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SPECTRA OF SOME SELFADJOINT JACOBI OPERATORS IN THE DOUBLE ROOT CASE

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Abstract. In this paper we prove a mixed spectrum of Jacobi operators defined by $\lambda_n = s(n)(1+x(n))$ and $q_n = -2s(n)(1+y(n))$, where $(s(n))$ is a real unbounded sequence, $(x(n))$ and $(y(n))$ are some perturbations.

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1. INTRODUCTION

Let $J = J(\lambda_n, q_n)$ be a Jacobi operator acting in $l^2 = l^2(\mathbb{N};\mathbb{C})$ and defined by

$$(Ju)(n) := \lambda_{n-1}u(n-1) + q_n u(n) + \lambda_n u(n+1), \quad n \geq 2, \tag{1.1}$$

$$(Ju)(1) := q_1 u(1) + \lambda_1 u(2). \tag{1.2}$$

We will always assume that the operator J acts on its maximal domain which is a set

$$\mathcal{D}_{max}(J) := \left\{u \in l^2(\mathbb{N};\mathbb{C}) : Ju \in l^2(\mathbb{N};\mathbb{C})\right\}.$$

In this paper we will describe spectral properties of a class of the Jacobi operators in the *critical case*. We will show that operators of this class have mixed spectra. We will do this using asymptotic behavior of generalized eigenvectors and the subordinacy theory [8]. The generalized eigenvectors are the solutions of a second-order difference equation

$$\lambda_{n-1}u(n-1) + q_n u(n) + \lambda_n u(n+1) = \lambda u(n), \qquad \lambda \in \mathbb{R},\ n \geq 2. \tag{1.3}$$

By the critical case (or *double root* case) we mean the situation when the characteristic equation associated with system (1.3) has only one (double) solution. It is the case when

$$\lim_{n\to+\infty}\frac{\lambda_{n-1}}{\lambda_n}=1,\qquad \lim_{n\to+\infty}\frac{1}{\lambda_n}=0,\qquad \lim_{n\to+\infty}\frac{q_n}{\lambda_n}=\pm 2. \tag{1.4}$$

Rewrite system (1.3) in the matrix form:

$$\vec{u}(n+1;\lambda)=B(n;\lambda)\vec{u}(n;\lambda),\qquad \lambda\in\mathbb{R}, n\geq 2, \tag{1.5}$$

where

$$\vec{u}(n;\lambda):=\begin{pmatrix} u(n-1;\lambda)\\ u(n;\lambda)\end{pmatrix},\quad B(n;\lambda):=\begin{pmatrix} 0 & 1\\ -\frac{\lambda_{n-1}}{\lambda_n} & \frac{\lambda-q_n}{\lambda_n}\end{pmatrix}.$$

If (1.4) is valid, then the limit of the transfer matrices $B(n;\lambda)$ has only one eigenvalue ∓ 1 (it is a Jordan box). Such operators have been studied e.g. in [3, 7, 9–11]. In all of the papers spectral properties of the considered Jacobi operators were obtained in the same way: the asymptotic behavior of the generalized eigenvectors + the subordinacy theory.

The critical case corresponds to spectral phase transition phenomena, where the spectral structure changes drastically. It is the exact *moment* where the spectrum shifts from discrete to absolutely continuous one. We give more precise description of this fact in Section 4, see also e.g. [6, 7, 10, 11].

Let us briefly describe the subordinacy theory. Let $\mu(\cdot)$ be a spectral measure associated with the Jacobi operator J,

$$\left((J-\lambda)^{-1}e_1,e_1\right)=\int_{\mathbb{R}}\frac{d\mu(t)}{t-\lambda},$$

where $e_1=(1,0,\ldots)$ is the first vector of the canonical base in $l^2(\mathbb{N};\mathbb{C})$. Let $u(\lambda)=(u(n;\lambda))_{n=1}^{+\infty}$ be a solution of system (1.3). We say that $u(\lambda)$ is subordinate if

$$\lim_{N\to+\infty}\frac{\sqrt{\sum_{n=1}^{N}|u(n;\lambda)|^2}}{\sqrt{\sum_{n=1}^{N}|v(n;\lambda)|^2}}=0, \tag{1.6}$$

for any other solution $v(\lambda)$ of (1.3) linearly independent with $u(\lambda)$.

Define disjoint subsets Σ_{ac}, Σ_s and Σ_0 of $\mathbb{R}$, such that $\mathbb{R}=\Sigma_{ac}\cup\Sigma_s\cup\Sigma_0$, by

$$\Sigma_{ac}:=\{\lambda\in\mathbb{R}\colon \text{no subordinate solution of (1.3) exists}\},$$

$$\Sigma_s:=\{\lambda\in\mathbb{R}\colon \text{a subordinate solution of (1.3) exists and satisfies initial conditions } q_1u(1)+\lambda_1u(2)=\lambda u(1)\},$$

$$\Sigma_0:=\{\lambda\in\mathbb{R}\colon \text{a subordinate solution of (1.3) exists but does not satisfy initial conditions } q_1u(1)+\lambda_1u(2)=\lambda u(1)\}.$$

Then $\mu(\Sigma_0) = 0$, and the decomposition of μ into its respective absolutely continuous and singular components is given by

$$\mu_{ac} = \mu|_{\Sigma_{ac}}, \qquad \mu_s = \mu|_{\Sigma_s}.$$

Moreover, the sets Σ_0, Σ_{ac} and Σ_s are optimal with respect to Lebesgue measure, in the sense that

(i) if $\Sigma_0 \subset \Sigma_0'$ and $\mu(\Sigma_0') = 0$, then Lebesgue measure of $\Sigma_0' \setminus \Sigma_0$ is zero,
(ii) if $\Sigma_{ac}' \subset \Sigma_{ac}$ and $\mu_{ac} = \mu|_{\Sigma_{ac}'}$, then Lebesgue measure of $\Sigma_{ac} \setminus \Sigma_{ac}'$ is zero,
(iii) Lebesgue measure of Σ_s is zero.

The order of the paper is the following. In Section 2 we present our main result - Theorem 2.5, which we prove in Section 3. Section 4 contains short spectral analysis of the non-critical situation.

