OPERATIONS RESEARCH AND DECISIONS

No. 1 2014
DOI: 10.5277/ord140105

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PROBABILITIES ON STREAMS AND REFLEXIVE GAMES

Probability measures on streams (e.g. on hypernumbers and p-adic numbers) have been defined. It was shown that these probabilities can be used for simulations of reflexive games. In particular, it can be proved that Aumann's agreement theorem does not hold for these probabilities. Instead of this theorem, there is a statement that is called the reflexion disagreement theorem. Based on this theorem, probabilistic and knowledge conditions can be defined for reflexive games at various reflexion levels up to the infinite level.

Keywords: *metagame, reflexive game, coinductive probabilities, p-adic probabilities, perlocutionary effect, Aumann's agreement theorem*

1. Introduction

Probabilities on streams were defined for the first time in papers [46, 47]. They are a natural generalization of probabilities on hypernumbers and p-adic numbers. Surveys and many details on the theory of p-adic valued probabilities may be found in [19–23, 41]. Some basic properties of non-Archimedean (p-adic, as well as hypernumber-valued) logical multiple-validity are considered in [41, 43–45, 48]. Recall that the fundamental work on non-Archimedean systems is [37].

All the results of this paper are obtained due to some basic features of streams and coinductive probabilities on them. Streams refer to mathematical objects which cannot be generated as inductive sets. For more details please see [15, 18, 31, 32, 35, 36–40]. Using these probabilities on streams, the reflexion disagreement theorem (theorem 2, section 3) can be readily proved, which contradicts Aumann's agreement theorem proved on standard real probabilities. It is possible to simulate reflexive games on coinductive probabilities. It is assumed that the reader knows some basic notions of

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speech act theory (see [50–52]), such as performative, illocutionary and perlocutionary effects. In this paper I propose a formalization of the notion "perlocutionary effect" to coinductively define knowledge operators in reflexive games.

This research is being fulfilled by the support of FP7-ICT-2011-8 and UMO-2012/07/B/HS1/00263.

2. Why can we reject Aumann's agreement theorem?

Aumann's agreement theorem [3, 4] actually says that two agents acting rationally (according to Bayesian formulas) and with common knowledge of each other's beliefs cannot agree to disagree. More specifically, if two people share common priors, and have common knowledge of each other's current probability assignments (their posteriors for a given event A are common knowledge), then they must have equal probability assignments (these posteriors must be equal). It is one of the most important statements of game theory, epistemic logic and so on. For example, according to this statement, any rational player has to behave in the same manner under the same circumstances. Rational players always have common knowledge, they know all the parameters of a game and are sure that their opponents know that they know the parameters of the game, that they know that they know and so on ad infinitum.

To prove his theorem, Aumann appeals to representing the possibility operator $\mathbf{P}_i(\omega)$ and the common knowledge operator K_i as least fixed points, i.e. as *inductive sets*. Let us remember their definitions.

Let Ω be a finite set of possible states of the world, which are called propositions, N be a set of agents, call them $i = 1, ..., N$. Agent's i knowledge structure is a function $\mathbf{P}_i$ which assigns to each $\omega \in \Omega$ a non-empty subset of Ω. $\mathbf{P}_i$ is a partition of Ω: each world ω belongs to exactly one element of each $\mathbf{P}_i$, i.e. Ω is a set of mutually disjoint subsets $\mathbf{P}_i$ whose union is Ω. Then $\mathbf{P}_i(\omega)$ is called i's knowledge state at ω. This means that if the true state is ω, the individual only knows that the true state is in $\mathbf{P}_i(\omega)$. We can interpret $\mathbf{P}_i(\omega)$ probabilistically as follows: $\mathbf{P}_i(\omega) = \{\omega' : P_i(\omega' \mid \omega) > 0\}$. Then all propositions in Ω in any of the N partitions form a σ-field A. $\mathbf{P}_i(\omega) \subseteq A$ is interpreted as meaning that at ω agent i knows that A has occurred, i.e. $\omega' \in A$ for all states ω' that i considers possible at ω.

For each i, the expression below defines the knowledge operator K_i which, applied to any set $A \in \mathsf{A}$, yields the set $K_i A \in \mathsf{A}$ of worlds in which i knows A:

$$K_i A = \{\omega : \mathbf{P}_i(\omega) \subseteq A\}$$

The most important property of the knowledge operator is $K_i A \subseteq A$, i.e. if an agent knows that event A has occurred whenever he is in state ω (i.e., $\omega \in K_i A$), then A is true in state ω (i.e., $\omega \in A$.)

We can prove the following statements:

$$K_i \Omega = \Omega \tag{1}$$

$$K_i (A \cap B) = K_i A \cap K_i B \tag{2}$$

$$A \subseteq B \Rightarrow K_i A \subseteq K_i B \tag{3}$$

$$K_i A \subseteq A \tag{4}$$

$$K_i K_i A = K_i A \tag{5}$$

$$\neg K_i \neg K_i A \subseteq K_i A \tag{6}$$

The properties of (1)–(6) are considered fundamental for defining knowledge operators in epistemic logic.

Nevertheless, we can define the possibility operator $\mathbf{P}_i(\omega)$ and the common knowledge operator K_i as the greatest fixed points as well, i.e. as coinductive sets. In this way, we cannot prove Aumann's agreement theorem. Instead, we prove the reflection disagreement theorem as an appropriate negation of Aumann's theorem. While for Aumann's theorem we need the property $\mathbf{P}_i(\omega) = \bigcap\{A : \omega \in K_i A\}$, for its negation we need the property $\mathbf{P}_i(\omega) = \bigcup\{A : \omega \in K_i A\}$. In other words, this new statement can be proven if we change some of the standard philosophical presuppositions in game theory for the following new assumptions: each rational agent can cheat (disagree in his heart with) other rational agents, no player can know everything prior to the game, each agent can try to foresee knowledge (beliefs) of his/her opponents and manipulate them, therefore common knowledge does not mean that an agent will agree with or be completely predictable to all others.

These philosophical presuppositions contradicting Aumann's ideas were first formulated by Lefebvre in his notion "reflexive games" in 1965 [25, 27, 28]. A game is called reflexive if when choosing an action an agent has to model (predict) the actions of his/her opponents [33, 34], e.g. (s)he can try to manipulate them or cheat them. In the earliest of these articles, Lefebvre formulated reflexive games assuming many reflexion levels [28]. At the zero level I ignore beliefs of opponents, at the first level I take into account their beliefs, at the second level I take into account that they try to predict my beliefs, at the third level I foresee their beliefs in which my beliefs are foreseen by them, etc. The gametheoretic mathematics for the early ideas of Lefebvre has been devoloped by Novikov and Chkhirtishvili [11–14, 33, 34]. The reflexion disagreement theorem that will be proved in the next section holds true for their approach, namely when we consider reflexive games at a reflexion level of any natural number. I am inspired and influenced by the early ideas of Lefebvre in the same measure as them.

