GLOBAL STABILITY OF POSITIVE DISCRETE-TIME TIME-VARYING NONLINEAR FEEDBACK SYSTEMS

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Abstract – The global stability of positive discrete-time time-varying nonlinear systems with time-varying scalar feedbacks is investigated. Sufficient conditions for the asymptotic stability of discrete-time positive time-varying linear systems are given. The new conditions are applied to discrete-time positive time-varying nonlinear systems with time-varying feedbacks. Sufficient conditions are established for the global stability of the discrete-time positive time-varying nonlinear systems with feedbacks.

Key words – global stability, discrete-time, positive, feedback, time-varying, nonlinear, system.

INTRODUCTION

In positive systems inputs, state variables and outputs take only nonnegative values for any nonnegative inputs and nonnegative initial conditions [1, 4, 7]. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollutions models. A variety of models having positive behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc. An overview of state of the art in positive systems theory is given in the monographs [1, 4, 7, 10, 11, 15, 16].

The stability of nonlinear continuous-time standard and fractional positive systems have been analyzed in [2, 3, 5, 6, 9, 11, 12, 14, 16, 18]. The positive different orders fractional linear systems has been introduced in [8, 9]. The dynamical properties of linear systems with state Metzler matrices have been analyzed in [17]. The decentralized stabilization of descriptor fractional positive systems with delays have been investigated in [21, 22] and the practical stability and robust stability of discrete-time fractional linear systems in [18, 20]. The global stability of positive time-varying nonlinear feedback time-varying systems has been considered in [13]. In this paper the results of [13] will be extended to positive discrete-time time-varying nonlinear systems.

The paper is organized as follows. The properties of positive discrete-time linear systems are recalled in section 2. New sufficient conditions for the global of positive time-varying discrete-time are established in section 3. The main result of the paper the sufficient conditions for the global stability of positive time-varying nonlinear is presented in section 4. Concluding remarks are given in section 5.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n \times m}$ - the set of $n \times m$ real matrices, $\Re_+^{n \times m}$ - the set of $n \times m$ real matrices with nonnegative entries and $\Re_+^n = \Re_+^{n \times 1}$, M_n - the set of $n \times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n - the $n \times n$ identity matrix.[1]

I. DISCRETE-TIME LINEAR SYSTEMS

Consider linear discrete-time system described by the equations

$$x_{i+1} = Ax_i + Bu_i, \ i \in Z_+ = \{0, 1, \dots\} \qquad (1a)$$

$$y_i = Cx_i, \qquad (1b)$$

where $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$, $C \in \Re^{p \times n}$.

Definition 1. [4, 7, 11] The system (1) is called (internally) positive if $x_i \in \Re_+^n$ and $y_i \in \Re_+^p$, $i \in Z_+$ for any initial conditions $x_0 \in \Re_+^n$ and all inputs $u_i \in \Re_+^m$, $i \in Z_+$.

Theorem 1. [4, 7, 11] The system (1) is positive if and only if

$$A \in \Re_+^{n \times n}, \ B \in \Re_+^{n \times m}, \ C \in \Re_+^{p \times n}. \qquad (2)$$

Definition 2. [11] The system (1) is called asymptotically stable (shortly stable) if the matrix A is a Schur matrix.

Theorem 2. [11] The positive system (1) is asymptotically stable if and only if:

1) All coefficients of the polynomial

$$p_A(z) = \det[I_n(z+1) - A] \qquad (3a)$$
$$= z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$$

are positive, i.e. $a_i > 0$ for $i = 0,1,...,n-1$.

2) There exists strictly positive vector $\lambda^T = [\lambda_1 \ \cdots \ \lambda_n]^T$, $\lambda_k > 0$, $k = 1,...,n$ such that

$$[A - I_n]\lambda < 0 \text{ or } \lambda^T[A - I_n] < 0. \qquad (3b)$$

Theorem 3. The positive system (1) is asymptotically stable if the sum of entries of each column (row) of the matrix A is less than one.