2. MAIN RESULTS

Let $J = J(\lambda_n, q_n)$ be a Jacobi operator defined by (1.1) and (1.2) with

$$\lambda_n = s(n)(1 + x(n)), \quad q_n = -2s(n)(1 + y(n)), \qquad n \geq 1, \tag{2.1}$$

where the sequences $(s(n))_{n=1}^{+\infty}$, $(x(n))_{n=1}^{+\infty}$ and $(y(n))_{n=1}^{+\infty}$ are such that, for $n \in \mathbb{N}$:

$$\text{(i)} \quad \lambda_n > 0, \quad q_n \in \mathbb{R}, \tag{2.2}$$

$$\text{(ii)} \quad \lim_{n\to+\infty} s(n) = +\infty,\ s(n) = \sum_{k=1}^{n} r(k),\ \left(\frac{r(n)}{s^{3/2}(n)}\right) \in l^1(\mathbb{N}; \mathbb{R}), \tag{2.3}$$

$$\text{(iii)} \quad \lim_{n\to+\infty} r(n) = 0, \quad (r(n)) \in \mathcal{D}^1, \quad r(n) > 0, \tag{2.4}$$

$$\text{(iv)} \quad \left(\sqrt{s(n)}x(n)\right) \in l^1(\mathbb{N}; \mathbb{R}), \quad \left(\sqrt{s(n)}y(n)\right) \in l^1(\mathbb{N}; \mathbb{R}). \tag{2.5}$$

Remark 2.1. We say that a sequence $(a(n))_{n=1}^{+\infty}$ belongs to $\mathcal{D}^1$ class iff

$$\sum_{n=1}^{+\infty} |a(n+1) - a(n)| < +\infty.$$

The above Jacobi operator induces second order difference equation (1.3) considered in [12]. We will use the asymptotic formulas of the generalized eigenvectors found in [12] to prove mixed nature of the spectrum of the Jacobi operator $J(\lambda_n, q_n)$. In [12, Section 2] the reader will find some properties and analysis of *regularity* of the sequences (λ_n) and (q_n). For example, sequences such like (n^α), for $\alpha \in (0, 1)$, and $(\ln n)$ satisfy the above assumptions. The case $s(n) = n^\alpha$ was studied in [7] and also in [10]. On the other hand the case $s(n) = \ln n$ has never been studied. Even more, the method from [10] can not be applied. We wanted to find the largest class of Jacobi operators in the double root case which contain the operators inducted by (n^α) and $(\ln n)$. In [12] we described the asymptotic behavior of the generalized eigenvectors of such operators. Here we are interested in their spectral properties.

For the reader's convenience we quote the results of [12] below.

Theorem 2.2. *Let* (λ_n) *and* (q_n) *be two real sequences defined by* (2.1)-(2.5). *If* $\lambda > 0$, *then recurrence system* (1.3) *has two linearly independent solutions* $(u_-(n;\lambda))_{n=1}^{+\infty}$ *and* $(u_+(n;\lambda))_{n=1}^{+\infty}$ *with the following asymptotic behavior:*

$$u_\pm(n;\lambda) \sim \left(\sqrt{s(n)}e^{-\varphi(n)}\right)^{1/2} \exp\left(\pm\sum_{k=1}^{n-1}\phi(k;\lambda)\right), \tag{2.6}$$

where

$$\begin{aligned}\varphi(n) &= \sum_{k=1}^{n} \frac{r(k)}{s(k-1)},\\ \phi(k;\lambda) &= \sqrt{\frac{\lambda}{s(k)}} + \frac{r(k)}{2\sqrt{\lambda s(k)}} - \frac{1}{24}\left(\frac{\lambda}{s(k)}\right)^{\frac{3}{2}}\\ &\quad + \mathcal{O}\left(s^{-2}(k)\right) + \mathcal{O}\left(\frac{r^2(k)}{s^{1/2}(k)}\right).\end{aligned} \tag{2.7}$$

All the $\mathcal{O}(\cdot)$ *terms in this formula are real.*

Theorem 2.3. *Let* (λ_n) *and* (q_n) *be two real sequences defined by* (2.1)–(2.5). *If* $\lambda < 0$, *then recurrence system* (1.3) *has two linearly independent solutions* $(u_-(n;\lambda))_{n=1}^{+\infty}$ *and* $(u_+(n;\lambda))_{n=1}^{+\infty}$ *with the following asymptotic behavior:*

$$u_\pm(n;\lambda) \sim \left(\sqrt{s(n)}e^{-\varphi(n)}\right)^{1/2} \exp\left(\pm i\sum_{k=1}^{n-1}\phi(k;\lambda)\right), \tag{2.8}$$

where $\varphi(n)$ *is given by* (2.7) *and*

$$\begin{aligned}\phi(k;\lambda) &= \sqrt{\frac{-\lambda}{s(k)}} - \frac{r(k)}{2\sqrt{-\lambda s(k)}} + \frac{1}{24}\left(\frac{-\lambda}{s(k)}\right)^{\frac{3}{2}}\\ &\quad + \mathcal{O}\left(s^{-5/2}(k)\right) + \mathcal{O}\left(r^2(k)s^{-1/2}(k)\right)\\ &\quad + i\mathcal{O}\left(s^{-2}(k)\right) + i\mathcal{O}\left(r^2(k)s^{-1}(k)\right).\end{aligned} \tag{2.9}$$

All the $\mathcal{O}(\cdot)$ *terms in this formula are real.*

Remark 2.4. We wrote the asymptotic formulas in the exponential form instead of the product form, as it is in [12], because it is suitable for us. This notation simplifies some calculations and allows us to use the Euler summation formula.

Using the above theorems we will prove the following.

Theorem 2.5. *Assume that* $J = J(\lambda_n, q_n)$ *is a Jacobi operator defined by* (1.1), (1.2). *If* (λ_n) *and* (q_n) *satisfy* (2.1)–(2.5), *then* J *is selfadjoint. Moreover:*

1. $\sigma(J) \cap (0, +\infty) \subset \sigma_p(J)$,
2. *if* $(s^{-2}(n))$ *and* $(r^2(n)s^{-1}(n))$ *are in* $l^1(\mathbb{N}, \mathbb{R})$, *then* $(-\infty, 0) \subset \sigma_{ac}(J)$ *and* J *is absolutely continuous on* $(-\infty, 0)$.

We see that in the double root case we have a pure point spectrum on the positive halfline and an absolutely continuous spectrum on the negative halfline. This result corresponds to [7, Theorem 5.1], but in our paper we deal with a much larger class of Jacobi operators. For example, the first part of the theorem is true for all the Jacobi operators $J = J(\lambda_n, q_n)$ with $s(n) = n^\alpha$ $(\alpha \in (0,1))$ or $s(n) = \ln n$. Unfortunately the second part of the theorem is not true for J with $s(n) = \ln n$ or $s(n) = n^\alpha$ where $\alpha \in \left(0, \frac{1}{2}\right)$. It is because $(s^{-2}(n))$ is not in l^1.

Generally, if $(s^{-2}(n))$ or $(r^2(n)s^{-1}(n))$ are not in l^1, then

$$\exp\left[\pm i \sum_{k=1}^{n-1} \left(i\mathcal{O}\left(\frac{1}{s^2(n)}\right) + i\mathcal{O}\left(\frac{r^2(n)}{s(n)}\right)\right)\right] \tag{2.10}$$

from (2.8) goes to infinity or to zero, depending on the sign. In this situation the solutions $u_\pm(\lambda)$ from Theorem 2.3 oscillate and one of them might be subordinate.

