 0, 1, 2, \ldots, 6$. In case of $n = 7$ we have one more orbit, for $n = 8$ we have four more orbits and the number of orbits increases simultaneously with n. These facts give the answer for the query formulated by us in [17].

9. Description of tables presenting the cycles of Kaprekar's transformations T_n

The tables included in this paper complete the tables presented in paper [17]. Each table is composed in the following way:

— in the first row the value of index n of the Kaprekar's transformation T_n is given,

— the second row presents the amount of minimal cycles of the given length of the given transformation T_n as well as the information whether the given transformation preserves the strong Sharkovsky's order or the Sharkovsky's order (see definitions 1 and 2 in [16]),

— the third row shows how many n-digit numbers is transformed by the given Kaprekar's transformation T_n (after the finite number of steps) onto the respective minimal cycle of this transformation,

— in the successive rows the successive cycles from the third row (except the trivial one, that is the zero cycle) are associated with: the order types (it concerns only the cycles of length greater than 1, see the proper definition in [16]); the sum of digits of particular elements of the cycle (in case when these sums are identical, we include them only once), the digit types (and again, in case when they are identical, we include them only once), the longest increasing interval of the given cycle, the longest increasing subsequence of the given cycle, the longest decreasing interval of the given cycle and the longest decreasing subsequence of the given cycle.

Table 1

Orbits of the Kaprekar's transformations T_n for $n = 16$

$n = 16$				
α	9 fix, 1 cyc length 2, 36 cyc. length 3, 1 cyc. length 7			
β	12453423840 → 6333333176666664 16151750334720 → 9753333086666421 288964555636736 → 9775531088644221 25833706778880 → 9975333086664201 63301395692080 → 9977551088442201 12672264084480 → 9997533086642001 1798167571200 → 9999753086420001 56054718720 → 9999975084200001 116754375618944 → (8764421997755322, 8765431997654322) 23320362175768 → (6431111088888654, 8777732087622222, 8655555264444432) 61059798199904 → (6433111088886654, 8777332087662222, 8655553266444432) 173962314566354 → (6433311088866654, 8773332087666222, 8655533266644432) 584232077559056 → (6433331088666654, 8733332087666622, 8655333266664432) 1562115147143000 → (6433333086666654, 8333332087666662, 8653333266666432) 20828871835200 → (6543111088886544, 8777321088762222, 8765555264444322) 247256035559542 → (6543311088866544, 8773321088766222, 8765553266444322) 503206745448336 → (6543331088666544, 8733321088766622, 8765533266644322) 647808143886664 → (6543333086666544, 8333321088766662, 8765333266664322) 148663568886560 → (6554311088865444, 8773211088876222, 8776555264443222) 209856407175472 → (6554331088665444, 8733211088876622, 8776553266443222) 300213325345888 → (6554333086665444, 8333211088876662, 8776533266643222) 113091617595840 → (6555431088654444, 8732111088887622, 8777655264432222) 129286456577040 → (6555433086654444, 8332111088887662, 8777653266432222) 46498994640358 → (6555543086544444, 8321111088888762, 8777765264322222) 9779663013120 → (9751111088888421, 9777775084222221, 9755555084444421) 103008180814792 → (9753111088886421, 9777753086422221, 9755553086444421) 305485387461632 → (9753311088866421, 9777533086642221, 9755533086644421) 279503795441776 → (9753331088666421, 9775333086664221, 9755333086664421) 56642138723040 → (9755111088884421, 9777751088422221, 9775555084444221) 439879201167232 → (9755311088864421, 9777531088642221, 9775553086444221) 709233455214416 → (9755331088664421, 9775331088664221, 9775533086644221) 55387159027200 → (9755511088844421, 9777511088842221, 9777555084442221) 255206998851840 → (9755531088644421, 9775311088864221, 9777553086442221) 40532740754800 → (9755551088444421, 9775111088884221, 9777755084422221) 219079113823280 → (9775511088844221, 9777551088442221, 9775551088444221) 13242806456880 → (9975111088884201, 9977775084222201, 9975555084444201) 103184533315240 → (9975311088864201, 9977753086422201, 9975553086444201) 193906021429120 → (9975331088664201, 9977533086642201, 9975533086644201) 78642569074160 → (9975511088844201, 9977751088422201, 9977555084442201) 169898689920960 → (9975531088644201, 9977531088642201, 9977553086442201) 45119851496720 → (9975551088444201, 9977511088842201, 9977755084422201) 8978941171200 → (9997511088842001, 9997775084222001, 9997555084442001) 53561712032400 → (9997531088642001, 9997753086422001, 9997553086442001) 17993317181840 → (9997551088442001, 9997751088422001, 9997755084422001) 2483222742000 → (9999751088420001, 9999775084220001, 9999755084420001) 1542305910431760 → (4333332087666666, 8533333176666642, 7533333086666643, 8433333086666652, 8633333086666632, 8633333266666632, 6433333266666654)			
$\beta_1 - \beta_8$		72	$(10, 9, 9, 9, 9, 9, 9, 8)$	
β_9		81	$(10, 9, 9, 9, 9, 9, 8, 18)$	2, 2, 1, 1
$\beta_{10} - \beta_{13}$	$(1, 3, 2)$	72	$(10, 9, 9, 9, 9, 9, 9, 8)$	
β_{14}	$(1, 2, 3)$	72	$(10, 9, 9, 9, 9, 9, 9, 8)$	
β_{15}	$(1, 3, 2)$	72	$(10, 9, 9, 9, 9, 9, 9, 8)$	
β_{16}	$(1, 3, 2)$	72	$(10, 9, 9, 9, 9, 9, 9, 8)$	
$\beta_{17} - \beta_{24}$	$(1, 2, 3)$	72	$(10, 9, 9, 9, 9, 9, 9, 8)$	