The later work of Lefebvre tried to simulate decision making in reflexive games by means of Boolean functions [26]. The main disadvantages of these approaches lie in the fact that reflexive levels are ignored and agents are presented as automata. However, an individual's assessment of a situation is variable. Reflexion varies depending on characteristic moods (illocutionary acts), as well as the persuasiveness and emotionality of our interlocutors (perlocutionary effects). In this sense, the dynamism of reflexion quite corresponds to the well-known paradox of Chevalley and Belzung [10], which is formulated as such: the emotional response of a given person in a given situation can vary at different points of time. Thus, the simulation of decision making in reflexive games by means of Boolean functions is too speculative and cannot help in analyzing everyday situations. Using this approach, we assume that the reactions and evaluations of a given agent remain the same forever. Nevertheless, this is false. This approach can be useful only in explaining some basic features of reflexive management that take place in a given situation (such as the case of Soviet and American ethical patterns [26]).

The early ideas of Lefebvre, which I try to develop in this paper, are very close to the idea of a metagame which was proposed by Nigel Howard [17]. According to him, for any game G and any player i there can be a metagame iG, in which player i chooses in full knowledge of the choices made by all the others. More formally, let $G = \langle S_1, S_2; M_1, M_2\rangle$ be the normal form of the game, where S_1 (S_2) is the set of strategies for player 1 (2) and M_1 (M_2) is his/her preference function. The set of outcomes is $S = S_1 \times S_2$, i.e. an outcome is an ordered pair $s = \langle s_1, s_2\rangle$. $M_i(s) = \{s' :$ is not preferred to s by player $i\}$, $i = 1, 2$. Let $B(S_1)$ ($B(S_2)$) be the set of non-null subsets of S_1 (S_2) and $K_1 \subseteq B(S_1)$ ($K_2 \subseteq B(S_2)$). Then the first level metagame KG is defined as the normal form $KG = \langle X_1, X_2;\ M_1', M_2'\rangle$, where $X_1 = \{x_1 : x_1 = \langle f_1, c_1\rangle;\ c_1 \in K_1;\ f_1 : K_2 \to c_1\}$, $X_2 = \{x_2 : x_2 = \langle f_2, c_2\rangle;\ c_2 \in K_2;\ f_2 : K_1 \to c_2\}$, and for $i = 1, 2$, M_1' satisfies the following property:

$$x' \in M_i'(x) \quad \text{iff} \quad \beta x' \in M_i'(\beta x)$$

where $\beta(f_1, c_1; f_2, c_2) = (f_1(c_2), f_2(c_1))$.

By induction, we can obtain the nth-level metagame $K_nK_{(n-1)} \dots K_1G$. The set of all metagames $K_nK_{(n-1)} \dots K_1G$ for any natural number n is called the infinite metagame based on G. This metagame corresponds to Lefebvre's reflexive game of infinite level.

Nigel Howard proposed to use metagames to have the possibility of requiring "more of rationality than that each player should optimize given its beliefs about the others' choices" [17]. Now (s)he should be able "to know the others' choices, and know how the other would choose to react to such knowledge, and know each others' reactions to such reactions, and so on" [17]. These ideas are almost the same as the ideas introduced in reflexive games [11–14, 33, 34]. The difference lies in other ways of defining preference functions.

The reflection disagreement theorem holds true for the infinite metagame in Howard's meaning, as well as for the reflexive game of infinite level in Lefebvre's meaning. This theorem shows limits in infinite mutual predictions of others' knowledge.

The mathematical meaning of the reflexion disagreement theorem is that we cannot prove the agreement theorem using probabilities running over streams (e.g. using probabilities with values on hypernumbers or p-adic numbers) in any way. In non-standard fields Aumann's theorem is false, because the powerset of any infinite set of streams is not a Boolean algebra and Bayes' theorem does not hold in general for streams [41, 46, 47]. Notice that we cannot avoid streams in the case of an infinite metagame or reflexive game of infinite level, because we face there an infinite data structure consisting of streams. Fuzzy and probability logic with values on streams is described in [41–47]. This logic can be used for developing a probability theory and epistemic logic for infinite metagames and reflexive games of infinite level.

Mathematically, an infinite metagame is a coalgebra [18, 32, 36, 39]. Graphically, coalgebras (e.g. processes or games) can be represented as infinite trees. As an example, let us refer to the following definition of binary trees labeled by x, y, … and whose interior nodes are either unary nodes labeled by u_1, u_2, … or binary nodes labeled by b_1, b_2, …:

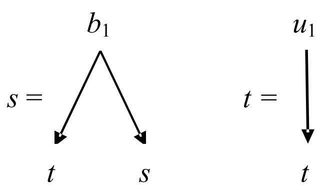
1. the variables x, y, … are trees,

2. if t is a tree, then adding a single node labeled by one of u_1, u_2, … as a new root with t as its only subtree gives a tree,

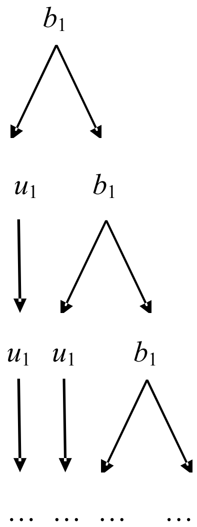
3. if s and t are trees, then adding a single node labeled by one of b_1, b_2, … as new root with s as the left subtree and t as the right subtree again gives a tree,

4. trees may go on forever (i.e. trees satisfy the greatest fixed point condition).

This definition allows us to define some binary trees by circular definitions such as:



to sum up, we obtain the following infinite binary tree:



Let Tr be the set of trees that we have defined. Then our definition introduces a coalgebra:

$$Tr = \{x, y, \ldots\} \cup (\{u_1, u_2, \ldots\} \times \mathrm{Tr}) \cup (\{b_1, b_2, \ldots\} \times Tr \times Tr)$$

The reflection disagreement theorem is valid for games presented in the form of a coalgebra. Recently, many researchers [5, 24, 29, 53] have focused on the idea that in economics, in particular in decision theory, we cannot avoid coalgebraic notions such as process dynamics, behavioral instability, self-reference, or circularity. There exist many more cases of non-equilibria in economics, because we engage coinductive databases more often as a matter of fact [5, 16]. For example, repeated games may be defined only coalgebraically [1, 30] and, as well, it is better to define epistemic games and belief functors as coalgebras [6–9].

Thus, the reflexion disagreement theorem can be proved if (1) we assume that rational agents can become unpredictable to and try to manipulate each other, (2) we define probabilities on streams (e.g. on hypernumbers or p-adic numbers), (3) games are presented as coalgebras. As we see, this new theorem is a very important statement within the new mathematics (coalgebras, transition systems, process calculi, etc.) which has been introduced into game theory recently. Sets of streams which have been modelled coalgebraically cannot generate inductive sets [2]. Therefore, Aumann's agreement theorem is meaningless on these sets, but we face just sets of streams in many kinds of games (e.g. if we deal with repeated games, games with infinite states, concurrent games, infinite metagames, reflexive games of infinite level, etc.). Instead of the agreement theorem, the reflexion disagreement theorem is valid if we cannot obtain inductive sets, e.g. in the case of sets of streams. Notice that according to Aczel [2], the universum of coinductive sets is much larger than the universum of inductive sets.