In non-critical situation it is easy to prove absolute continuity via asymptotic analysis and the subordinacy theory. In the critical situation we have a new phenomenon. The asymptotic formulas of the linearly independent solutions of system (1.5) contain vectors which are equal. This obstacle forced us to find a new (not trivial) way to estimate the quotient in (1.6). Precise description of this problem and its solution (in our setting) the reader will find in the next section. In [9] M. Moszyński investigates this problem in general situation.

3. PROOF OF THEOREM 2.5

First let us prove that the operator J is selfadjoint. It is obvious that from assumptions (2.4) and (2.5) one can obtain boundness of the sequences $(r(n))$ and $(x(n))$. We have that there are constants $C_1 > 0$ and $C_2 > 0$ such that for all $n \in \mathbb{N}$,

$$\lambda_n = s(n)(1 + x(n)) \leq (1 + C_2) \sum_{k=1}^{n} r(k) \leq (1 + C_2) \sum_{k=1}^{n} C_1 = C_1(1 + C_2)n.$$

The above inequality guarantees that the Carleman's condition is fulfilled, so J must be selfadjoint. For details see [1, Chapter VII, Theorem 1.3].

For the proof of the first part of Theorem 2.5 let us fix $\lambda > 0$. In this situation Theorem 2.2 implies existence of a solution $u_-(\lambda) = (u(n;\lambda))_{n=1}^{+\infty}$ of equation (1.3) with the asymptotics given by (2.6). From (2.3) and (2.4) we see that the main (dominant) part in asymptotic formula (2.6) is

$$\sum_{k=1}^{n} \sqrt{\frac{\lambda}{s(k)}}.$$

Because, as we noticed before, $s(n) \leq C_1 n$ for every $n \in \mathbb{N}$ the above sum goes to infinity for $n \to +\infty$, so

$$\exp\left(-\sum_{k=1}^{n}\sqrt{\frac{\lambda}{s(k)}}\right) \in l^2(\mathbb{N};\mathbb{R}).$$

The above observation and Lemma 3.1 below prove that there exist constants $C_1, C_2 > 0$ such that for large enough n

$$|u_-(n;\lambda)| \leq \frac{C_1}{\sqrt[4]{n}}\exp\left(-C_2\sqrt{n}\right)$$

which implies $u_-(\lambda) \in l^2$. The subordinacy theory [8, Theorem 3] finishes the proof of the first claim.

Now, let us turn to the proof of the second part of the theorem. Let λ be a fixed negative real number. According to Theorem 2.3, recurrence system (1.3) with $\lambda < 0$ has a base of solutions $\{u_+(\lambda), u_-(\lambda)\}$ with the asymptotic behavior given by (2.8). It means that system (1.5) has a base of solutions $\{\vec{u}_+(\lambda), \vec{u}_-(\lambda)\}$ with the following asymptotic behavior:

$$\vec{u}_\pm(n;\lambda) \sim \psi_\pm(n;\lambda)\vec{e}_\pm,$$

where $\psi_+(n;\lambda) = \overline{\psi_-(n;\lambda)}$ and $\vec{e}_\pm \in \mathbb{C}^2$. If the vectors $\vec{e}_+$ and $\vec{e}_-$ were linearly independent (which we normally have in the non-critical situation), then the proof of the second part of the theorem would be quite simple. Unfortunately, we are in the critical situation and the vectors $\vec{e}_\pm$ are linearly dependent, even more they are equal! To verify this see [12, Section 5].

This kind of situation was studied by M.Moszyński in [9]. Unfortunately particular results of his paper do not cover Jacobi operators considered in this paper. Of course we could use general result [9, Theorem 0.3] but then we would end up with the same kind of estimations which we present below.

We want to show that for $(u(n;\lambda))$ and $(v(n;\lambda))$, two linearly independent solutions of (1.3) with $\lambda < 0$, there exists a constant $\rho > 0$ such that

$$\frac{\sum_{n=n_0}^{N}|u(n;\lambda)|^2}{\sum_{n=n_0}^{N}|v(n;\lambda)|^2} \geq \rho > 0, \tag{3.1}$$

for all sufficiently large N.

Let $\lambda < 0$ be fixed. From Theorem 2.3 we know that for such λ there are two linearly independent solutions $u_\pm(\lambda) = (u_\pm(n;\lambda))_{n=1}^{+\infty}$ with the following asymptotics:

$$u_\pm(n;\lambda) \sim f(n)\exp\left(\pm i\Phi(n;\lambda)\right),$$

where

$$f(n) = \left(\sqrt{s(n)}e^{-\varphi(n)}\right)^{1/2}, \quad \Phi(n;\lambda) = \sum_{k=1}^{n-1}\sqrt{\frac{-\lambda}{s(k)}}(1+\gamma(k;\lambda)), \tag{3.2}$$

$\varphi(n)$ is given by (2.7) and $\gamma(n;\lambda)$ is a sequence of the order $\mathcal{O}(r(n)) + \mathcal{O}(s^{-1}(n))$. To be precise, the sequences $u_\pm(\lambda)$ in here differ from the ones in Theorem 2.3 by a constant factor. Because $(s^{-2}(n))$ and $(r^2(n)s^{-1}(n))$ are in $l^1(\mathbb{N}, \mathbb{R})$, so the terms (2.10) are convergent and can be omitted in the asymptotics of $u_\pm(n;\lambda)$ (see formula (2.8)). In this situation, the sequences in (3.2) are real! For the simplicity we leave the same notion.

Let $u(\lambda) = (u(n;\lambda))_{n=1}^{+\infty}$ and $v(\lambda) = (v(n;\lambda))_{n=1}^{+\infty}$ be linearly independent solutions of (1.3). Then

$$u(n;\lambda) = c_+ u_+(n;\lambda) + c_- u_-(n;\lambda), \quad n \in \mathbb{N},$$

$$v(n;\lambda) = d_+ u_+(n;\lambda) + d_- u_-(n;\lambda), \quad n \in \mathbb{N},$$

with some complex constants $c_\pm$ and $d_\pm$. If we assume that $c_\pm = r_\pm e^{i\theta_\pm}$ and $d_\pm = s_\pm e^{i\zeta_\pm}$, then

$$|u(n;\lambda)|^2 = \left|\left(r_+ f(n) e^{i(\theta_+ + \Phi(n;\lambda))}(1+o(1)) + r_- f(n) e^{i(\theta_- - \Phi(n;\lambda))}(1+o(1))\right)\right|^2,$$

$$|v(n;\lambda)|^2 = \left|\left(s_+ f(n) e^{i(\zeta_+ + \Phi(n;\lambda))}(1+o(1)) + s_- f(n) e^{i(\zeta_- - \Phi(n;\lambda))}(1+o(1))\right)\right|^2.$$

