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AN ALTERNATIVE APPROACH TO JERK IN MOTION ALONG A SPACE CURVE WITH APPLICATIONS

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Jerk is the time derivative of an acceleration vector and, hence, the third time derivative of the position vector. In this paper, we consider a particle moving in the three dimensional Euclidean space and resolve its jerk vector along the tangential direction, radial direction in the osculating plane and the other radial direction in the rectifying plane. Also, the case for planar motion in space is given as a corollary. Furthermore, motion of an electron under a constant magnetic field and motion of a particle along a logarithmic spiral curve are given as illustrative examples. The aforementioned decomposition is a new contribution to the field and it may be useful in some specific applications that may be considered in the future.

Keywords: jerk, kinematics of a particle, plane and space curves

1. Introduction

In Newtonian physics, the force acting on the particle is related to its acceleration through $\mathbf{F} = m\mathbf{a}$. The jerk $\mathbf{J}$ is the time derivative of the acceleration. Therefore, if the mass is constant, we have $\mathbf{J} = (1/m)d\mathbf{F}/dt$. If the time derivative of the force is nonzero, we have a nonzero jerk vector. The time derivative of the force may be nonzero even if it does not depend on time explicitly, e.g. the force in a simple harmonic oscillator $\mathbf{F} = -k\mathbf{r}$.

The concepts of the jerk and jerk vector were proved to be useful in many areas, especially in motion control and mechanics. For example, "the jerk, as a predictor of large accelerations of short duration, has important practical engineering applications in the design of intermittent--motion mechanisms such as cams and genevas (Bickford, 1972; Faires, 1965; Freudenstein and Sandor, 1964)" (Schot, 1978). In the applications such as gymnastics, railways and streetcar design, the physiological aspects of the jerk that is felt were investigated in the work (Melchior, 1928).

For a particle moving along a curve in the 3D Euclidean space, decomposition of the acceleration vector along the tangential and normal components is known in the literature. However, in the situation that the angular momentum is constant, the decomposition (which is considered by Siacci (1879)) of the acceleration vector along the tangential and radial components is more beneficial than the aforementioned decomposition. In Siacci's study, a component of acceleration lies along the tangent to the path while the other one is directed from the particle towards the foot of the perpendicular that is from an arbitrary fixed origin to the instantaneous osculating plane to the path (Casey, 2011). Later on, Casey studied Siacci's resolution of the acceleration vector in space which is equipped with the Serret-Frenet frame (Casey, 2011). Inspired by Siacci's and Casey's studies, in the present paper we obtain a new resolution of the jerk vector for the aforementioned particle.

This paper is organized as follows. In Section 2, the Serret-Frenet frame of a curve has been reviewed to disambiguate the ensuing Sections. Also, some necessary knowledge on the jerk vector of a particle moving along a space curve, and backgrounds on motion of this particle have been recalled. In Section 3, we have obtained a new resolution of the jerk vector for the aforementioned particle. There exist three components of the jerk vector in our resolution. One component lies along the tangent line of the path, the second and the third component lie along the lines which pass through the particle and the foots of the perpendiculars that are from the origin of the space to the instantaneous osculating plane and rectifying plane, respectively. Additionally, we have discussed the planar case in the space as a corollary. In Section 4, we have given illustrative examples for our resolution. Finally, in Section 5 we conclude the results obtained in our paper.

2. Preliminaries

Let the 3-dimensional Euclidean space E^3 be equipped with the standard inner product

$$\langle \mathbf{A}, \mathbf{B} \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{2.1}$$

where $\mathbf{A} = [a_1, a_2, a_3]$, $\mathbf{B} = [b_1, b_2, b_3]$ are arbitrary vectors in E^3. The norm of the vector $\mathbf{A} \in E^3$ is expressed by $|\mathbf{A}| = \langle \mathbf{A}, \mathbf{A} \rangle^{1/2}$. A curve $\alpha = \alpha(s) : I \subseteq \mathbb{R} \to E^3$ is a unit speed curve if $|\alpha'(s)| = 1$ for every $s \in I$. Then, s is called the arc-length parameter of the curve $\alpha(s)$.