3. Reflexion disagreement theorem

Let A be any set. We define the set A^ω of all streams over A as $A^\omega = \{\sigma : \{0, 1, 2, ...\} \to A\}$. For more details on stream calculus, see [15, 35, 38–40]. For a stream σ, we call $\sigma(0)$, the initial value of σ. We define the derivative $\sigma'(0)$ of a stream σ, for all $n \geq 0$, by $\sigma'(n) = \sigma(n + 1)$. For any $n \geq 0$, $\sigma(n)$ is called the nth element of σ. This can also be expressed in terms of higher-order stream derivatives, defined, for all $k \geq 0$, by $\sigma^{(0)} = \sigma$, $\sigma^{(k+1)} = (\sigma^{(k)})'$. In this case, the nth element of a stream σ is given by $\sigma(n) = \sigma^{(n)}(0)$. Also, the stream can be understood as an infinite sequence of derivatives. It will be denoted by an infinite sequence of values or by an infinite tuple:

$$\sigma = \sigma(0) :: \sigma(1) :: \sigma(2) :: \ \dots \ :: \sigma(n-1) :: \ ...$$

$$\sigma = \langle \sigma(0), \sigma(1), \sigma(2), \dots \rangle$$

A bisimulation on A^ω is a relation $R \subseteq A^\omega \times A^\omega$ such that, for all σ and τ in A^ω, if $\langle \sigma, \tau \rangle \in R$ then (i) $\sigma(0) = \tau(0)$ and (ii) $\langle \sigma', \tau' \rangle \in R$.

Theorem 1 (coinduction). For all σ, $\tau \in A^\omega$, if there exists a bisimulation relation $R \subseteq A^\omega \times A^\omega$ with $\langle \sigma, \tau \rangle \in R$, then $\sigma = \tau$. This principle is called coinduction.

The repeated stream at each step is denoted by $[\sigma(0)]$ or by $[a]$. We can define the addition and multiplication of streams as follows. The sum $\sigma + \tau$ and the product $\sigma \times \tau$ of streams σ and τ are defined element-wise:

$$\forall n \in \mathbf{N}, (\sigma + \tau)(n) = \sigma(n) + \tau(n)$$

$$\forall n \in \mathbf{N}, (\sigma \times \tau)(n) = \sum_{k=0}^{n} \sigma(k)\tau(n-k)$$

To define addition and multiplication by coinduction, we should use the following facts about the differentiation of sums and products by applying the basic operations: $(\sigma + \tau)' = \sigma' + \tau'$, $(\sigma \times \tau)' = (|\sigma(0)| \times \tau') + (\sigma' \times \tau)$, where $|\sigma(0)| = \langle \sigma(0), 0, 0, 0, \ ...\rangle$. We see that the sum behaves exactly as in classical calculus. However, multiplication does not. Now we can define them, as well as two other stream operations as follows:

Differential equation	Initial value	Name
$(\sigma+\tau)' = \sigma' + \tau'$	$(\sigma+\tau)(0) = \sigma(0)+\tau(0)$	sum
$(\sigma\times\tau)' = (\lvert\sigma(0)\rvert\times\tau') + (\sigma'\times\tau)$	$(\sigma\times\tau)(0) = \sigma(0)\times\tau(0)$	product
$(\sigma^{-1})' = \lvert -1\rvert \times \lvert\sigma(0)^{-1}\rvert \times \sigma' \times \sigma^{-1}$	$(\sigma^{-1})(0) = \sigma(0)^{-1}$	inverse

We can embed the set of real numbers into the set of streams by defining the following constant stream. Let $r \in \mathbf{R}$. Then $|r| = \langle r, 0, 0, 0, \ldots\rangle$ is defined so that: its differential equation is $|r|' = [0]$, its initial value is $|r|(0) = r$. This allows us to add and multiply real numbers and streams:

$$|r| + \sigma = \langle r + \sigma(0), \sigma(1), \sigma(2), \ldots\rangle$$

$$|r| \times \sigma = \langle r\sigma(0),\ r\sigma(1),\ r\sigma(2), \ldots\rangle$$

Taking into account these equalities, we are able to rely on our intuition that it would be natural to define any positive real number of $[0,\ 1]$ to be less than any positive inconstant stream of $[0, 1]^{\omega}$, e.g. $|1|$ is less than $[1]$ and $|r + 1|$ is less than $[r]$.

Consider the set of streams $[0,1]^{\omega}$ and extend the standard order structure on $[0,1]$ to a partial order structure on $[0,1]^{\omega}$. Further define this order as follows:

$O_{[0,1]^{\omega}}$ (1) for any streams $\sigma, \tau \in [0,1]^{\omega}$, we set $\sigma \leq \tau$ if $\sigma(n) \leq \tau(n)$ for every $n \in \mathbf{N}$. For any streams $\sigma, \tau \in [0,1]^{\omega}$, we set $\sigma = \tau$ if σ, τ are bisimilar. For any streams $\sigma, \tau \in [0,1]^{\omega}$, we set $\sigma < \tau$ if $\sigma(n) \leq \tau(n)$ for every $n \in \mathbf{N}$ and there exists n_0 such that $\sigma(n_0) \neq \tau(n_0)$. (2) each stream of the form $|r| \in [0,1]^{\omega}$ (i.e. constant stream) is less than an inconstant stream σ.

This ordering relation is not linear, but partial, because there exist streams $\sigma, \tau \in [0,1]^{\omega}$, which are incomparible.

We introduce two operations: sup, inf in the partial order structure $O_{[0,1]^{\omega}}$. Assume that $\sigma, \tau \in [0,1]^{\omega}$ are either both constant streams or both inconstant streams. Then their supremum and infimum are defined by coinduction: the differential equation for the supremum is

$$(\sup(\sigma, \tau))' = \sup(\sigma', \tau')$$

and its initial value is

$$(\sup(\sigma,\tau))(0) = \sup(\sigma(0), \tau(0))$$

the differential equation for the infimum is

$$(\inf(\sigma, \tau))' = \inf(\sigma', \tau')$$

and its initial value is

$$(\inf(\sigma, \tau))(0) = \inf(\sigma(0), \tau(0))$$

Suppose now that one and only one of $\sigma, \tau \in [0,1]^{\omega}$ is constant, then an inconstant stream is greater than a constant one, therefore their supremum gives an inconstant stream, but their infimum gives a constant stream.

According to $\mathsf{O}_{[0,1]^{\omega}}$, there exist the maximal stream $[1] \in [0,1]^{\omega}$ and the minimal stream $[0] = |0| \in [0,1]^{\omega}$. Each p-adic number has a unique expansion $n = \sum_{k=-N}^{+\infty} \alpha_k p^k$, where $\alpha_k \in \{0, 1, \ldots, p-1\}$, $\forall k \in \mathbf{Z}$, and $\alpha_{-N} \neq 0$ that is called the canonical expansion of the p-adic number n (or the p-adic expansion for n). p-adic numbers can be identified with sequences of digits:

$$n = \ldots \alpha_2 \alpha_1 \alpha_0, \alpha_{-1} \ldots \alpha_{-N}$$

or with infinite tuples:

$$n = \langle \alpha_{-N}, \ \ldots, \ \alpha_{-2}, \ \alpha_{-1}, \ \alpha_0, \ \alpha_1, \ \alpha_2, \ \ldots \rangle$$

The set of such numbers is denoted by $\mathbf{Q}_p$.