If in the above equalities we rewrite all the complex numbers in the polar form, then we have

$$|u(n;\lambda)|^2 = |f(n)|^2 \left(r_+^2 + r_-^2 + 2r_+r_- \cos\left(\theta + 2\Phi(n;\lambda)\right) + \epsilon_u(n;\lambda)\right), \tag{3.3}$$

$$|v(n;\lambda)|^2 = |f(n)|^2 \left(s_+^2 + s_-^2 + 2s_+s_- \cos\left(\zeta + 2\Phi(n;\lambda)\right) + \epsilon_v(n;\lambda)\right), \tag{3.4}$$

where $\theta := \theta_+ - \theta_-$, $\zeta := \zeta_+ - \zeta_-$ and $(\epsilon_\sigma(n;\lambda))_{n=1}^{+\infty}$ (for $\sigma = u, v$) are some real sequences decaying to zero in infinity, such that the terms in the brackets in (3.3) and (3.4) are not negative. From the above we have that for any $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\left((r_+ - r_-)^2 - \varepsilon\right)|f(n)|^2 \leq |u(n;\lambda)|^2, \tag{3.5}$$

$$\left((s_+ + s_-)^2 - \varepsilon\right)|f(n)|^2 \geq |v(n;\lambda)|^2. \tag{3.6}$$

If $r_+ \neq r_-$ and ε is small enough, e.g. $\varepsilon \leq \min\left\{\frac{(r_+-r_-)^2}{2}, \frac{(s_++s_-)^2}{2}\right\}$, then

$$\frac{\sum_{n=n_0}^N |u(n;\lambda)|^2}{\sum_{n=n_0}^N |v(n;\lambda)|^2} \geq \frac{\left((r_+-r_-)^2 - \varepsilon\right)\sum_{n=n_0}^N |f(n)|^2}{\left((s_++s_-)^2 + \varepsilon\right)\sum_{n=n_0}^N |f(n)|^2} \geq \frac{1}{3}\frac{(r_+-r_-)^2}{(s_++s_-)^2} > 0$$

and (3.1) is true. If $r_+ = r_-$, then we have negative number in the left hand side of (3.5). To deal with this situation we need more delicate approach.

Without lost of generality we may assume that $r_+ = r_- = 1$. Then (3.3) reads as

$$|u(n;\lambda)|^2 = |f(n)|^2 \left(2 + 2\cos\left(\theta + 2\Phi(n;\lambda)\right) + \epsilon_u(n;\lambda)\right). \tag{3.7}$$

Because θ and $\Phi(n;\lambda)$ are real, so

$$|u(n;\lambda)|^2 \leq C|f(n)|^2, \quad n \geq n_0,$$

with some constant $C > 0$ (independent on n and θ). This estimation is also true for $v(\lambda)$ But it may happen that, for some n, $\cos(\theta + 2\Phi(n;\lambda))$ will be equal (or almost equal) to -1, so we cannot estimate (3.7) from below by $C'|f(n)|^2$ with $C' > 0$. This situation occurs when, for some $k \in \mathbb{Z}$,

$$\left|\Phi(n;\lambda) - \frac{\pi - \theta}{2} - k\pi\right| < \varepsilon,$$

where ε is a small real number.

Let us fix $\varepsilon > 0$ and define, for all $k \in \mathbb{Z}$, numbers

$$\rho_k = \frac{\pi - \theta}{2} + k\pi,$$

and sets

$$A_k = \{n \in \mathbb{N} : \ \rho_k - \varepsilon \leq \Phi(n;\lambda) < \rho_k + \varepsilon\},$$
$$B_k = \{n \in \mathbb{N} : \ \rho_k + \varepsilon \leq \Phi(n;\lambda) < \rho_{k+1} - \varepsilon\}.$$

If a natural number n is in $\bigcup_{k\in\mathbb{Z}} A_k$, then (3.7) is very small and $u(n;\lambda)$ is almost zero. On the other hand, $u(n;\lambda)$ is separated from zero for every $n \in \bigcup_{k\in\mathbb{Z}} B_k$.

Because A_k and B_k are disjoint sets we can decompose the set of all natural numbers in the following way:

$$\mathbb{N} = \bigcup_{k\in\mathbb{Z}} (A_k \cup B_k).$$

Take $n_0 \in \mathbb{N}$ such that

$$0 < \sqrt{\frac{-\lambda}{s(n)}}(1 + \gamma(n;\lambda)) < \varepsilon, \qquad n \geq n_0, \tag{3.8}$$

and define

$$k_0 := \min\{k \in \mathbb{Z} : \exists n > n_0 : \rho_k - \varepsilon \leq \Phi(n;\lambda) < \rho_k + \varepsilon\}. \tag{3.9}$$

Simply, k_0 is the number of the first A_k set which is not empty.

Let $\overline{n} \in A_{k_0}$, then we may have $\Phi(\overline{n};\lambda) \leq \rho_{k_0}$ or $\rho_{k_0} < \Phi(\overline{n};\lambda)$. In the former case, using (3.2) and (3.8),

$$\rho_{k_0} - \varepsilon \leq \Phi(\overline{n};\lambda) < \Phi(\overline{n}+1;\lambda) = \Phi(\overline{n};\lambda) + \sqrt{\frac{-\lambda}{s(\overline{n}+1)}}(1 + \gamma(\overline{n}+1;\lambda)) < \rho_{k_0} + \varepsilon,$$

which means that $\overline{n}+1 \in A_{k_0}$ also. In the latter case, again using (3.2) and (3.8),

$$\rho_{k_0} + \varepsilon > \Phi(\overline{n};\lambda) > \Phi(\overline{n}-1;\lambda) = \Phi(\overline{n};\lambda) - \sqrt{\frac{-\lambda}{s(\overline{n})}}(1+\gamma(\overline{n};\lambda)) \geq \rho_{k_0} - \varepsilon,$$

which implies that $\overline{n}-1 \in A_{k_0}$. In either situation, $\#A_{k_0} > 1$.

Assumptions (2.3) and (2.4) imply that for n large enough $s(n) < n$ which proves that $\Phi(n;\lambda)$ diverges to infinity. Even more, by (3.2) and (3.8),

$$\Phi(n+1;\lambda) - \Phi(n;\lambda) < \varepsilon, \qquad n \geq n_0.$$

So, if $\#A_{k_0} > 1$, then it must be

$$\#A_k > 1, \quad \#B_k > 1, \qquad k \geq k_0, \tag{3.10}$$

because the sets A_k and B_k are getting bigger if $k \to +\infty$.