Table 2

Orbits of the Kaprekar's transformations T_n for $n = 16$, continuation

$n = 16$, continuation				
$\beta_{25} - \beta_{30}$	$(1, 3, 2)$	72	$(10, 9, 9, 9, 9, 9, 9, 8)$	
$\beta_{31} - \beta_{34}$	$(1, 2, 3)$	72	$(10, 9, 9, 9, 9, 9, 9, 8)$	
$\beta_{35} - \beta_{39}$	$(1, 3, 2)$	72	$(10, 9, 9, 9, 9, 9, 9, 8)$	
β_{40}	$(1, 2, 3)$	72	$(10, 9, 9, 9, 9, 9, 9, 8)$	
β_{41}	$(1, 2, 3)$	72	$(10, 9, 9, 9, 9, 9, 9, 8)$	
β_{42}	$(1, 3, 2)$	72	$(10, 9, 9, 9, 9, 9, 9, 8)$	
β_{43}	$(1, 3, 2)$	72	$(10, 9, 9, 9, 9, 9, 9, 8)$	
β_{44}	$(1, 2, 3)$	72	$(10, 9, 9, 9, 9, 9, 9, 8)$	
β_{45}	$(1, 3, 2)$	72	$(10, 9, 9, 9, 9, 9, 9, 8)$	
β_{46}	$(1, 7, 3, 4, 2, 5, 6)$	72	$(10, 9, 9, 9, 9, 9, 9, 8)$	