The expansion

$$n = \alpha_0 + \alpha_1 p + \ldots + \alpha_k p^k + \ldots = \sum_{k=0}^{\infty} \alpha_k p^k$$

where $\alpha_k \in \{0, 1, \ldots, p-1\}$, $\forall k \in \mathbf{N}$, is called the expansion of the p-adic integer n.

This number sometimes has the following notation: $n = \ldots \alpha_3\alpha_2\alpha_1\alpha_0$ or $n = \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3, \ldots \rangle$. The set of such numbers is denoted by $\mathbf{Z}_p$.

It can be easily shown that p-adic numbers may be represented as potentially infinite data structures such as streams. Each stream of the form

$$\sigma = \sigma(0) :: \sigma(1) :: \sigma(2) :: \ldots :: \sigma(n-1) :: \ldots$$

where $\sigma(n) \in \{0, 1, \ldots, p-1\}$ for every $n \in \mathbf{N}$, may be converted into a p-adic integer.

It is easily shown that the set A^{ω} of all p-adic streams includes the set of natural numbers. Let n be a natural number. It has a finite p-adic expansion $n = \sum_{k=0}^{m} \alpha_k p^k$. Thus we can identify n with a p-adic stream $\sigma = \sigma(0) :: \sigma(1) :: \ldots :: \sigma(m) :: \sigma^{(m+1)}$, where $\sigma(i) = \alpha_i$ for $i = \overline{0, m}$ and $\sigma^{(m+1)} = [0]$.

Extend the standard order structure on $\mathbf{N}$ to a partial order structure on p-adic streams (i.e. on $\mathbf{Z}_p$).

- for any p-adic streams σ, $\tau \in \mathbf{N}$ we have $\sigma \leq \tau$ in $\mathbf{N}$ iff $\sigma \leq \tau$ in $\mathbf{Z}_p$,
- each p-adic stream $\sigma = \sigma(0) :: \sigma(1) :: \ldots :: \sigma(m) :: \sigma^{(m+1)}$, where $\sigma^{(m+1)} = [0]$ (i.e. each finite natural number), is less than any infinite number τ, i.e. $\sigma < \tau$ for any $\sigma \in \mathbf{N}$ and $\tau \in \mathbf{Z}_p \setminus \mathbf{N}$.

Define this partial order structure on $\mathbf{Z}_p$ as follows:

$\mathcal{O}_{Z_p}$ Let

$$\sigma = \sigma(0) :: \sigma(1) :: \ldots :: \sigma(n-1) :: \sigma^{(n)}$$

and

$$\tau = \tau(0) :: \tau(1) :: \ldots :: \tau(n-1) :: \tau^{(n)}$$

be p-adic streams. (1) We set $\sigma < \tau$ if the following three conditions hold: (i) there exists n such that $\sigma(n) < \tau(n)$, (ii) $\sigma(k) \leq \tau(k)$ for all $k > n$, (iii) σ is a finite integer, i.e. there exists m such that $\sigma^{(m)} = [0]$. (2) We set $\sigma = \tau$ if σ and τ are bisimilar (see Theorem 1). (3) Suppose that σ, τ are infinite integers. We set $\sigma \leq \tau$ by coinduction: $\sigma \leq \tau$ iff $\sigma(n) \leq \tau(n)$ for every $n \in \mathbf{N}$. We set $\sigma < \tau$ if we have $\sigma \leq \tau$ and there exists $n_0 \in \mathbf{N}$ such that $\sigma(n_0) < \tau(n_0)$.

The ordering relation $\mathcal{O}_{Z_p}$ is not linear but partial, because there exist p-adic streams $\sigma, \tau \in \mathbf{Z}_p$, which are incomparible. As an example, let $p = 2$ and let σ represents the

p-adic integer $-\frac{1}{3} = \ldots 10101 \ldots 101$ and τ the p-adic integer $-\frac{2}{3} = \ldots 01010 \ldots 010$. Then the p-adic streams σ and τ are incomparible.

Now we introduce two operations sup, inf in the partial order structure on $\mathbf{Z}_p$. Suppose that the p-adic streams σ, τ represent infinite p-adic integers. Then their sup and inf may be defined by coinduction as follows: the differential equation for the supremum is $(\sup(\sigma, \tau))' = \sup(\sigma', \tau')$ and its initial value is $(\sup(\sigma, \tau))(0) = \sup(\sigma(0), \tau(0))$, the differential equation for the infimum is $(\inf(\sigma, \tau))' = \inf(\sigma', \tau')$ and its initial value is $(\inf(\sigma, \tau))(0) = \inf(\sigma(0), \tau(0))$. Now suppose that at most one of two streams σ, τ represents a finite p-adic integer. In this case, $\sup(\sigma, \tau) = \tau$ if and only if $\sigma \leq \tau$ under condition O_{Z_p} and $\inf(\sigma, \tau) = \sigma$ if and only if $\sigma \leq \tau$ under condition O_{Z_p}.

It is important to remark that there exists the maximal p-adic stream $N_{\max} \in \mathbf{Z}_p$ under condition O_{Z_p}. It is easy to see that:

$$N_{\max} = [p-1] = -1 = (p-1) + (p-1)p + \ldots + (p-1)p^k + \ldots$$

Now, using the given notion of streams, let us prove the reflexion disagreement theorem. For Ω, the finite set of possible states of the world, and N, the set of agents, we can unconventionally define agent i's accepted performances as a function $\mathbf{Q}_i$ which assigns to each $o \in \Omega$ a non-empty subset of Ω, so that each world o belongs to one or more elements of each $\mathbf{Q}_i$, i.e. Ω is contained in the union of the $\mathbf{Q}_i$, but the $\mathbf{Q}_i$ are not mutually disjoint. Thus $\mathbf{Q}_i(o)$ is called i's accepted performative state at o. If the successful performance is o, the individual knows (accepts) that the performative state is in $\mathbf{Q}_i(o)$. The elements of $\mathbf{Q}_i(o)$ are those states of the world that are considered to be types of situations for performative states making the latter successful at o.

We can propose a stream interpretation of $\mathbf{Q}_i(o)$ and construct Ω^ω. We know that the set Ω^ω is much larger than Ω. According to the orders $\mathrm{O}_{[0,1]^\omega}$ and O_{Z_p}, we can identify all members of Ω with some streams of Ω^ω. Let the set of these streams be denoted by ${}^\delta\Omega$. Evidently, ${}^\delta\Omega \subset \Omega^\omega$. Assume that Ω^ω is a union of $\mathbf{Q}_i$. Therefore, $\mathbf{Q}_i$ contains ${}^\delta\Omega$ (and hence by assumption Ω).