Let $K > k_0$ be a natural number and set

$$A_k = \{\underline{a}_k, \ldots, \overline{a}_k\}, \quad B_k = \{\underline{b}_k, \ldots, \overline{b}_k\}, \qquad k = k_0, \ldots, K,$$

$$n_1 := \underline{a}_{k_0}, \qquad N := \overline{b}_K.$$

With this notations we have

$$n_1 = \underline{a}_{k_0} < \ldots < \overline{b}_{k-1} < \underline{a}_k < \overline{a}_k < \underline{b}_k < \overline{b}_k < \ldots < \overline{b}_K = N, \tag{3.11}$$

$$\{n_1, \ldots, N\} = A_{k_0} \cup B_{k_0} \cup \ldots \cup A_K \cup B_K.$$

The idea is to estimate (3.7) separately for $n \in A_k$ and $n \in B_k$ $(k = k_0, \ldots, K)$. Let $k \in \{k_0, \ldots, K\}$. From the definition of A_k we have

$$\Phi(\overline{a}_k;\lambda) < \rho_k + \varepsilon, \quad \Phi(\underline{a}_k;\lambda) \geq \rho_k - \varepsilon,$$

which implies

$$\begin{aligned} 2\varepsilon &> \Phi(\overline{a}_k;\lambda) - \Phi(\underline{a}_k;\lambda) = \sum_{l=\underline{a}_k}^{\overline{a}_k-1} \sqrt{\frac{-\lambda}{s(l)}}(1+\gamma(l;\lambda)) \\ &\geq C_1 \sum_{l=\underline{a}_k}^{\overline{a}_k-1} \sqrt{\frac{-\lambda}{s(l)}} \geq C_1 \sqrt{\frac{-\lambda}{s(\overline{a}_k-1)}}(\#A_k - 1). \end{aligned} \tag{3.12}$$

Here C_1 is a positive constant.

Now, for $k \in \{k_0, \ldots, K\}$, from the definition of B_k, we have

$$\Phi(\underline{a}_k - 1;\lambda) < \rho_k - \varepsilon, \quad \Phi(\overline{a}_k + 1;\lambda) \geq \rho_k + \varepsilon,$$

and

$$\begin{aligned} 2\varepsilon &\leq \Phi(\overline{a}_k+1;\lambda)-\Phi(\underline{a}_k-1;\lambda)=\sum_{l=\underline{a}_k}^{\overline{a}_k+1}\sqrt{\frac{-\lambda}{s(l)}}(1+\gamma(l;\lambda)) \\ &\leq C_2\sum_{l=\underline{a}_k}^{\overline{a}_k+1}\sqrt{\frac{-\lambda}{s(l)}}\leq C_2\sqrt{\frac{-\lambda}{s(\underline{a}_k)}}(\#A_k+1). \end{aligned} \tag{3.13}$$

The constant C_2 is positive.

If we combine together (3.12), (3.13), (3.11) and recall that $s(n+1)>s(n)$ ($n\in\mathbb{N}$) we will have the following estimation of sizes of the sets A_k:

$$\frac{2\varepsilon}{C_2\sqrt{-\lambda}}\sqrt{s(\underline{a}_k)}-1\leq \#A_k\leq \frac{2\varepsilon}{C_1\sqrt{-\lambda}}\sqrt{s(\overline{a}_k-1)}+1,\quad k=k_0,\dots,K.$$

Because $s(n)$ increases to infinity and $\underline{b}_k=\overline{a}_k+1$, $\overline{b}_{k-1}=\underline{a}_k-1$, we can rewrite the above estimation as:

$$\underline{C}_A(\varepsilon;\lambda)\sqrt{s(\overline{b}_{k-1})}\leq \#A_k\leq \overline{C}_A(\varepsilon;\lambda)\sqrt{s(\overline{b}_k)},\quad k=k_0,\dots,K. \tag{3.14}$$

The constants $\underline{C}_A(\varepsilon;\lambda)$ and $\overline{C}_A(\varepsilon;\lambda)$ are strictly positive for all $\varepsilon>0$ and $\lambda<0$, and they go to zero if $\varepsilon\to 0$.

In the same simple way we can estimate sizes of the sets B_k. We have, that there are some positive constants C_3 and C_4 such that

$$\frac{\pi-2\varepsilon}{C_4\sqrt{-\lambda}}\sqrt{s(\underline{b}_k)}-1\leq \#B_k\leq \frac{\pi-2\varepsilon}{C_3\sqrt{-\lambda}}\sqrt{s(\overline{b}_k-1)}+1,\quad k=k_0,\dots,K.$$

Again, because $s(n)$ goes to infinity, we have

$$\underline{C}_B(\varepsilon;\lambda)\sqrt{s(\overline{b}_{k-1})}\leq \#B_k\leq \overline{C}_B(\varepsilon;\lambda)\sqrt{s(\overline{b}_k)},\quad k=k_0,\dots,K. \tag{3.15}$$

Here, $\underline{C}_B(\varepsilon;\lambda)$ and $\overline{C}_B(\varepsilon;\lambda)$ are also strictly positive for all $\varepsilon>0$ and $\lambda<0$, but they tend to some positive constants if $\varepsilon\to 0$.

In the next step of this proof we will estimate the sequence $(f(n))$, given by (3.2), which appears in the asymptotic formulas of the solutions of generalized eigenequation (1.3).

Lemma 3.1. *Let*

$$f(n)=\left[\sqrt{s(n)}\exp\left(-\sum_{k=1}^{n}\frac{r(k)}{s(k-1)}\right)\right]^{1/2},\qquad n\in\mathbb{N}.$$

There are some positive constants $\underline{C}_f$ and $\overline{C}_f$ such that, for sufficiently large n,

$$\frac{\underline{C}_f}{\sqrt{s(n)}}\leq |f(n)|^2\leq \frac{\overline{C}_f}{\sqrt{s(n)}}. \tag{3.16}$$

Proof. Define a function $\tilde{r} : [1, +\infty) \to \mathbb{R}$,

$$\tilde{r}(x) = \begin{cases} r(n), & x = n \in \mathbb{N}, \\ r(n+1)(x-n) + r(n)(n+1-x), & x \in (n, n+1). \end{cases} \tag{3.17}$$

Remark 3.2. Here we could use any other continuous function $\tilde{r}$ monotonic on every interval $[n, n+1]$ and such that $\tilde{r}(n) = r(n)$ for all $n \in \mathbb{N}$.

From Euler summation formula we have

$$\sum_{k=1}^{n} r(k) = \int_{1}^{n} \tilde{r}(x)\, dx + d_r(n) + r(n), \qquad n > 1,$$

where

$$d_r(n) = \sum_{k=1}^{n-1} \int_{k}^{k+1} (r(k) - \tilde{r}(x))\, dx, \qquad n > 1.$$

We assumed that $(r(n)) \in \mathcal{D}^1$. It implies

$$\begin{aligned} |d_r(n)| &\le \sum_{k=1}^{n-1} \int_{k}^{k+1} |r(k) - \tilde{r}(x)|\ dx \le \sum_{k=1}^{n-1} \int_{k}^{k+1} |r(k+1) - r(k)|\ dx \\ &= \sum_{k=1}^{n-1} |r(k+1) - r(k)| < \sum_{k=1}^{+\infty} |r(k+1) - r(k)| < +\infty. \end{aligned}$$

From the above inequality we have

$$s(n) = \sum_{k=1}^{n} r(k) = \int_{1}^{n} \tilde{r}(t)\, dt + C_r(n), \qquad n > 1, \tag{3.18}$$

where $(C_r(n))$ is a real bounded sequence.