Table 3

Orbits of the Kaprekar's transformations T_n for $n = 17$

$n = 17$				
α	3 fix, 1 cyc length 2, 6 cyc. length 5; Sharkovsky's order			
β	113545244138720 $\to$ 86433331976666532 2242260814424720 $\to$ 98765420987543211 420053260194588 $\to$ (87333320987666622, 96654332966654331) 1880904701228820 $\to$ (86433320987666532, 96643331976665331, 88433331976666512, 87643331976665322, 86543331976665432) 2914077854756922 $\to$ (86543320987665432, 96643320987665331, 98643331976665311, 88743331976665212, 87654331976654322) 6647662007373128 $\to$ (87643320987665322, 96654331976654331, 88433320987666512, 97664331976653321, 88543331976665412) 19495759008608962 $\to$ (87654320987654322, 96654320987654331, 98643320987665311, 98764331976653211, 88754331976654212) 27971772407003112 $\to$ (88543320987665412, 97664320987653321, 98654331976654311, 88743320987665212, 97665431976543321) 38313964702271018 $\to$ (88754320987654212, 97665420987543321, 98654320987654311, 98764320987653211, 98765431976543211)			
β_1		81	$(10, 9, 9, 9, 9, 9, 9, 8, 9)$	
β_2		81	$(10, 9, 9, 9, 9, 9, 9, 8, 9)$	
β_3		81	$(10, 9, 9, 9, 9, 9, 9, 8, 9)$	2, 2, 1, 1
β_4	$(1, 5, 4, 3, 2)$	81	$(10, 9, 9, 9, 9, 9, 9, 8, 9)$	
β_5	$(1, 5, 4, 2, 3)$	81	$(10, 9, 9, 9, 9, 9, 9, 8, 9)$	
β_6	$(1, 3, 5, 2, 4)$	81	$(10, 9, 9, 9, 9, 9, 9, 8, 9)$	
β_7	$(1, 5, 2, 3, 4)$	81	$(10, 9, 9, 9, 9, 9, 9, 8, 9)$	
β_8	$(1, 4, 2, 5, 3)$	81	$(10, 9, 9, 9, 9, 9, 9, 8, 9)$	
β_9	$(1, 2, 3, 4, 5)$	81	$(10, 9, 9, 9, 9, 9, 9, 8, 9)$	