Let us denote by $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ the moving Serret-Frenet frame along the unit speed curve $\alpha(s)$. $\mathbf{T}(s)$, $\mathbf{N}(s)$ and $\mathbf{B}(s)$ are called the tangent, principal normal and binormal vectors, respectively. They can be defined as follows

$$\mathbf{T}(s) = \alpha'(s) \qquad \mathbf{N}(s) = \frac{\alpha''(s)}{|\alpha''(s)|} \qquad \mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s) \tag{2.2}$$

On the other hand, the derivative formulas of Serret-Frenet frame in the matrix form are given by

$$\begin{bmatrix} \mathbf{T}'(s) \\ \mathbf{N}'(s) \\ \mathbf{B}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{bmatrix} \tag{2.3}$$

where the functions $\kappa(s)$, $\tau(s)$ are defined as follows

$$\kappa(s) = |\mathbf{T}'(s)| \qquad \tau(s) = -\langle \mathbf{B}'(s), \mathbf{N}(s) \rangle \tag{2.4}$$

and also called as the curvature and the torsion of the curve $\alpha(s)$, respectively (Shifrin, 2015).

Take into consideration a particle whose mass is m and which moves in E^3 under the influence of arbitrary forces. Choose an arbitrary fixed origin O in the space and denote by $\mathbf{x}$ the position vector of P at time t. Let the curve C, parameterized by the arc-length parameter s, be the oriented curve traced out by P. Here, the arc-length of C corresponds to the time t. In this case, the unit tangent vector of the curve C can be written as

$$\mathbf{T} = \frac{d\mathbf{x}}{ds} \tag{2.5}$$

With the aid of (2.3) and (2.5), the velocity vector $\mathbf{v}$ and the acceleration vector $\mathbf{a}$ of P at the time t are given as follows

$$\begin{aligned} \mathbf{v} &= \frac{d\mathbf{x}}{dt} = \frac{ds}{dt}\mathbf{T} \\ \mathbf{a} &= \frac{d\mathbf{v}}{dt} = \frac{d^2 s}{dt^2}\mathbf{T} + \kappa\Big(\frac{ds}{dt}\Big)^2\mathbf{N} \end{aligned} \tag{2.6}$$

Thus, the jerk vector (derivative of acceleration with respect to time) is obtained after some calculations as (see Tsirlin (2017) for more details)

$$\mathbf{J} = \Big[\frac{d^3s}{dt^3} - \kappa^2\Big(\frac{ds}{dt}\Big)^3\Big]\mathbf{T} + \Big[3\kappa\frac{ds}{dt}\frac{d^2s}{dt^2} + \frac{d\kappa}{ds}\Big(\frac{ds}{dt}\Big)^3\Big]\mathbf{N} + \Big[\kappa\tau\Big(\frac{ds}{dt}\Big)^3\Big]\mathbf{B} \tag{2.7}$$