Let us define a function $\tilde{s} : [1, +\infty) \to \mathbb{R}$,

$$\tilde{s}(x) := \int_{1}^{x} \tilde{r}(t)\, dt + 1, \qquad x \ge 1. \tag{3.19}$$

We added the 1 in order to $\tilde{s}(1) \ne 0$ but still $\tilde{s}'(x) = \tilde{r}(x)$ for all $x \ge 1$. The definition of $\tilde{s}$ and (3.18) imply

$$s(n) = \tilde{s}(n) + C_r(n) - 1, \qquad n > 1. \tag{3.20}$$

Now we can define continuous version of $(f(n))$. Let $\tilde{f} : [1, +\infty) \to \mathbb{R}$ be defined as

$$\tilde{f}(x) := \left[\sqrt{\tilde{s}(x)} \exp\left(- \int_{1}^{x} \frac{\tilde{r}(t)}{\tilde{s}(t)}\, dt \right) \right]^{1/2}, \qquad x \ge 1.$$

Simple calculus shows that, for $x \geq 1$,

$$\tilde{f}(x) = \left[\sqrt{\tilde{s}(x)} \exp\left(-\ln \tilde{s}(x) + \ln \tilde{s}(1)\right)\right]^{1/2} = \left[\sqrt{\tilde{s}(x)} \frac{\tilde{s}(1)}{\tilde{s}(x)}\right]^{1/2} = \left[\frac{\tilde{s}(1)}{\sqrt{\tilde{s}(x)}}\right]^{1/2}.$$

This observation implies, for n sufficiently large,

$$\left|\tilde{f}(n)\right|^2 = \frac{\tilde{s}(1)}{\sqrt{\tilde{s}(n)}} = \frac{\tilde{s}(1)}{\sqrt{s(n)}} \left(1 + \mathcal{O}\left(s^{-1}(n)\right)\right). \tag{3.21}$$

On the other hand

$$\left|\tilde{f}(n)\right|^2 = \sqrt{\tilde{s}(n)} \exp\left(-\int_1^n \frac{\tilde{r}(t)}{\tilde{s}(t)}\, dt\right), \qquad n > 1. \tag{3.22}$$

Using Euler summation formula once again we can *rewrite* the integral in the above formula as

$$\int_1^n \frac{\tilde{r}(x)}{\tilde{s}(x)}\, dx = \sum_{k=1}^{n} \frac{\tilde{r}(k)}{\tilde{s}(k)} - d_{r/s}(n) - \frac{\tilde{r}(n)}{\tilde{s}(n)}, \qquad n > 1,$$

where

$$d_{r/s}(n) = \sum_{k=1}^{n-1} \int_k^{k+1} \left(\frac{\tilde{r}(k)}{\tilde{s}(k)} - \frac{\tilde{r}(x)}{\tilde{s}(x)}\right) dx, \qquad n > 1.$$

We can estimate the above integral in the following way:

$$\begin{aligned}\left|\int_k^{k+1} \left(\frac{\tilde{r}(k)}{\tilde{s}(k)} - \frac{\tilde{r}(x)}{\tilde{s}(x)}\right) dx\right| &\leq \int_k^{k+1} \left|\frac{\tilde{r}(k)}{\tilde{s}(k)} - \frac{\tilde{r}(x)}{\tilde{s}(x)}\right| dx \\ &= \int_k^{k+1} \left|\frac{\tilde{s}(x)\tilde{r}(k) - \tilde{s}(x)\tilde{r}(x) + \tilde{s}(x)\tilde{r}(x) - \tilde{s}(k)\tilde{r}(x)}{\tilde{s}(k)\tilde{s}(x)}\right| dx \\ &\leq \max_{x \in [k,k+1]} \left(\frac{|\tilde{r}(k) - \tilde{r}(x)|}{\tilde{s}(k)} + \frac{\tilde{r}(x)\,|\tilde{s}(x) - \tilde{s}(k)|}{\tilde{s}^2(k)}\right).\end{aligned}$$

From (3.17), (3.19) and (3.20) we have, for k large enough, that the right-hand side of the above inequality is

$$\mathcal{O}\left(\frac{(\Delta r)(k)}{s(k)}\right) + \mathcal{O}\left(\frac{r^2(k)}{s^2(k)}\right),$$

and is summable, according to assumptions (2.3) and (2.4). So, $(d_{r/s}(n))$ is convergent and

$$\int_1^n \frac{\tilde{r}(x)}{\tilde{s}(x)}\, dx = \sum_{k=1}^{n} \frac{\tilde{r}(k)}{\tilde{s}(k)} + C_{r/s}(n), \qquad n > 1,$$

where $C_{r/s}(n)$ tends to a real constant $C_{r/s}$, if $n \to +\infty$. If we recall that $\tilde{r}(k) = r(k)$, for all $k \in \mathbb{N}$, and use (3.20), then for n sufficiently large

$$\begin{aligned}\int_1^n \frac{\tilde{r}(x)}{\tilde{s}(x)}\,dx &= \sum_{k=1}^{n} \frac{r(k)}{s(k) - C_r(k) + 1} + C_{r/s}(n)\\ &= \sum_{k=1}^{n} \left[\frac{r(k)}{s(k)} + \mathcal{O}\left(\frac{r(k)}{s^2(k)}\right)\right] + C_{r/s}(n)\\ &= \sum_{k=1}^{n} \left[\frac{r(k)}{s(k-1)} + \mathcal{O}\left(\frac{r(k)}{s^2(k)}\right)\right] + C_{r/s}(n)\\ &= \sum_{k=1}^{n} \frac{r(k)}{s(k-1)} + C(n),\end{aligned}$$

where $(C(n))$ is a convergent real sequence. In the last two equalities we used assumption (2.3). If we put the above equality into (3.22), then for n large enough

$$\begin{aligned}\left|\tilde{f}(n)\right|^2 &= \sqrt{\tilde{s}(n)}\exp\left(-\sum_{k=1}^{n}\frac{r(k)}{s(k-1)} - C(n)\right)\\ &= \sqrt{s(n) - C_r(n) + 1}\exp\left(-\sum_{k=1}^{n}\frac{r(k)}{s(k-1)} - C(n)\right)\\ &= \epsilon(n)\sqrt{s(n)}\exp\left(-\sum_{k=1}^{n}\frac{r(k)}{s(k-1)}\right) = \epsilon(n)|f(n)|^2.\end{aligned}$$

Here $(\epsilon(n))$ is real and converges to some positive constant. This observation and (3.21) finish the proof of Lemma 3.1. □

Remark 3.3. Lemma 3.1 gives us something more. Under conditions of Theorem 2.5 we can prove that $f(n) \sim C(s(n))^{-1/4}$. This simplifies the form of the asymptotics (2.6) and (2.8), making them look more "WKB-like". In order to prove Theorem 2.5 we only needed to estimate $|f(n)|^2$, that is why we formulated Lemma 3.1 this way.