Table 4

Orbits of the Kaprekar's transformations T_n for $n = 18$

	$n = 18$
α	13 fix, 60 cyc length 3, 1 cyc. length 7; Sharkovsky's order
β	343248360 $\rightarrow$ 555554999999444445 193060004280 $\rightarrow$ 633333331766666664 40682728645629000 $\rightarrow$ 886644219977553312 462548810471040 $\rightarrow$ 975333330866666421 32758226653688364 $\rightarrow$ 977553310886644221 1045242025027200 $\rightarrow$ 997533330866664201 11993957371770192 $\rightarrow$ 997755310886442201 810435000735360 $\rightarrow$ 999753330866642001 1815284433159288 $\rightarrow$ 999775510884422001 230753248165440 $\rightarrow$ 999975330866420001 21279200342400 $\rightarrow$ 999997530864200001 465242057280 $\rightarrow$ 999999750842000001 417192011665180 $\rightarrow$ (643111110888888654, 877777320876222222, 865555552644444432) 1673925407129460 $\rightarrow$ (643311110888886654, 877773320876622222, 865555532664444432) 5006518660845124 $\rightarrow$ (643331110888866654, 877733320876662222, 865555332666444432) 14609158747913340 $\rightarrow$ (643333110888666654, 877333320876666222, 865553332666644432) 26954900351649616 $\rightarrow$ (643333310886666654, 873333320876666622, 865533332666664432) 112288180545441560 $\rightarrow$ (643333330866666654, 833333320876666662, 865333332666666432) 415099293052356 $\rightarrow$ (654311110888886544, 877773210887622222, 876555552644444322) 3172702814525340 $\rightarrow$ (654331110888866544, 877733210887662222, 876555532664444322) 15791141249243468 $\rightarrow$ (654333110888666544, 877333210887666222, 876555332666444322) 21827337061506816 $\rightarrow$ (654333310886666544, 873333210887666622, 876553332666644322) 37884329753093292 $\rightarrow$ (654333330866666544, 833333210887666662, 876533332666664322) 5104875400454544 $\rightarrow$ (655431110888865444, 877732110888762222, 877655552644443222) 17611050472067172 $\rightarrow$ (655433110888665444, 877332110888766222, 877655532664443222) 20720837004971100 $\rightarrow$ (655433310886665444, 873332110888766622, 877655332666443222) 20413502503210920 $\rightarrow$ (655433330866665444, 833332110888766662, 877653332666643222) 7665399521143776 $\rightarrow$ (655543110888654444, 877321110888876222, 877765552644432222) 10670008500612792 $\rightarrow$ (655543310886654444, 873321110888876622, 877765532664432222) 9664365651123504 $\rightarrow$ (655543330866654444, 833321110888876662, 877765332666432222) 4615754089561056 $\rightarrow$ (655554310886544444, 873211110888887622, 877776552644322222) 4405165352325876 $\rightarrow$ (655554330866544444, 833211110888887662, 877776532664322222) 1325127359483640 $\rightarrow$ (655555430865444444, 832111110888888762, 877777652643222222) 250697264684472 $\rightarrow$ (975111110888888421, 977777750842222221, 975555550844444421) 2390765619058848 $\rightarrow$ (975311110888886421, 977777530864222221, 975555530864444421) 14368726942559760 $\rightarrow$ (975331110888866421, 977775330866422221, 975555330866444421) 21731724146954172 $\rightarrow$ (975333110888666421, 977753330866642221, 975553330866644421) 11515994937142104 $\rightarrow$ (975333310886666421, 977533330866664221, 975533330866664421) 1001919175110984 $\rightarrow$ (975511110888884421, 977777510884222221, 977555550844444221) 20898333269792568 $\rightarrow$ (975531110888864421, 977775310886422221, 977555530864444221) 50870232777464298 $\rightarrow$ (975533110888664421, 977753310886642221, 977555330866444221) 67209914965351896 $\rightarrow$ (975533310886664421, 977533310886664221, 977553330866644221) 2201220396976032 $\rightarrow$ (975551110888844421, 977775110888422221, 977755550844442221) 12888794397934224 $\rightarrow$ (975553110888644421, 977753110888642221, 977755530864442221) 26301173380953480 $\rightarrow$ (975553310886644421, 977533110888664221, 977755330866442221) 2180598897630816 $\rightarrow$ (975555110888444421, 977751110888842221, 977775550844422221) 9803497619570592 $\rightarrow$ (975555310886444421, 977531110888864221, 977775530864422221) 902221889350176 $\rightarrow$ (975555510884444421, 977511110888884221, 977777550844222221) 9924142664688144 $\rightarrow$ (977551110888844221, 977775510884422221, 977555510884444221) 44780476859549520 $\rightarrow$ (977553110888644221, 977755310886442221, 977555310886444221) 13448849194329000 $\rightarrow$ (977555110888444221, 977755110888442221, 977755510884442221) 393351449572224 $\rightarrow$ (997511110888884201, 997777750842222201, 997555550844444201) 3990289674793968 $\rightarrow$ (997531110888864201, 997777530864222201, 997555530864444201) 12341187080996904 $\rightarrow$ (997533110888664201, 997775330866422201, 997555330866444201) 9414249141726000 $\rightarrow$ (997533310886664201, 997753330866642201, 997553330866644201) 2176728436209600 $\rightarrow$ (997551110888844201, 997777510884222201, 997755550844442201) 17441176197149328 $\rightarrow$ (997553110888644201, 997775310886422201, 997755530864442201)