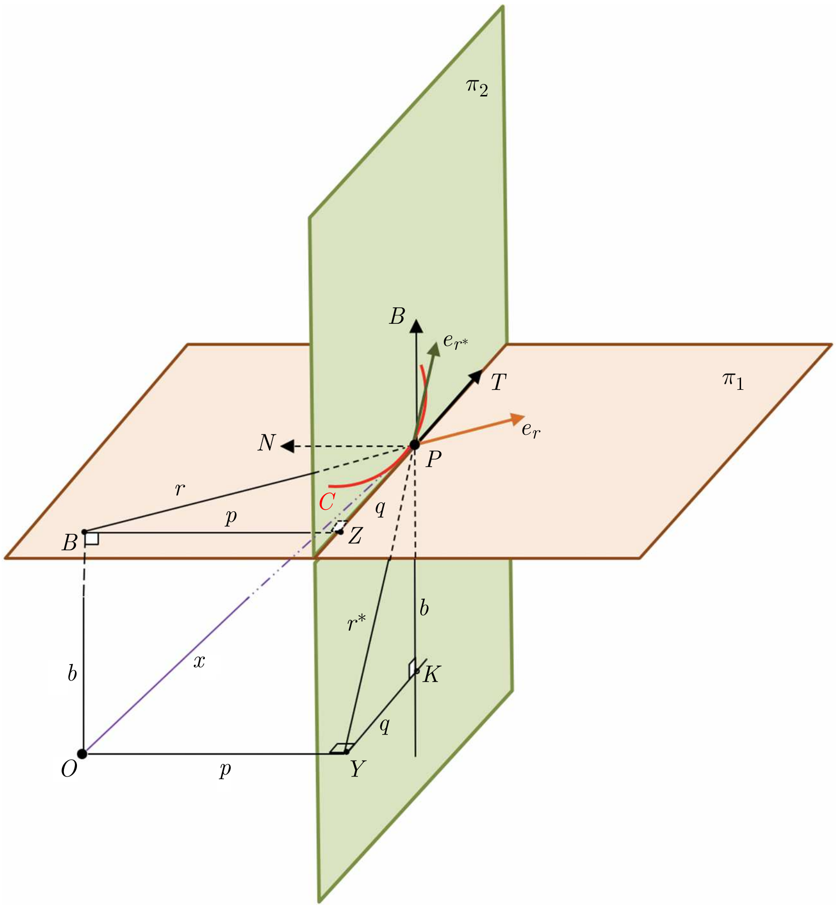

Fig. 1. A particle P moves on a curve C in the 3D Euclidean space E^3. π_1 and π_2 are the osculating and rectifying planes at P, respectively. B is the foot of the perpendicular line segment which is from the origin O to plane π_1. Y is the foot of the perpendicular line segment which is from the origin to plane π_2. BZ and YK are perpendicular line segments to the tangent and binormal axes, respectively. $\mathbf{r}$ and $\mathbf{r}^*$ are the position vectors of P relative to B and Y. $\mathbf{e}_r$ is the unit vector in the direction of BP and $\mathbf{e}_{r^*}$ is the unit vector in the direction of YP

A particle moving along a space curve may be seen as a point of this curve. So, the aforementioned particle P has a position vector in terms of the Serret-Frenet basis of the aforementioned curve C. Let the position vector of P on the Serret-Frenet basis be resolved as follows

$$\mathbf{x} = q\mathbf{T} - p\mathbf{N} + b\mathbf{B} \tag{2.8}$$

where

$$q = \langle \mathbf{x}, \mathbf{T}\rangle \qquad -p = \langle \mathbf{x}, \mathbf{N}\rangle \qquad b = \langle \mathbf{x}, \mathbf{B}\rangle \tag{2.9}$$

Denote by $\mathbf{r}$ the vector

$$\mathbf{r} = -p\mathbf{N} + q\mathbf{T} \tag{2.10}$$

which lies in the osculating plane π_1 to C at P. Then, the equality

$$r^2 = \langle \mathbf{r}, \mathbf{r} \rangle = p^2 + q^2 \tag{2.11}$$

is obtained where r is length of the vector $\mathbf{r}$ (see Fig.1).

Also, the angular momentum of the particle P about O is obtained by vector multiplication of the position vector of P and the linear momentum vector of P as

$$\mathbf{H}_O = (q\mathbf{T} - p\mathbf{N} + b\mathbf{B}) \times m\frac{ds}{dt}\mathbf{T} = m\frac{ds}{dt}b\mathbf{N} + m\frac{ds}{dt}p\mathbf{B} \tag{2.12}$$

On the other hand, on the physical assumption that the binormal component of the angular momentum is nonzero, it is ensured that p never vanishes. So, from the perspective of (2.11), r is nonzero. In this case, the equalities

$$\mathbf{e}_r = \frac{1}{r}\mathbf{r} \tag{2.13}$$

and

$$\mathbf{N} = \frac{1}{p}(-r\mathbf{e}_r + q\mathbf{T}) \tag{2.14}$$

can be written where $\mathbf{e}_r$ is the unit vector in the direction of the radial vector $\mathbf{r}$ (see Casey (2011) for more details).

3. An alternative resolution of the jerk vector for a space curve

Let us define a vector $\mathbf{r}^*$ in the rectifying plane π_2 to C at P by

$$\mathbf{r}^* = q\mathbf{T} + b\mathbf{B} \tag{3.1}$$

Then we get the equality (see Fig.1)

$$(r^*)^2 = \langle \mathbf{r}^*, \mathbf{r}^* \rangle = q^2 + b^2 \tag{3.2}$$

where r^* is length of the vector $\mathbf{r}^*$.

We aim to resolve the jerk vector $\mathbf{J}$ in (2.7) along the tangential direction, radial direction BP in the osculating plane π_1 and the radial direction YP in the rectifying plane π_2. We know that the normal vector $\mathbf{N}$ is expressed in terms of $\mathbf{e}_r$ and $\mathbf{T}$ as in (2.14) on the physical assumption that the binormal component of angular momentum is nonzero. We will make use of this relation to realize our aim in the next step. But now, let us try to express the vector $\mathbf{B}$ in terms of $\mathbf{r}^*$ and $\mathbf{T}$. In view of (3.1), we can conclude that this is possible if and only if $b \neq 0$. By making the second physical assumption that the normal component of angular momentum never vanishes, we can ensure that b is nonzero. By considering this assumption, we can write the equation

$$\mathbf{B} = \frac{1}{b}(-q\mathbf{T} + \mathbf{r}^*) \tag{3.3}$$

While $b \neq 0$, it can be immediately seen from (3.2) that $r^* \neq 0$. So the unit vector $\mathbf{e}_{r^*}$ can be defined by