In the above calculations we used the phrase "for sufficiently large n" several times. In every case it means that there exists a natural number n' such that for $n \geq n'$ some expression is true. If we take n'' as the largest n', then all the above estimations are true for $n \geq n''$. In (3.8) we fixed n_0 and later we used it to define k_0 in (3.9), and n_1 in (3.11). Without lost of generality we can always assume that $n_0 = n''$. In the remaining part of this paper whenever some expression is true "for n large enough" we always assume that n_0, and by this, k_0 and n_1 are large enough.

We have estimated the lengths of the sets A_k and B_k, and the values of $f(n)$. Now we can estimate $|u(n;\lambda)|^2$ and $|v(n;\lambda)|^2$ separately on A_k and B_k. Let us start with the latter.

From (3.6) we know that $|v(n;\lambda)|^2$ is bounded from above by $\kappa|f(n)|^2$, with some positive constant κ. If $n \in A_k$ $(k = k_0, \dots, K)$, then (3.16) and (3.11) imply

$$|v(n;\lambda)|^2 \le \frac{\overline{C}_f \kappa}{\sqrt{s(\overline{b}_{k-1})}}, \qquad n \in A_k,\ k = k_0, \dots, K. \tag{3.23}$$

The above inequality and (3.14) give us the following estimation:

$$\sum_{n \in A_k} |v(n;\lambda)|^2 \le C_1(\varepsilon) \sqrt{\frac{s(\overline{b}_k)}{s(\overline{b}_{k-1})}}, \qquad k = k_0, \dots, K. \tag{3.24}$$

Here $C_1(\varepsilon) > 0$ and goes to zero, if $\varepsilon \to 0$.

Similarly for $n \in B_k$ $(k = k_0, \dots, K)$, (3.16) and (3.11) imply

$$|v(n;\lambda)|^2 \le \frac{\overline{C}_f C}{\sqrt{s(\overline{b}_{k-1})}}, \qquad n \in B_k,\ k = k_0, \dots, K. \tag{3.25}$$

Now, by this estimation and (3.15) we have:

$$\sum_{n \in B_k} |v(n;\lambda)|^2 \le C_2(\varepsilon) \sqrt{\frac{s(\overline{b}_k)}{s(\overline{b}_{k-1})}}, \qquad k = k_0, \dots, K. \tag{3.26}$$

where $C_2(\varepsilon) > 0$ tends to some strictly positive constant, if $\varepsilon \to 0$.

If we notice that

$$\sum_{n=n_1}^{N} |v(n;\lambda)|^2 = \sum_{n=k_0}^{K} \left(\sum_{n \in A_k} |v(n;\lambda)|^2 + \sum_{n \in B_k} |v(n;\lambda)|^2 \right), \tag{3.27}$$

and use (3.24) and (3.26), we will get the following:

$$\sum_{n=n_1}^{N} |v(n;\lambda)|^2 \le C_U(\varepsilon) \sum_{k=k_0}^{K} \sqrt{\frac{s(\overline{b}_k)}{s(\overline{b}_{k-1})}}. \tag{3.28}$$

where $C_U(\varepsilon) \to C_U > 0$, if $\varepsilon \to 0$.

Now let us turn to $|u(n;\lambda)|^2$. In order to estimate $\sum_{n_1}^{N} |u(n;\lambda)|^2$ from below we only need to consider $u(n;\lambda)$ for $n \in B_k$. If $n \in A_k$, then $|u(n;\lambda)|$ is very small and we can omit it. From (3.27), with u instead of v, we have:

$$\sum_{n=n_1}^{N} |u(n;\lambda)|^2 \ge \sum_{k=k_0}^{K} \sum_{n \in B_k} |u(n;\lambda)|^2. \tag{3.29}$$

Let $n \in B_k$ $(k = k_0, \dots, K)$. From the definition of the sets B_k, there exists a positive constant $\tilde{\kappa} = \tilde{\kappa}(\varepsilon)$ such that

$$2 + 2\cos\left(\theta + 2\Phi(n;\lambda)\right) + \epsilon_u(n;\lambda) \ge \tilde{\kappa}(\varepsilon), \qquad n \ge n_0,$$

which implies (see also (3.7), (3.11) and (3.16)):

$$|u(n;\lambda)|^2 \geq \frac{C_f\tilde{\kappa}(\varepsilon)}{\sqrt{s(\bar{b}_k)}}, \qquad n \in B_k,\ k = k_0,\dots,K. \tag{3.30}$$

Here we remind that ε is fixed.

The estimations (3.15), (3.29) and (3.30) imply:

$$\sum_{n=n_1}^{N} |u(n;\lambda)|^2 \geq C_L(\varepsilon) \sum_{k=k_0}^{K} \sqrt{\frac{s(\bar{b}_{k-1})}{s(\bar{b}_k)}}, \tag{3.31}$$

where $C_L(\varepsilon)$ is some positive constant which goes to zero, if $\varepsilon \to 0$.

The last thing which requires a proof is the fact that $\sqrt{\frac{s(\bar{b}_{k-1})}{s(\bar{b}_k)}}$ in (3.31) is not too small,

$$\sqrt{\frac{s(\bar{b}_{k-1})}{s(\bar{b}_k)}} = \left(\frac{s(\bar{b}_k) + s(\bar{b}_{k-1}) - s(\bar{b}_k)}{s(\bar{b}_k)}\right)^{1/2} = \left(1 - \frac{s(\bar{b}_k) - s(\bar{b}_{k-1})}{s(\bar{b}_k)}\right)^{1/2}.$$

Define

$$\phi(k) := \frac{s(\bar{b}_k) - s(\bar{b}_{k-1})}{s(\bar{b}_k)}, \qquad k = k_0,\dots,K.$$

From assumptions (2.2) and (2.3), and (3.11) we have that $\phi(k) > 0$ and

$$\phi(k) = \frac{r(\bar{b}_{k-1}+1) + \ldots + r(\bar{b}_k)}{s(\bar{b}_k)} \leq \frac{(\#A_k + \#B_k)}{s(\bar{b}_k)} \max_{t\in A_k\cup B_k} r(t).$$