Now let us define probabilities on streams as follows: a *finitely* additive probability measure is a nonnegative set function $P(\cdot)$ defined on the sets $A \subseteq \Omega^\omega$, into the set $[0, 1]^\omega$, and satisfying the following properties:

(i) $P(\varnothing) \geq |0|$ for all $A \subseteq \Omega^\omega$,

(ii) $P(\Omega^\omega) = [1]$ and $P(\varnothing) = |0|$,

(iii) If $A \subseteq \Omega^\omega$ and $B \subseteq \Omega^\omega$ are disjoint, i.e. $\inf(P(A), P(B)) = |0|$, then $P(A \cup B) = P(A) + P(B)$. Otherwise, $P(A \cup B) = P(A) + P(B) - \inf(P(A), P(B)) = \sup(P(A), P(B))$.

(iv) $P(\neg A) = [1] - P(A)$ for all $A \subseteq \Omega^\omega$, where $\neg A = \Omega^\omega \setminus A$.

(v) Relative probability functions $P(A|B) \in [0, 1]^\omega$ are characterized by the following constraint:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \tag{7}$$

where $P(B) \neq |0|$ and $P(A \cap B) = \inf(P(A), P(B))$. Note that since there are no partitions of sets of streams in the general case [41, 47], there are also some problems in defining the conditional relation $P(A|B)$ between events. There are many more dependent events than in the usual σ-field. For example, any real number in [0, 1] is less than any inconstant stream in $[0, 1]^\omega$. Let $P(B) = a$, $P(A) = b$, where a is a number in [0, 1] and b is an inconstant stream in $[0, 1]^\omega$. Then according to $\mathsf{O}_{[0,1]^\omega}$, $P(A|B) = 1$. However, this case cannot be defined using the traditional condition for independence $P(B) = P(B|A)$. Instead of this, we use the following condition: $P(A)P(B) = \inf(P(A), P(B))$.

The main originality of such probabilities is that conditions (iii), (iv) are independent. As a result, in some probability spaces $\langle \Omega^\omega, P \rangle$ the Bayes formulas may not hold [41, 47] and Aumann's theorem cannot be proven.

A particular case of stream-valued probabilities is presented by p-adic probabilities. Let us define them on any subsets of Ω^ω as follows: a finitely additive probability measure is a set function $P(\cdot)$ defined for sets $A \subseteq \Omega^\omega$, into the set $\mathbf{Z}_p$ and satisfying the following properties:

(i′) $P(\Omega^\omega) = -1$ and $P(\varnothing) = 0$.

(ii′) If $A \subseteq \Omega^\omega$ and $B \subseteq \Omega^\omega$ are disjoint, i.e. $\inf(P(A), P(B)) = 0$, then $P(A \cup B) = P(A) + P(B)$. Otherwise, $P(A \cup B) = P(A) + P(B) - \inf(P(A), P(B)) = \sup(P(A), P(B))$. Let us illustrate this property using 7-adic probabilities. Let $P(A) = \ldots 323241$ and $P(B) = \ldots 354322$ in 7-adic metrics. Then $P(A) + P(B) = \ldots 010563$, $\inf(P(A), P(B)) = \ldots 323221$; $P(A) + P(B) - \inf(P(A), P(B)) = \sup(P(A), P(B)) = \ldots 354342$.

(iii′) $P(\neg A) = -1 - P(A)$ for all $A \subseteq \Omega^\omega$, where $\neg A = \Omega^\omega \setminus A$.

(iv′) Relative probability functions $P(A|B) \in \mathbf{Z}_p$ are characterized by the following constraint:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

where $P(B) \neq 0$ and $P(A \cap B) = \inf(P(A), P(B))$.

Now we can interpret $\mathbf{Q}_i(|o|)$, where $|o| \in {}^{\sigma}\Omega$, probabilistically as follows: $\mathbf{Q}_i(|o|) = \{\tau : P_i(\tau \,||\, o|) > [0]\}$. These relative probabilities cannot determine a partition of Ω^{ω}. In other words, using them we cannot define an equivalence relation corresponding to a partition. Instead of this the following properties hold, as we can prove on the basis of the orders $\mathrm{O}_{[0,1]^{\omega}}$ and O_{Z_p}:

- If $P_i(\tau \mid \pi) > [0]$, then $P_i(\rho \mid \pi) > [0]$ and $P_i(\tau \mid \rho) > [0]$. This property holds instead of the usual transitivity in real probability logic: if $P_i(\rho \mid \pi) > 0$ and $P_i(\tau \mid \rho) > 0$, then $P_i(\tau \mid \pi) > 0$.
- $P_i(\tau \,||\, \omega|) = |1|$, where $|\omega| \in {}^{\sigma}\Omega$ and $\tau \in \Omega^{\omega} \setminus {}^{\sigma}\Omega$.
- $P_i(\tau \mid \tau) > [0]$.

Thus, the possibility operator $\mathbf{Q}_i$ has the following properties: for all $\tau, \pi \in \Omega^{\omega}$:

$$\tau \in \mathbf{Q}_i(\tau) \tag{8}$$

$$\pi \in \mathbf{Q}_i(\tau) \Rightarrow \mathbf{Q}_i(\pi) = \mathbf{Q}_i(\tau) \tag{9}$$

Now we consider the relation $A \subseteq \mathbf{Q}_i(o)$, where $A \subseteq {}^{\sigma}\Omega$, as the statement that at o agent i accepts the performance A, i.e. $|o| \in A$ for all states $|\upsilon|$ that i considers possible at $|o|$:

$$K_i A = \{|o| : A \subseteq \mathbf{Q}_i(|o|)\} \tag{10}$$

This set is another interpretation of the knowledge operator which is coinductive now. If $A \subseteq \mathbf{Q}_i(|o|)$, an individual i who observes $|o|$, will accept a state of the performance A. The most important property of the knowledge operator is $A \subseteq K_i A$, i.e. if A is successful in state $|o|$ (i.e. $|o| \in A$), then an agent accepts the performance A in state $|o|$ (i.e. $|o| \in K_i A$).

The following statements can be proved in relation to the coinductive knowledge operator defined in (10):

$${}^{\sigma}\Omega \subseteq K_i\, {}^{\sigma}\Omega \subseteq \Omega^{\omega} \tag{11}$$

$$(K_i A \cap K_i B) \Rightarrow K_i(A \cap B) \tag{12}$$

$$K_i(A \cup B) \Rightarrow (K_i A \cup K_i B) \tag{13}$$

$$K_i(A \cup B) = (K_i A \cap K_i B) \tag{14}$$

$$A \subseteq B \Rightarrow K_i A \subseteq K_i B \tag{15}$$

$$A \subseteq K_i A \tag{16}$$

$$K_i K_i A = K_i A \tag{17}$$

We can compare Aumann's statements (1)–(6) with statements (11)–(17) to notice that the latter assume a new epistemic logic with a stream interpretation. For example, it is possible to build up a kind of multi-valued illocutionary logic [50–52], where streams are values for perlocutionary effects. So, the coinductive knowledge operator K_i of (11)–(17) designates perlocutionary effects of illocutionary acts, i.e. it just takes into account successful performative propositions (it defines what influence was made on the hearer's behavior). Let K_jA (K_iA) denote agent j's (i's) performative (cognitive or emotional) assessments of the state of affairs A with the expected perlocutionary effect of these assessments on agent j (i). So, K_jA means j + performative verb + A (e.g. j thinks A, j likes A, j hates A, etc.) and agent j follows this statement in his (her) behavior.