$$\mathbf{e}_{r^*} = \frac{1}{r^*}\mathbf{r}^* \tag{3.4}$$

With the aid of (3.3) and (3.4), we get

$$\mathbf{B} = \frac{1}{b}(-q\mathbf{T} + r^*\mathbf{e}_{r^*}) \tag{3.5}$$

Now, if we substitute equations (2.14) and (3.5) into equation (2.7), we obtain the jerk vector $\mathbf{J}$ of the particle P as follows

$$\begin{aligned}\mathbf{J} &= \Big[\frac{d^3s}{dt^3} - \kappa^2\Big(\frac{ds}{dt}\Big)^3 + 3\kappa\frac{q}{p}\frac{ds}{dt}\frac{d^2s}{dt^2} + \frac{q}{p}\frac{d\kappa}{ds}\Big(\frac{ds}{dt}\Big)^3 - \kappa\tau\frac{q}{b}\Big(\frac{ds}{dt}\Big)^3\Big]\mathbf{T} \\ &\quad + \Big[-3\kappa\frac{r}{p}\frac{ds}{dt}\frac{d^2s}{dt^2} - \frac{r}{p}\frac{d\kappa}{ds}\Big(\frac{ds}{dt}\Big)^3\Big]\mathbf{e}_r + \Big[\kappa\tau\frac{r^*}{b}\Big(\frac{ds}{dt}\Big)^3\Big]\mathbf{e}_{r^*} = T_t\mathbf{T} + T_r\mathbf{e}_r + T_{r^*}\mathbf{e}_{r^*}\end{aligned} \tag{3.6}$$

on the physical assumption that each of the normal and binormal components of its angular momentum vector never vanishes.

In general, the unit vectors $\mathbf{T}$, $\mathbf{e}_r$ and $\mathbf{e}_{r^*}$ do not constitute an orthonormal set. So, the three components of $\mathbf{J}$ in (3.6) are not equal to the orthogonal projections of $\mathbf{J}$ onto $\mathbf{T}$, $\mathbf{e}_r$ and $\mathbf{e}_{r^*}$. We call T_t, T_r, T_{r^*} the tangential, first radial and second radial components of the jerk, respectively. By taking into account of the above derivation, we can express the following theorem.

Theorem 3.1. *Let a particle P of mass m travel along a curve C in 3D Euclidean space E^3, and assume that each of the normal and binormal components of its angular momentum vector never vanishes. In this case, the jerk vector of P can be expressed as in equation (3.6). In here T_t, T_r, T_{r^*} are the tangential, first radial and second radial components of the jerk, respectively. T_t lies along the tangent line of C. T_r lies along the line which passes through the particle P and the foot of the perpendicular that is from the origin of the space to the osculating plane, while T_{r^*} lies along the line which passes through the particle P and the foot of the perpendicular that is from the origin of the space to the rectifying plane.*

Corollary 3.1. *Let the motion of P be restricted to a fixed plane and assume that the binormal component of its angular momentum vector never vanishes. Then, the jerk vector of P is expressed as in the following*

$$\begin{aligned}\mathbf{J} &= \Big[\frac{d^3s}{dt^3} - \kappa^2\Big(\frac{ds}{dt}\Big)^3 + 3\kappa\frac{q}{p}\frac{ds}{dt}\frac{d^2s}{dt^2} + \frac{q}{p}\frac{d\kappa}{ds}\Big(\frac{ds}{dt}\Big)^3\Big]\mathbf{T} \\ &\quad + \Big[-3\kappa\frac{r}{p}\frac{ds}{dt}\frac{d^2s}{dt^2} - \frac{r}{p}\frac{d\kappa}{ds}\Big(\frac{ds}{dt}\Big)^3\Big]\mathbf{e}_r\end{aligned} \tag{3.7}$$

Proof. In the 3-dimensional Euclidean space E^3, let the oriented curve C, traced out by particle P be restricted to a fixed plane which does not contain the origin O. We know that in the planar case $\tau = 0$, $\mathbf{B}$ is constant and orthogonal to the plane. Also, since the plane does not pass through O, b is a non-zero constant. If equations (2.8) (2.10), (2.13), (3.1) and (3.4) are considered, the position vector of P is given by

$$\mathbf{x} = r\mathbf{e}_r + b\mathbf{B} = -p\mathbf{N} + r^*\mathbf{e}_{r^*} \tag{3.8}$$

Now, (3.6) holds in the plane of motion. If we take into consideration $\tau = 0$, from (3.6), we obtain

$$T_t = \frac{d^3s}{dt^3} - \kappa^2\Big(\frac{ds}{dt}\Big)^3 + 3\kappa\frac{q}{p}\frac{ds}{dt}\frac{d^2s}{dt^2} + \frac{q}{p}\frac{d\kappa}{ds}\Big(\frac{ds}{dt}\Big)^3 \tag{3.9}$$

and also we conclude that T_r does not change and T_{r^*} vanishes. It means that the jerk vector is expressed as in equation (3.7) for the fixed plane which does not contain the origin O.