Table 5

Orbits of the Kaprekar's transformations T_n for $n = 18$, continuation

$n = 18$, continuation				
β	18009598159150416 → (997553310886644201, 997753310886642201, 997755330866442201) 2665782114515520 → (997555110888444201, 997775110888422201, 997775550844422201) 10795265281272240 → (997555310886444201, 997753110888642201, 997775530864422201) 1693205117280432 → (997555510884444201, 997751110888842201, 997777550844222201) 9164642341681944 → (997755110888442201, 997775510884422201, 997755510884442201) 355749617013336 → (999751110888842001, 999777750842222001, 999755550844442001) 2684116560135036 → (999753110888642001, 999777530864222001, 999755530864442001) 5685547633260768 → (999753310886642001, 999775330866422001, 999755330866442001) 1989409814323776 → (999755110888442001, 999777510884222001, 999775550844422001) 4691018366080488 → (999755310886442001, 999775310886422001, 999775530864422001) 1187183159259552 → (999755510884442001, 999775110888422001, 999777550844222001) 150660139646016 → (999975110888420001, 999977750842220001, 999975550844420001) 939005034577896 → (999975310886420001, 999977530864220001, 999975530864420001) 301152102149808 → (999975510884420001, 999977510884220001, 999977550844220001) 28556497404864 → (999997510884200001, 999997750842200001, 999997550844200001) 149175155897380652 → (433333320876666666, 853333331766666642, 753333330866666643, 843333330866666652, 863333330866666632, 863333332666666632, 643333332666666654)			
β_1		108	(10, 9, 9, 9, 9, 8, 18, 18, 18)	
β_2		81	(10, 9, 9, 9, 9, 9, 9, 9, 8)	
β_3		90	(10, 9, 9, 9, 9, 9, 9, 8, 18)	
$\beta_4 - \beta_{12}$		81	(10, 9, 9, 9, 9, 9, 9, 9, 8)	
$\beta_{13} - \beta_{15}$	(1, 3, 2)	81	(10, 9, 9, 9, 9, 9, 9, 9, 8)	2,2,2,2
β_{16}	(1, 2, 3)	81	(10, 9, 9, 9, 9, 9, 9, 9, 8)	3,3,1,1
β_{17}, β_{18}	(1, 3, 2)	81	(10, 9, 9, 9, 9, 9, 9, 9, 8)	2,2,2,2
β_{19}	(1, 2, 3)	81	(10, 9, 9, 9, 9, 9, 9, 9, 8)	3,3,1,1
$\beta_{20} - \beta_{22}$	(1, 3, 2)	81	(10, 9, 9, 9, 9, 9, 9, 9, 8)	2,2,2,2
β_{23}, β_{24}	(1, 2, 3)	81	(10, 9, 9, 9, 9, 9, 9, 9, 8)	3,3,1,1
β_{25}	(1, 3, 2)	81	(10, 9, 9, 9, 9, 9, 9, 9, 8)	2,2,2,2
$\beta_{26} - \beta_{34}$	(1, 2, 3)	81	(10, 9, 9, 9, 9, 9, 9, 9, 8)	3,3,1,1
$\beta_{35} - \beta_{42}$	(1, 3, 2)	81	(10, 9, 9, 9, 9, 9, 9, 9, 8)	2,2,2,2
β_{43}	(1, 2, 3)	81	(10, 9, 9, 9, 9, 9, 9, 9, 8)	3,3,1,1
β_{44}	(1, 3, 2)	81	(10, 9, 9, 9, 9, 9, 9, 9, 8)	2,2,2,2
$\beta_{45} - \beta_{49}$	(1, 2, 3)	81	(10, 9, 9, 9, 9, 9, 9, 9, 8)	3,3,1,1
β_{50}, β_{51}	(1, 3, 2)	81	(10, 9, 9, 9, 9, 9, 9, 9, 8)	2,2,2,2
β_{52}	(1, 2, 3)	81	(10, 9, 9, 9, 9, 9, 9, 9, 8)	3,3,1,1
$\beta_{53} - \beta_{58}$	(1, 3, 2)	81	(10, 9, 9, 9, 9, 9, 9, 9, 8)	2,2,2,2
$\beta_{59} - \beta_{62}$	(1, 2, 3)	81	(10, 9, 9, 9, 9, 9, 9, 9, 8)	3,3,1,1
$\beta_{63} - \beta_{67}$	(1, 3, 2)	81	(10, 9, 9, 9, 9, 9, 9, 9, 8)	2,2,2,2
β_{68}, β_{69}	(1, 2, 3)	81	(10, 9, 9, 9, 9, 9, 9, 9, 8)	3,3,1,1
β_{70}, β_{71}	(1, 3, 2)	81	(10, 9, 9, 9, 9, 9, 9, 9, 8)	2,2,2,2
β_{72}	(1, 2, 3)	81	(10, 9, 9, 9, 9, 9, 9, 9, 8)	3,3,1,1
β_{73}	(1, 3, 2)	81	(10, 9, 9, 9, 9, 9, 9, 9, 8)	2,2,2,2
β_{74}	(1, 7, 3, 4, 2, 5, 6)	81	(10, 9, 9, 9, 9, 9, 9, 9, 8)	3,5,2,3

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