Based on the standard propositional language L on the set of values [0, 1] (or $\{0, 1, \ldots, p-1\}$), we can construct an extension L'' containing new modal operators $\mathbf{E}$, $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ said to be perlocutionary effects. The semantics of L'' is defined in the following way. Assume that V is a valuation of well-formed formulas of L and it takes values in [0, 1] (or $\{0, 1, \ldots, p-1\}$). Let us extend V to V_e as follows:

A) If for $\varphi \in \mathsf{L}$, $V(\varphi) = r$, then $V_e(\mathbf{E}_i(\varphi)) = \langle \sigma(0) = r, \sigma(1), \sigma(2), \ldots\rangle$, i.e. $V_e(\mathbf{E}_i(\varphi))$ is a mapping from $V(\varphi)$ to an inconstant stream σ starting with $V(\varphi)$.

B) If for $\varphi \in \mathsf{L}$, $V(\varphi) = r$, then $V_e(\varphi) = |r|$.

C) For all $\varphi \in \mathsf{L}$, $V_e(\mathbf{E}_i(-\varphi)) \leq V_e(-\mathbf{E}_i(\varphi))$.

D) For all φ, $\psi \in \mathsf{L}$, $V_e(\mathbf{E}_i(\varphi) \wedge V_e(\mathbf{E}_i(\psi)) = \inf(V_e(\mathbf{E}_i(\varphi), V_e(\mathbf{E}_i(\psi)))$, $V_e(\mathbf{E}_i(\varphi) \vee V_e(\mathbf{E}_i(\psi)) = \sup(V_e(\mathbf{E}_i(\varphi), V_e(\mathbf{E}_i(\psi)))$, $V_e(\mathbf{E}_i(\varphi) \Rightarrow V_e(\mathbf{E}_i(\psi)) = [1] - \sup(V_e(\mathbf{E}_i(\varphi), V_e(\mathbf{E}_i(\psi))) + V_e(\mathbf{E}_i(\psi))$ for streams and $V_e(\mathbf{E}_i(\varphi) \Rightarrow V_e(\mathbf{E}_i(\psi)) = [p-1] - \sup(V_e(\mathbf{E}_i(\varphi), V_e(\mathbf{E}_i(\psi))) + V_e(\mathbf{E}_i(\psi))$ for p-adic integers, $V_e(-\mathbf{E}_i(\varphi)) = [1] - V_e(\mathbf{E}_i(\varphi))$ for streams and $V_e(-\mathbf{E}_i(\varphi)) = [p-1] - V_e(\mathbf{E}_i(\varphi))$ for p-adic integers.

Using these semantics, the following propositions will be perlocutionary tautologies, i.e. they will be true:

$$\varphi \Rightarrow \mathbf{E}(\varphi) \tag{18}$$

$$\neg\varphi \Rightarrow \mathbf{E}(\neg\varphi) \tag{19}$$

$$\mathbf{E}(\neg\varphi) \Rightarrow \neg\mathbf{E}(\varphi) \tag{20}$$

$$(\mathbf{E}(\varphi) \wedge \mathbf{E}(\psi)) \Rightarrow \mathbf{E}(\varphi \wedge \psi) \tag{21}$$

$$\mathbf{E}(\varphi \vee \psi) \Rightarrow (\mathbf{E}(\varphi) \vee \mathbf{E}(\psi)) \tag{22}$$

$$(\mathbf{E}(\varphi \Rightarrow \psi)) \Rightarrow (\mathbf{E}(\varphi) \Rightarrow \mathbf{E}(\psi)) \tag{23}$$

The epistemic logic closed under tautologies (18)–(23) is a kind of many-valued logic with values in the set of non-Archimedean numbers [41, 43–45, 48].

Aumann's understanding of common knowledge satisfies the classical intuition of the inductive behavior of all logical entities, i.e. the presupposition that we can appeal only to inductive sets in our reasoning. For example, we can always find an infinite intersection according to the knowledge operators of different people. However, this intuition contradicts the possibility of reflexive games where I can cheat or make false public announcements and should detect whether I am cheated by other people.

Under the conditions of reflexive games, I cannot define common perlocutionary effects as the infinite intersection κA. An infinite mutual reflexion between two individuals assumes an infinite union: both have mutual knowledge of A or both know that both know A or both know that both know that both know A etc. ad infinitum. In other words, the common perlocutionary operator $\overline{KA}$ is defined as follows:

$$\overline{KA} = K_1 A \cup K_2 A \cup K_1 K_2 A \cup K_2 K_1 A \cup K_1 K_2 K_1 A \cup \ldots$$

For each natural number n an operator $\overline{M_n}$ expressing nth degree mutual reflexion for perlocutionary effects is defined as follows:

$$\overline{M_0 A} = A, \qquad \overline{M_{n+1} A} = \bigcup_{i=1}^{N} K_i \overline{M_n A}$$

The common perlocutionary effect $\overline{\kappa}$ is understood as the mutual reflexion of perlocutionary effects of all finite degrees:

$$\overline{\kappa A} = \bigcup_{n=0}^{+\infty} \overline{M_n A}$$

Also, let us define for each natural number n an operator M_n expressing nth degree mutual reflexion:

$$M_0 A = A, \qquad M_{n+1} A = \bigcap_{i=1}^{N} K_i M_n A$$

and common knowledge, κ, as the mutual reflexion of common knowledge of all finite degrees:

$$\kappa A = \bigcap_{n=0}^{+\infty} M_n A$$

Lemma 1. If $|o| \in \kappa A$, then for any i, $\kappa A \subseteq \mathbf{Q}_i(|o|)$ and if $|o| \in \overline{\kappa A}$, then for some i, $\overline{\kappa A} \subseteq \mathbf{Q}_i(|o|)$.

Proof. If $|o| \in \kappa A$, then $|o| \in K_i M_n A$ for all agents i and degrees n of the mutual reflexion of common knowledge. Therefore, $\kappa A \subseteq \mathbf{Q}_i(|o|)$ for any i. If $|o| \in \overline{\kappa A}$, then $|o| \in K_i \overline{M_n A}$ for some agent i and degree n of the mutual reflexion of perlocutionary effects. Therefore, $\overline{M_n A} \subseteq \mathbf{Q}_i(|o|)$ for some n, and thus $\overline{\kappa A} \subseteq \mathbf{Q}_i(|o|)$ for some i. Q.E.D.