In the second part of the proof, we consider a plane that passes through the origin. In the space E^3, let the oriented curve C, traced out by the particle P, be restricted to a fixed plane which contains the origin O. Similarly $\tau = 0$, $\mathbf{B}$ is constant and orthogonal to the plane. Also, since the plane passes through O, $b = 0$. Thus each of the statements

$$-\frac{q}{b}\kappa(s)\tau(s)\Big(\frac{ds}{dt}\Big)^3 \tag{3.10}$$

and

$$\frac{r^*}{b}\kappa(s)\tau(s)\Big(\frac{ds}{dt}\Big)^3 \tag{3.11}$$

have indefiniteness $0/0$. So, we will examine this motion in the limit case. The plane of motion in E^3 passes through the origin if and only if $b = 0$. For this reason, assume that $b \to 0$. Then, the distance between the foot of the perpendicular that is from the origin to the rectifying plane and the line which is determined by the binormal vector is about to become equal to the distance between the foot of the perpendicular, which is from the origin to the rectifying plane and the particle. It means that $r^*/q \to 1$ (or $r^* \approx q$). Also for $b \to 0$, YK is about to coincide with YP: that is the vector $\mathbf{T}$ is about to coincide with the vector $\mathbf{e}_{r^*}$ (see Fig. 2).

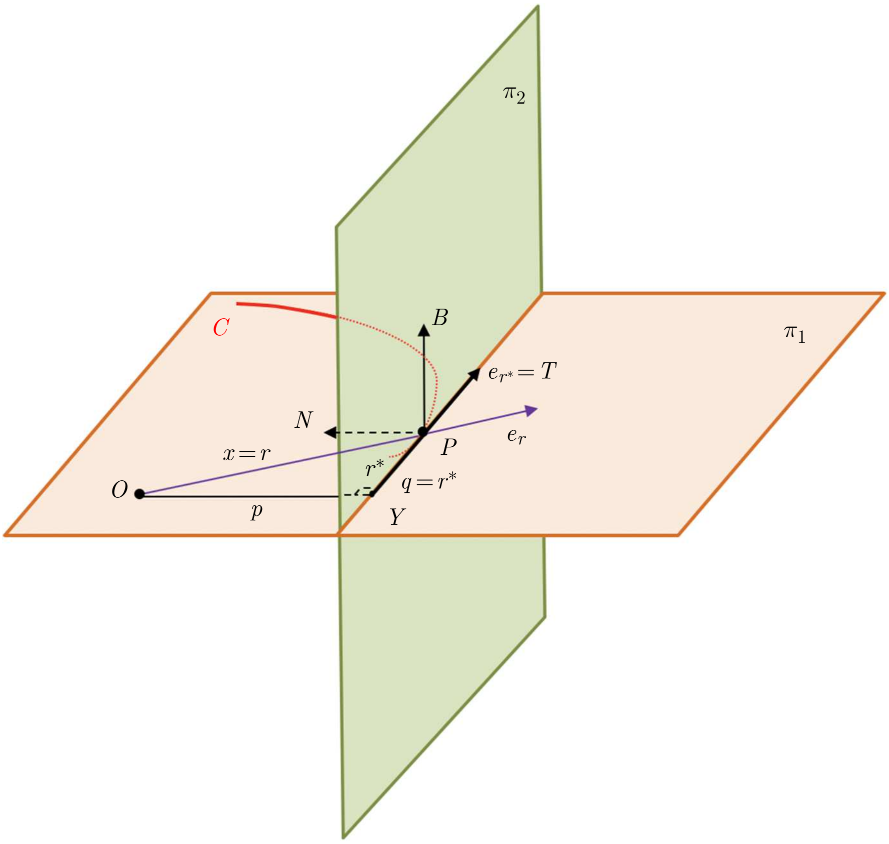

Fig. 2. A particle P moves on a curve C which is in a fixed plane, containing the origin O of the 3D Euclidean space E^3 (this plane is the osculating plane of C according to the theory of differential geometry). The osculating and rectifying planes of the curve C at P are denoted by π_1 and π_2, respectively. OY is perpendicular to the tangent axis. YP is the perpendicular to the binormal axis. $\mathbf{r}$ and $\mathbf{r}^*$ are the position vectors of P relative to O and Y. $\mathbf{e}_r$ and $\mathbf{e}_{r^*}$ are the unit vectors in the directions OP and YP, respectively