Proof. The proof follows from condition (3b) for $\lambda^T = [1 \ ... \ 1]$ since $[1 \ ... \ 1]^T A < [1 \ ... \ 1]$ if the sum of entries of each column of the matrix A is less than 1. Proof for rows is similar. □

II. ASYMPTOTIC STABILITY OF POSITIVE TIME-VARYING DISCRETE-TIME LINEAR SYSTEMS

Consider the time-varying discrete-time linear system

$$x_{i+1} = A(i)x_i + B(i)u_i, \ i \in Z_+ = \{0, 1, ...\} \quad (4a)$$

$$y_i = C(i)x_i, \qquad (4b)$$

where $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state, input and output vectors and $A(i) \in \Re^{n \times n}$, $B(i) \in \Re^{n \times m}$, $C(i) \in \Re^{p \times n}$ are bounded matrices with entries depending of i.

Definition 3. The matrix $A(i)$ is called positive one if all entries are not-negative functions of i.

Definition 4. The system (4) is called positive if its matrices $A(i)$, $B(i)$ and $C(i)$ are positive.

Theorem 4. The solution of equation (4a) for known initial condition $x_0 \in \Re^n$ and input $u_i \in \Re^m$, $i \in Z_+$ is given by

$$x_i = \Phi(i,0)x_0 + \sum_{j=0}^{i-1} \Phi(i, j+1)B(j)u_j, \ i \in Z_+ \qquad (5a)$$

and

$$y_i = C(i)\Phi(i,0)x_0 + \sum_{j=0}^{i-1} C(i)\Phi(i, j+1)B(j)u_j, \ i \in Z_+ \text{ (5b)}$$

where

$$\Phi(i, j) = \begin{cases} I_n & \text{for } i = j \geq 0 \\ A(i-1)A(i-2)...A(j) & \text{for } i > j \geq 0 \end{cases}. \text{ (5c)}$$

Definition 5. The positive system (4) for $u_i = 0$ is called asymptotically stable if $\lim_{i \to \infty} x_i = 0$ for all nonzero initial conditions $x_0 \in \Re_+^n$.

Consider the autonomous positive time-varying linear system

$$x_{i+1} = A(i)x_i, \ x_i \in \Re^n, \ i \in Z_+ = \{0, 1, ...\} \qquad (6)$$

where $x_i \in \Re^n$ is the state vector, $A(i) \in \Re_+^{n \times n}$, and the positive time-invariant system

$$x_{i+1} = Ax_i, \ x_i \in \Re_+^n, \ i \in Z_+ = \{0, 1, ...\} \qquad (7)$$

where $x_i \in \Re^n$ is the state vector and $A \in \Re_+^{n \times n}$.

The matrix $A = [a_{jk}]$ of the system (7) with the matrix $A(i) = [a_{jk}(i)]$ of the system (6) are related by

$$a_{jk} = \max_{0 \leq i \leq \infty} a_{jk}(i) \text{ for } j, k = 1,...,n. \qquad (8)$$

Theorem 5. The positive time-varying linear system (6) is asymptotically stable if the positive time-invariant system (7) is asymptotically stable.

Proof. From (8) and Theorem 3 it follows that the asymptotic stability of the linear time-invariant system (7) implies the asymptotic stability of the time-varying system (6). □

Example 1. Consider the positive time-varying linear system (6) with the state matrix

$$A(i) = \begin{bmatrix} 0.4e^{-i} & 0.3\sin\dfrac{\pi}{2}(i+1) \\ 0.5\cos\dfrac{\pi}{2}i & 0.2\left(e^{-i} + \cos\dfrac{\pi}{2}i\right) \end{bmatrix}. \qquad (9)$$