Theorem 2 (reflexion disagreement theorem). Let us consider a hypothesis H in coinductive probability logic [46, 47] for which the various agents' coinductive probabilities are $q_1, \ldots, q_N$ after they condition $P(\cdot)$ on priors. The propositions C and $\overline{C}$ of coinductive probability logic are defined as follows:

$$C = \bigcap_{i=1}^{N} \{|o| : P(H \mid \mathbf{Q}_i(|o|)) = q_i\}$$

$$\overline{C} = \bigcup_{i=1}^{N} \{|o| : P(H \mid \mathbf{Q}_i(\omega)) = q_i\}$$

Let the coinductive probability space $\langle \Omega^{\omega}, P \rangle$ be closed under all of the operators K_i, M_n, κ, $\overline{K_i}$, $\overline{M_n}$, and $\overline{\kappa}$ and let P be the standard probability measure that is common to all the agents. Assume that the probability of C and $\overline{C}$ becoming common knowledge or common perlocution is not equal to zero, i.e. $P(\kappa C) \neq [0]$ and $P(\overline{\kappa C}) \neq [0]$, then

$$P(H \mid \kappa C) \neq q_i \qquad \text{for some } i$$

$$P(H \mid \overline{\kappa C}) \neq q_i \text{ for some } i$$

Proof: By lemma 1, $\kappa C = \bigcup_j D_{ij}$, where $\bigcup_j D_{ij}$ covers $\mathbf{Q}_i$ but is not a partition of $\mathbf{Q}_i$ because of the basic properties of coinductive probabilities. Thus

$$P(H \mid \kappa C) = \frac{P(H \cap \bigcup_j D_{ij})}{P(\bigcup_j D_{ij})} = \frac{\inf(P(H), \sup_j P(D_{ij}))}{\sup_j P(D_{ij})}$$

$$\neq \frac{\sum_j P(H \mid D_{ij}) P(D_{ij})}{\sum_j P(D_{ij})} = \frac{\sum_j q_i P(D_{ij})}{\sum_j P(D_{ij})} = q_i$$

Thus, $P(H \mid \kappa C) \neq q_i$ in general. In the same way we can show that $P(H \mid \overline{\kappa C}) \neq q_i$ in general. Q.E.D.

4. Cellular-automatic reflexive games

The reflexion disagreement theorem is valid for games presented in coalgebraic form. There are many kinds of such games: repeated, concurrent, etc. In this section, a new way of presentating a game in coalgebraic form will be proposed on the basis of prooftheoretic cellular automata [49]. These automata can be used in formulating context-based decision rules in games. Usually, payoff matrices are involved in representing databases of games, (see Fig. 1). However, in the case of coinductive databases, we cannot appeal to payoff matrices. For example, we cannot appeal to them if we are dealing with games limited by certain contexts or with infinite games. Some kinds of coinductive databases for making decisions could be presented by payoff cellular automata. These automata are constructed as follows: The cells of the automata belong to the set $\mathbf{Z}^d$ and they take their values in S. The set S of states consists of the payoffs for all n players. The cardinality, $|S|$, is equal to $i_1 i_2 \ldots i_n$, where i_j is the number of all pure strategies available to the jth player, $j = 1, \ldots, n$. Each state has the form of an n-tuple $\langle a_{ij\ldots k}, b_{ij\ldots k}, \ldots, c_{ij\ldots k}\rangle$, where

(1) $a_{ij\ldots k}$ is the payoff to player 1 when (1) he plays a_i (2) player 2 plays b_j, …, (n) player n plays c_k,

(2) $b_{ij\ldots k}$ is the payoff to player 2 when (1) player 1 plays a_i (2) player 2 plays b_j, …, (n) and player n plays c_k, etc.,

…

(n) $c_{ij\ldots k}$ is the payoff to player n when (1) player 1 plays a_i (2) player 2 plays b_j, …, (n) and player n plays c_k.

$$\begin{array}{ccc} & & \text{Player 1} \\ & & \begin{array}{cc} a_1 & a_2 \end{array} \\ \text{Player 2} & \begin{array}{c} b_1 \\ \\ b_2 \end{array} & \left| \begin{array}{cc} \langle a_{11}, b_{11}\rangle & \langle a_{21}, b_{21}\rangle \\ & \\ \langle a_{12}, b_{12}\rangle & \langle a_{22}, b_{22}\rangle \end{array} \right| \end{array}$$

Fig. 1. An example of a payoff matrix showing the possible strategies available to player 1 (a_1 and a_2) and player 2 (b_1 and b_2) and the payoff that each player receives for his choice, depending on what other players do. The payoff is in the form $\langle a_{ij}, b_{ij}\rangle$, where a_{ij} is the payoff to player 1 when he plays a_i and player 2 plays b_j and b_{ij} is the payoff to player 2 when he plays b_j and player 2 plays a_i

The local transition function, δ_j, for player j, where $j = 1, \ldots, n$, is presented by a decision rule based on the past payoffs of all the players. The rule δ_j can be the same for all the players or different. The initial configuration of a payoff cellular automaton is a set of premises which, together with the decision rule, fully determines the future behaviour of the automaton. These premises may be understood as players' assumptions regarding the the expected payoff vector for different contexts before the game. The *game context* is defined by the neighbourhood $N(z)$ of the cell z. The number of premises (payoff vectors that we can take into account) cannot exceed the number $n = |N(z) \cup z|$. The decision rule δ_j is a mapping from the set of premises for $N(z) \cup z$ to a conclusion. This rule generates the sequence $\boldsymbol{a}^0(z)$, $\boldsymbol{a}^1(z)$, …, $\boldsymbol{a}^t(z)$, … for any $z \in \mathbf{Z}^d$, where $\boldsymbol{a} = \langle a_{ij\ldots k}, b_{ij\ldots k}, \ldots, c_{ij\ldots k}\rangle$ and $\boldsymbol{a}^i(z)$ denotes the state of z at the ith step of the application of δ_j to $\boldsymbol{a}^0(z)$, the state of z at step 0. This sequence is called a derivation trace from an initial state $\boldsymbol{a}^0(z)$. Obviously, this sequence is an infinite stream.

Example 1 (saddle point)

Let us consider a simple payoff cellular automaton for the zero-sum game with two players, 1 and 2. Let a_{ij} be the payoff of player 1 when the ith strategy of player 1 and jth strategy of player 2 are played. If $\max_i \min_j a_{ij} = \min_j \max_i a_{ij}$ for $z \cup N(z)$ at step t, then a_{ij} is called a saddle point for z at time t. Thus, a saddle point is an element of the payoff cellular automaton at time t which is both a maximum of the minimums of each row within the neighborhood $N(z)$ and a minimum of the maximums of each column within the same neighborhood. The cells $z \cup N(z)$ may have no saddle points, one saddle point, or multiple saddle points. Let the payoff states pictured in Table 1 be the initial configuration of the automaton. The set S of states consists of the integers –5, –4, –3, …, 7, 8.