Consequently, for $b \to 0$, we can say that the vector

$$-\kappa\tau\frac{q}{b}\Big(\frac{ds}{dt}\Big)^3\mathbf{T} + \kappa\tau\frac{r^*}{b}\Big(\frac{ds}{dt}\Big)^3\mathbf{e}_{r^*} \tag{3.12}$$

is about to coincide with the zero vector. Thus, by considering (3.6), we can conclude that the jerk vector is expressed as in equation (3.7) for the fixed plane which contains the origin O, as well. □

4. Illustrative examples

In this Section, we give illustrative examples to calculate the components of jerk according to Theorem 3.1 and Corollary 3.1.

4.1. Motion of an electron along a right-handed circular helix

Assume that an electron travels along a right-handed circular helix lying on a cylinder which has radius R. This is motion of an electron (with electrical charge $-e$, and mass m) under a constant magnetic field $(0, 0, B)$ along the z-axis. In Cartesian coordinates, the position vector of the electron is given as follows

$$\mathbf{x} = [R\cos(\omega t), R\sin(\omega t), v_z t] \tag{4.1}$$

where $\omega = eB/m$ and R, v_z are positive constants (see Fig.3).

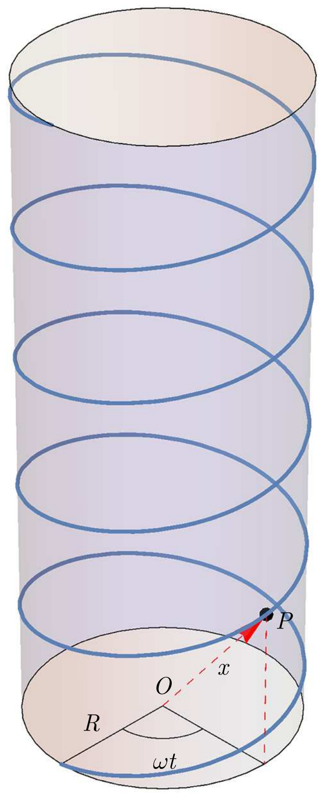

Fig. 3. The right-handed circular helix on a cylinder as the path of an electron under a constant magnetic field. The electron is denoted by the letter P

Let the helix axis be the z-axis, and α be the helix angle determined by $\tan\alpha = R\omega/v_z$. The velocity, acceleration and jerk vector of the electron can be obtained as

$$\begin{aligned}
\mathbf{v} &= [-R\omega\sin(\omega t), R\omega\cos(\omega t), v_z] \\
\mathbf{a} &= [-R\omega^2\cos(\omega t), -R\omega^2\sin(\omega t), 0] \\
\mathbf{J} &= [R\omega^3\sin(\omega t), -R\omega^3\cos(\omega t), 0]
\end{aligned} \tag{4.2}$$

We can immediately write the following

$$dx = -R\omega \sin(\omega t)dt \qquad dy = R\omega \cos(\omega t)dt \qquad dz = v_z dt \tag{4.3}$$

Using $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$, the speed of electron and its first and second derivatives can be found as follows

$$\frac{ds}{dt} = \sqrt{R^2\omega^2 + v_z^2} \qquad \frac{d^2s}{dt^2} = 0 \qquad \frac{d^3s}{dt^3} = 0 \tag{4.4}$$

As can be seen easily, the oriented curve traced out by the electron can be parameterized by the arc-length $s = s(t) = \beta t$ where $\beta = \sqrt{R^2\omega^2 + v_z^2}$ as in the following

$$\boldsymbol{\gamma}(s) = \Big[R\cos\frac{\omega s}{\beta}, R\sin\frac{\omega s}{\beta}, \frac{v_z s}{\beta}\Big] \tag{4.5}$$

Also, if we use equations (2.2) and (4.5) while keeping the equality $\tan\alpha = R\omega/v_z$ in mind, we can easily find the Serret-Frenet basis as in the following

$$\begin{aligned}
\mathbf{T} &= \Big[-\sin\alpha\sin\frac{\omega s}{\beta}, \sin\alpha\cos\frac{\omega s}{\beta}, \cos\alpha\Big]\\
\mathbf{N} &= \Big[-\cos\frac{\omega s}{\beta}, -\sin\frac{\omega s}{\beta}, 0\Big]\\
\mathbf{B} &= \Big[\cos\alpha\sin\frac{\omega s}{\beta}, -\cos\alpha\cos\frac{\omega s}{\beta}, \sin\alpha\Big]
\end{aligned} \tag{4.6}$$