Taking into account (8) and (9) we obtain

$$A = \max_{0 \leq i \leq \infty} \begin{bmatrix} 0.4e^{-i} & 0.3\sin\dfrac{\pi}{2}(i+1) \\ 0.5\cos\dfrac{\pi}{2}i & 0.2\left(e^{-i} + \cos\dfrac{\pi}{2}i\right) \end{bmatrix} \qquad (10)$$

$$= \begin{bmatrix} 0.4 & 0.3 \\ 0.5 & 0.2 \end{bmatrix}.$$

From Theorem 3 applied to the time invariant linear system with the matrix (10) it follows that the system is asymptotically stable. Therefore, by Theorem 5 the time-varying system with the matrix (9) is also asymptotically stable.

III. GLOBAL STABILITY OF POSITIVE NONLINEAR TIME-VARYING SYSTEMS

Consider the positive system shown in Fig. 1 which consists of positive discrete-time time-varying linear part, positive discrete-time time varying linear feedback, nonlinear element
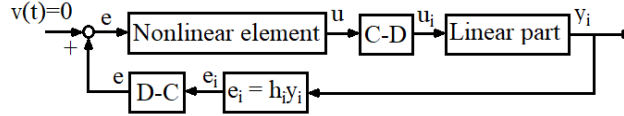

Fig. 1. Positive nonlinear feedback system

The positive time-varying discrete-time linear system is described by the equations

$$x_{i+1} = A(i)x_i + B(i)u_i, \quad i \in Z_+ = \{0, 1, ...\} \quad (11a)$$

$$y_i = C(i)x_i, \quad (11b)$$

where $x_i \in \Re_+^n$, $u_i \in \Re_+$, $y_i \in \Re_+$ are the state, input and output vectors and $A(i) \in \Re_+^{n \times n}$, $B(i) \in \Re_+^{n \times 1}$, $C(i) \in \Re_+^{1 \times n}$.

The characteristic of the nonlinear element is shown in Fig. 2 and satisfies the condition
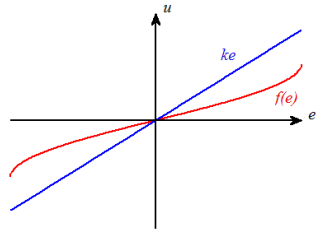
$$0 \le \frac{f(e)}{e} \le k < \infty. \quad (12)$$

Fig. 2. Characteristic of single-input single-output nonlinear element

The single-input single-output discrete-time feedback element is described by the equation

$$e_i = h_i y_i, \quad i \in Z_+ = \{0, 1, ...\} . \quad (13)$$

Substituting (11b) into (13) we obtain

$$e_i = h_i C(i)x_i, \quad i \in Z_+ = \{0, 1, ...\} . \quad (14)$$

From (12) we have $u = f(e) \le ke$ and taking into account (14) we obtain

$$u_i \le ke_i = kh_i C(i)x_i, \quad i \in Z_+ = \{0, 1, ...\} . \quad (15)$$

Substitution of (15) into (11a) yields

$$x_{i+1} \le A(i) + kh_i B(i)C(i)x_i = A_c(i)x_i,$$
$$i \in Z_+ = \{0, 1, ...\} \quad (16a)$$

where

$$A_c(i) = A(i) + kh_i B(i)C(i)x_i \in \Re_+^{n \times n} . \quad (16b)$$

and continuous to discrete (C-D) and discrete to continuous (D-C) converters.

Applying to the positive discrete-time system (16) Theorem 5 we obtain the following.

Theorem 6. The positive discrete-time nonlinear system shown in Fig. 1 is globally stable if the matrix (16b) satisfies Theorem 5.

For example applying Theorems 6 and 4 to the positive nonlinear system we may find the minimal value of k satisfying the condition (12) for which the positive time-varying nonlinear system is globally stable.