This inequality, (3.14), (3.15) and (2.4) shows that

$$\phi(k) \leq \frac{C(\varepsilon)}{\sqrt{s(\bar{b}_k)}} \max_{t\in A_k\cup B_k} r(t), \qquad k = k_0,\dots,K.$$

The right hand side of this inequality goes to zero, if $k \to +\infty$. So we can always assume there is a constant $C > 0$ such that, for $k = k_0,\dots,K$, the following is true:

$$\sqrt{\frac{s(\bar{b}_{k-1})}{s(\bar{b}_k)}} = (1-\phi(k))^{1/2} = 1 - \frac{1}{2}\phi(k) + \mathcal{O}\left(\phi^2(k)\right) \geq C > 0. \tag{3.32}$$

In order to end the proof we need to use (3.28), (3.31) and (3.32):

$$\frac{\sum_{n=n_1}^{N}|u(n;\lambda)|^2}{\sum_{n=n_1}^{N}|v(n;\lambda)|^2} \geq \frac{C_L(\varepsilon)\sum_{k=k_0}^{K}\sqrt{\frac{s(\bar{b}_{k-1})}{s(\bar{b}_k)}}}{C_U(\varepsilon)\sum_{k=k_0}^{K}\sqrt{\frac{s(\bar{b}_k)}{s(\bar{b}_{k-1})}}} \geq \frac{C_L(\varepsilon)C(K-k_0)}{C_U(\varepsilon)\frac{1}{C}(K-k_0)} \geq \rho(\varepsilon) > 0.$$

In this formula $\rho(\varepsilon)$ is a constant because ε was fixed, but $N = \bar{b}_K$ goes to infinity, because K was chosen to be an arbitrarily large natural number. With this sentence we finish the proof.

□

4. NON-CRITICAL SITUATION

In this section we consider a Jacobi operator $J_\delta = J_\delta(\lambda_n, q_n)$ with

$$\lambda_n := s(n)(1 + x(n)) \quad q_n = \delta s(n)(1 + y(n)), \qquad n \geq 1, \tag{4.1}$$

where the sequences $(s(n))_{n=1}^{+\infty}$, $(x(n))_{n=1}^{+\infty}$ and $(y(n))_{n=1}^{+\infty}$ are defined by (2.2)-(2.5), and δ is a real parameter. We will describe $\sigma(J_\delta)$ for $\delta \neq \pm 2$. This result will justify the name "critical situation" for $\delta = \pm 2$.

Theorem 4.1. *Let $J_\delta = J_\delta(\lambda_n, q_n)$ be a Jacobi operator defined by* (1.1), (1.2). *If (λ_n) and (q_n) satisfy* (4.1) *and* (2.2)-(2.5), *then J_δ is selfadjoint and*

1. *if $|\delta| < 2$, then $\sigma_{ac}(J_\delta) = \mathbb{R}$ and J_δ is absolutely continuous on the real line,*
2. *if $|\delta| > 2$, then $\sigma(J_\delta) = \sigma_d(J_\delta)$.*

Proof. In order to prove selfadjointness of J_δ one need to repeat the reasoning from the proof of Theorem 2.5.

Let us first deal with the case $|\delta| < 2$. Simple calculations show that, for n large enough,

$$\begin{aligned}
\frac{\lambda_{n-1}}{\lambda_n} &= 1 - \frac{r(n)}{s(n-1)} + \mathcal{O}\left(\frac{r^2(n)}{s^2(n)}\right) + \mathcal{O}(x(n)),\\
\frac{1}{\lambda_n} &= \frac{1}{s(n)} + \mathcal{O}(x(n)),\\
\frac{q_n}{\lambda_n} &= \delta + \mathcal{O}(x(n)) + \mathcal{O}(y(n)).
\end{aligned}$$

So the forward differences of the above sequences must be equal respectively:

$$\begin{aligned}
\frac{\lambda_n}{\lambda_{n+1}} - \frac{\lambda_{n-1}}{\lambda_n} &= \frac{r(n) - r(n+1)}{s(n)s(n+1)} + \mathcal{O}\left(\frac{r^2(n)}{s^2(n)}\right) + \mathcal{O}(x(n)),\\
\frac{1}{\lambda_{n+1}} - \frac{1}{\lambda_n} &= \frac{s(n) - s(n+1)}{s(n)s(n+1)} + \mathcal{O}(x(n)) = \frac{-r(n+1)}{s(n)s(n+1)} + \mathcal{O}(x(n))\\
&= \frac{-r(n+1)}{s^2(n+1)} + \mathcal{O}\left(\frac{r^2(n)}{s^3(n)}\right) + \mathcal{O}(x(n)),\\
\frac{q_{n+1}}{\lambda_{n+1}} - \frac{q_n}{\lambda_n} &= \mathcal{O}(x(n)) + \mathcal{O}(y(n)).
\end{aligned}$$

The above equalities and assumpions (2.2)-(2.5) prove that the sequences $(\frac{\lambda_{n-1}}{\lambda_n})$, $(\frac{1}{\lambda_n})$ and $(\frac{q_n}{\lambda_n})$ are of bounded variation. It is also easy to check that

$$\lim_{n\to+\infty} \frac{\lambda_{n-1}}{\lambda_n} = 1, \qquad \lim_{n\to+\infty} \frac{1}{\lambda_n} = 0, \qquad \lim_{n\to+\infty} \frac{q_n}{\lambda_n} = \delta.$$

Now Theorem 1.6 of [4] finishes the proof of the first part of the theorem.

In order to prove the second claim we will use Theorem 4.1 of [5] which says that if

$$\lim_{n\to+\infty} |q_n| = +\infty, \quad \liminf_{n\to+\infty} \frac{q_n^2}{\lambda_n^2+\lambda_{n-1}^2} > 2, \tag{4.2}$$

then a Jacobi operator $J = J(\lambda_n, q_n)$ has only discrete spectrum. In our situation

$$\begin{aligned} \frac{q_n^2}{\lambda_n^2+\lambda_{n-1}^2} &= \frac{\delta^2 s^2(n)(1+y(n))^2}{2s^2(n)\left((1+x(n))^2+\left(1-\frac{r(n)}{s(n)}\right)^2(1+x(n-1))^2\right)} \\ &= \frac{\delta^2}{2}\left(1+\mathcal{O}\left(\frac{r(n)}{s(n)}\right)+\mathcal{O}\left(x(n)\right)+\mathcal{O}\left(y(n)\right)\right) \longrightarrow \frac{\delta^2}{2}. \end{aligned}$$

The last statement proves (4.2), if $|\delta| > 2$. □

Now we see that $\delta = -2$ and $\delta = 2$ are border lines between absolute continuity and discreteness of the operator J_δ. In some sense it explains why the operators $J_{\pm 2}$ have mixed spectrum, as we stated in Theorem 2.5.

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