On the other hand, from (2.4) it is easy to see that the curvature κ and the torsion τ are constants

$$\kappa = \frac{R}{R^2 + v_z^2/\omega^2} = \frac{\sin^2\alpha}{R} \qquad \tau = \frac{v_z/\omega}{R^2 + v_z^2/\omega^2} = \frac{\omega\cos^2\alpha}{v_z} \tag{4.7}$$

From (2.9), (4.5) and (4.6)

$$q = \frac{s v_z \cos\alpha}{\beta} = t v_z \cos\alpha \qquad p = R \qquad b = \frac{s v_z \sin\alpha}{\beta} = t v_z \sin\alpha \tag{4.8}$$

Thus from (2.11) and (3.2), the following equalities can be written

$$r = \sqrt{R^2 + \frac{s^2 v_z^2 \cos^2\alpha}{\beta^2}} = \sqrt{R^2 + t^2 v_z^2 \cos^2\alpha} \qquad r^* = \frac{s v_z}{\beta} = t v_z \tag{4.9}$$

Consequently, for the aforementioned electron, the components of the jerk vector are found as

$$T_t = \frac{-R^2\omega^4 - \omega^2 v_z^2}{\sqrt{R^2\omega^2 + v_z^2}} \qquad T_r = 0 \qquad T_{r^*} = \omega^2 v_z \tag{4.10}$$

by applying Theorem 3.1.

4.2. Motion of a particle along a logarithmic spiral curve

Let a particle P move on the logarithmic spiral curve

$$\delta(t) = [\mathrm{e}^{\omega t}\cos(\omega t), \mathrm{e}^{\omega t}\sin(\omega t), 0]$$

in the 3D Euclidean space E^3. In this case, the position vector of P is given as

$$\mathbf{x} = [\mathrm{e}^{\omega t}\cos(\omega t), \mathrm{e}^{\omega t}\sin(\omega t), 0] \tag{4.11}$$

where ω indicates the angular frequency and t indicates time. The position vector $\mathbf{x}$ is given in the MKS unit system. The velocity, acceleration and jerk vectors related to the particle P have been obtained as follows

$$\begin{aligned}
\mathbf{v} &= \omega[\mathrm{e}^{\omega t}\cos(\omega t) - \mathrm{e}^{\omega t}\sin(\omega t), \mathrm{e}^{\omega t}\sin(\omega t) + \mathrm{e}^{\omega t}\cos(\omega t), 0] \\
\mathbf{a} &= 2\omega^2[-\mathrm{e}^{\omega t}\sin(\omega t), \mathrm{e}^{\omega t}\cos(\omega t), 0] \\
\mathbf{J} &= 2\omega^3[-\mathrm{e}^{\omega t}\sin(\omega t) - \mathrm{e}^{\omega t}\cos(\omega t), \mathrm{e}^{\omega t}\cos(\omega t) - \mathrm{e}^{\omega t}\sin(\omega t), 0]
\end{aligned} \tag{4.12}$$

The equalities

$$dx = \omega\mathrm{e}^{\omega t}(\cos(\omega t) - \sin(\omega t))dt \qquad dy = \omega\mathrm{e}^{\omega t}(\sin(\omega t) + \cos(\omega t))dt \tag{4.13}$$

can be easily written. In this case, we get the speed of P and its first and second derivatives as

$$\frac{ds}{dt} = \sqrt{2}\omega\mathrm{e}^{\omega t} \qquad \frac{d^2s}{dt^2} = \sqrt{2}\omega^2\mathrm{e}^{\omega t} \qquad \frac{d^3s}{dt^3} = \sqrt{2}\omega^3\mathrm{e}^{\omega t} \tag{4.14}$$

by means of the equality $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$.