Example 2. Consider the system shown in Fig. 1 with the discrete-time time-varying object (11) with the matrices

$$A(i) = \begin{bmatrix} 0.2e^{-i}\sin\frac{\pi}{2}i & 0.1e^{-i}\cos\frac{\pi}{2}(i+1) \\ 0.2\cos\frac{\pi}{2}i & 0.3e^{-i} \end{bmatrix},$$

$$B(i) = \begin{bmatrix} 0.3\sin\frac{\pi}{2}i \\ 0.2e^{-i} \end{bmatrix}, \quad (17)$$

$$C(i) = [0.2(1+e^{-i}) \quad 0.4(1-0.5e^{-i})]$$

and the feedback gain

$$h_i = 0.6(1+e^{-i}), \quad i \in Z_+ = \{0, 1, ...\} . \quad (18)$$

Find the minimal value of k satisfying the condition (12) for which the positive nonlinear time-varying system is globally stable. Using (16b), (17) and (18) we obtain

$$A_c(i) = A(i) + kh_i B(i)C(i)$$

$$= \begin{bmatrix} 0.2e^{-i}\sin\frac{\pi}{2}i & 0.1e^{-i}\cos\frac{\pi}{2}(i+1) \\ 0.2\cos\frac{\pi}{2}i & 0.3e^{-i} \end{bmatrix} + k0.6(1+e^{-i})$$

$$\times \begin{bmatrix} 0.3\sin\frac{\pi}{2}i \\ 0.2e^{-i} \end{bmatrix}[0.2(1+e^{-i}) \quad 0.4(1-0.5e^{-i})]$$

$$= \begin{bmatrix} A_{c11}(i) & A_{c12}(i) \\ A_{c21}(i) & A_{c22}(i) \end{bmatrix},$$

$$A_{c11}(i) = 0.2e^{-i}\sin\frac{\pi}{2}i + k0.6(1 + e^{-i})$$

$$\times (0.3\sin\frac{\pi}{2}i)0.2(1 + e^{-i}),$$

$$A_{c12}(i) = 0.1e^{-i}\cos\frac{\pi}{2}(i+1) + k0.6(1 + e^{-i})$$

$$\times (0.3\sin\frac{\pi}{2}i)0.4(1 - 0.5e^{-i}), \qquad (19b)$$

$$A_{c21}(i) = 0.2\cos\frac{\pi}{2}i + k0.6(1 + e^{-i})$$

$$\times (0.2e^{-i})0.2(1 + e^{-i}),$$

$$A_{c22}(i) = 0.3e^{-i} + k0.6(1 + e^{-i})(0.2e^{-i})$$

$$\times 0.4(1 - 0.5e^{-i}).$$

Applying to (19) Theorem 4 we obtain

$$A_c(i) \leq \begin{bmatrix} 0.2 + k0.144 & 0.1 + k0.072 \\ 0.2 + k0.096 & 0.3 + k0.048 \end{bmatrix} \qquad (20)$$

and using Theorem 3 for columns: column $1 - k \leq \dfrac{0.6}{0.24} = 2.5$, column $2 - k \leq \dfrac{0.6}{0.12} = 5$ for rows: row $1 - k \leq \dfrac{0.7}{0.216} = 3.241$, row $2 - k \leq \dfrac{0.5}{0.144} = 3.472$.

Therefore, the positive nonlinear system is globally stable for $k \leq 2.5$.

IV. CONCLUDING REMARKS

The global stability of positive discrete-time time-varying nonlinear systems with positive time-varying scalar feedbacks has been investigated. Sufficient conditions for the asymptotic stability of discrete-time positive time-varying linear systems have been proposed (Theorem 5). The new conditions have been applied to discrete-time positive time-varying nonlinear systems with time-varying feedbacks. Sufficient conditions are established for the global stability of the discrete-time positive time-varying nonlinear systems with feedbacks (Theorem 6) and illustrated by numerical example. The considerations can be extended to positive descriptor time –varying nonlinear systems. An open problem is an extension of these considerations to the fractional orders nonlinear feedback systems.

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