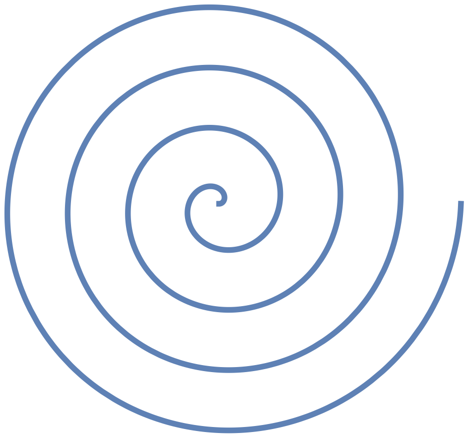

Fig. 4. An illustration of the logarithmic spiral. The radial direction is log-scaled. In this plot $\omega = 1$ and $0 < t \leqslant 8\pi$ in MKS unit system

As can be seen easily, the oriented curve traced out by the particle can be parameterized by the arc-length $s = s(t) = \sqrt{2}\mathrm{e}^{\omega t} - \sqrt{2}$ as in the following

$$\delta^*(s) = \Big(\frac{s}{\sqrt{2}} + 1\Big)\Big[\cos\ln\Big(\frac{s}{\sqrt{2}} + 1\Big), \sin\ln\Big(\frac{s}{\sqrt{2}} + 1\Big), 0\Big] \tag{4.15}$$

Moreover, if we use equations (2.2) and (4.15), we can easily find the elements of the Serret--Frenet basis

$$\begin{aligned}
\mathbf{T} &= \frac{1}{\sqrt{2}}\Big[\cos\ln\Big(\frac{s}{\sqrt{2}} + 1\Big) - \sin\ln\Big(\frac{s}{\sqrt{2}} + 1\Big), \cos\ln\Big(\frac{s}{\sqrt{2}} + 1\Big) + \sin\ln\Big(\frac{s}{\sqrt{2}} + 1\Big), 0\Big] \\
\mathbf{N} &= \frac{1}{\sqrt{2}}\Big[-\cos\ln\Big(\frac{s}{\sqrt{2}} + 1\Big) - \sin\ln\Big(\frac{s}{\sqrt{2}} + 1\Big), \cos\ln\Big(\frac{s}{\sqrt{2}} + 1\Big) - \sin\ln\Big(\frac{s}{\sqrt{2}} + 1\Big), 0\Big] \\
\mathbf{B} &= [0, 0, 1]
\end{aligned} \tag{4.16}$$

Since the motion is planar, we know that $\tau = 0$. On the other hand, from equation (2.4), it is easy to see that the curvature is equal to $1/(s + \sqrt{2})$. From (2.9), (2.11), (4.15) and (4.16) we obtain

$$q = p = \frac{s + \sqrt{2}}{2} \qquad r = \frac{s + \sqrt{2}}{\sqrt{2}} \tag{4.17}$$

Consequently, the jerk vector is obtained for the particle P in the tangential and radial components as

$$\mathbf{J} = (2\sqrt{2}\omega^3 \mathrm{e}^{\omega t})\mathbf{T} + (-4\omega^3 \mathrm{e}^{\omega t})\mathbf{e}_r \tag{4.18}$$

by applying Corollary 3.1.

5. Conclusion

Along a space curve in the 3-dimensional Euclidean space, the resolution of the jerk vector (time derivative of acceleration vector) is well known thanks to ref. (Résal, 1862). In this resolution, the jerk vector lies along the tangential, normal and binormal components.

In this paper, a new resolution of the jerk vector for a particle which moves on a space curve in 3-dimensional Euclidean space is obtained. Our resolution comprises the tangential and radial components in the osculating plane and the radial component in the rectifying plane. Additionally, the case for planar motion is discussed as a corollary.

This resolution is a new contribution to the field. Just like in the case of Siacci's resolution of the acceleration vector which comprises the tangential and radial component in the osculating plane, in the future, it may be needed for some specific applications in many areas of science.

References

1. Bickford J.H., 1972, *Mechanisms for Intermittent Motion*, Krieger Pub. Co.
2. Casey J., 2011, Siacci's resolution of the acceleration vector for a space curve, *Meccanica*, **46**, 2, 471-476
3. Faires V.M., 1965, Design of machine elements. Technical report, Macmillan New York
4. Freudenstein F., Sandor G.N., 1964, *Mechanical Design and Systems Handbook*, chap. 4, p. 13, McGraw-Hill Companies
5. Melchior P., 1928, *Zeitschrift Ver. Dtsch. Ing.*, **72**, 1842
6. Résal H., 1862, *Traité de cinématique pure*, Mallet-Bachelier
7. Schot S.H., 1978, Jerk: the time rate of change of acceleration, *American Journal of Physics*, **46**, 11, 1090-1094
8. Shifrin T., 2015, *Differential Geometry: a First Course in Curves and Surfaces*, University of Georgia
9. Siacci F., 1879, Moto per una linea gobba, *Atti della Reale Accademia della Scienze*, **14**, 946-951
10. Tsirlin M., 2017, Jerk by axes in motion along a space curve, *Journal of Theoretical and Applied Mechanics*, **55**, 4, 1437-1441

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