Table 1. Initial configuration of a payoff cellular automaton A with the neighborhood consisting of 8 members in 2-dimensional space and with players 1 and 2

8	2	3	2	3
–3	0	2	–5	–4
–2	–1	6	–1	8
4	1	9	2	4
5	–2	3	0	2

The local transition function is defined as follows:

$$\boldsymbol{a}^{t+1}(z) = \begin{cases} a_{ij}, & \text{if } a_{ij} = \max(a_{kl}, a_{mn}),\ a_{kl}, a_{mn} \text{ are saddle points of } N(z) \\ \boldsymbol{a}^{t}(z) & \text{otherwise} \end{cases}$$

At time $t = 1$, the configuration of Table 1 the following form:

Table 2. Values for A given in Fig. 2 at $t = 1$

2	2	2	2	2
2	2	2	2	2
1	1	1	2	2
1	1	1	2	2
1	1	1	2	2

Now let us define reflexive games within payoff cellular automata. Denote the reflexive players by 1 and 2. Let A, a state of affairs, be identified with a set of payoffs within a game context (i.e. within a neighborhood). In other words, let $A_{\langle aij,\, bij\rangle}$ be a set of payoffs at the point $z \in \mathbf{Z}^d$ consisting of all the payoffs in $N(z) \cup z$, where z has state $\langle a_{ij}, b_{ij}\rangle$, see Table 3.

Table 3. The initial configuration of a payoff cellular automaton with the neighbourhood consisting of 8 members in 2-dimensional space and with players 1 and 2

⟨3,3⟩	⟨12,−12⟩	⟨13,−15⟩
⟨−12,12⟩	⟨−5,−5⟩	⟨−2,−2⟩
⟨−1,1⟩	⟨0,−6⟩	⟨2,−3⟩

The configuration shown in Table 3 presents 9 states of affairs:

$$A_{\langle 3,\,3\rangle} = \{\langle 3, 3\rangle, \langle 12, -12\rangle, \langle -12, 12\rangle, \langle -5, -5\rangle\}$$

$$A_{\langle 12,\,-12\rangle} = \{\langle 3, 3\rangle, \langle 12, -12\rangle, \langle 13, -15\rangle, \langle -12, 12\rangle, \langle -5, -5\rangle, \langle -2, -2\rangle\}$$

$$A_{\langle 13,\,-15\rangle} = \{\langle 12, -12\rangle, \langle 13, -15\rangle, \langle -5, -5\rangle, \langle -2, -2\rangle\},$$

$$A_{\langle -12,\,12\rangle} = \{\langle 3, 3\rangle, \langle 12, -12\rangle, \langle -12, 12\rangle, \langle -5, -5\rangle, \langle -1, 1\rangle, \langle 0,-6\rangle\}, \ldots$$

Let $\mathbf{B}_1^1 A_{\langle aij,\, bij\rangle}$ (accordingly, $\mathbf{B}_1^2 A_{\langle aij,\, bij\rangle}$) denote agent 1's (accordingly, agent 2's) Boolean superpositions of 1's payoffs of $A_{\langle aij,\, bij\rangle}$ (accordingly, 2's payoffs) for each first (accordingly, second) projection of all the points of $A_{\langle aij,\, bij\rangle}$. Then $K_1 A_{\langle aij,\, bij\rangle} = A_{\langle aij,\, bij\rangle} \cup \mathbf{B}_1^1 A_{\langle aij,\, bij\rangle}$ and $K_2 A_{\langle aij,\, bij\rangle} = A_{\langle aij,\, bij\rangle} \cup \mathbf{B}_1^2 A_{\langle aij,\, bij\rangle}$. Let $\mathbf{B}_2^1 A_{\langle aij,\, bij\rangle}$ (accordingly, $\mathbf{B}_2^2 A_{\langle aij,\, bij\rangle}$) denote agent 1's (accordingly, agent 2's) Boolean superpositions of $\mathbf{B}_1^1 A_{\langle aij,\, bij\rangle}$ and $\mathbf{B}_1^2 A_{\langle aij,\, bij\rangle}$ for each first (accordingly, second) projection of all the points of $A_{\langle aij,\, bij\rangle}$. Then $K_1 K_2 A_{\langle aij,\, bij\rangle} = A_{\langle aij,\, bij\rangle} \cup \mathbf{B}_1^2 A_{\langle aij,\, bij\rangle} \cup \mathbf{B}_2^1 A_{\langle aij,\, bij\rangle}$ and $K_2 K_1 A_{\langle aij,\, bij\rangle} = A_{\langle aij,\, bij\rangle} \cup \mathbf{B}_1^1 A_{\langle aij,\, bij\rangle} \cup \mathbf{B}_2^2 A_{\langle aij,\, bij\rangle}$. Let $\mathbf{B}_3^1 A_{\langle aij,\, bij\rangle}$ (accordingly, $\mathbf{B}_3^2 A_{\langle aij,\, bij\rangle}$) denote agent 1's (accordingly, agent 2's) Boolean superpositions of $\mathbf{B}_2^1 A_{\langle aij,\, bij\rangle}$ and $\mathbf{B}_2^2 A_{\langle aij,\, bij\rangle}$ for each first (accordingly, second) projection of all the points of $A_{\langle aij,\, bij\rangle}$. Then $K_2 K_1 K_2 A_{\langle aij,\, bij\rangle} = A_{\langle aij,\, bij\rangle} \cup \mathbf{B}_1^2 A_{\langle aij,\, bij\rangle} \cup \mathbf{B}_2^1 A_{\langle aij,\, bij\rangle} \cup \mathbf{B}_3^2 A_{\langle aij,\, bij\rangle}$ and $K_1 K_2 K_1 A_{\langle aij,\, bij\rangle} = A_{\langle aij,\, bij\rangle} \cup \mathbf{B}_1^1 A_{\langle aij,\, bij\rangle} \cup \mathbf{B}_2^2 A_{\langle aij,\, bij\rangle} \cup \mathbf{B}_3^1 A_{\langle aij,\, bij\rangle}$, and so on.

Example 2 (reflexive game of the second level)

Let us consider the payoff cellular automaton in Table 3 where the set S of states consists of all pairs $\langle a_{ij}^t, b_{ij}^t\rangle$, where a_{ij}^t, b_{ij}^t at time $t = 0, 1, 2, \ldots$ are integers in [–15, 13] and the local transition function is as follows: $a^{t+1}(z) = \langle a_{ij}^{t+1}, b_{ij}^{t+1}\rangle$, where $a_{ij}^{t+1} = ((\bigvee_m b_m^t \Rightarrow \bigvee_k a_k^t) \wedge (a_{ij}^t \wedge b_{ij}^t))$ and $b_{ij}^{t+1} = (\bigvee_k b_k^t \Rightarrow \bigvee_m a_m^t)$ and $\bigvee_k a_k^t$, $\bigvee_m b_m^t$ are the maximal payoffs of player 1 and player 2, respectively from a cell in $N(z) \cup z$ at time t, the logical operations are understood thus: $a \Rightarrow b := 13 - \max(a, b) + b$, $a \vee b := \max(a, b)$.

This automaton simulates a reflexive game, where player 1 is at the second level of reflexion, while player 2 is at the first level of reflexion. Its evolution at time $t = 1$ is shown in Table 4.

Table 4. The configuration of the payoff cellular automaton of figure 4 at $t = 1$

⟨3, 13⟩	⟨–12, 13⟩	⟨–15, 13⟩
⟨–12, 13⟩	⟨–5, 13⟩	⟨–2, 13⟩
⟨–1, 1⟩	⟨–6, 3 ⟩	⟨–3, 13⟩

Reflexion by agent i at the nth level in bimatrix games is expressed by $(n + 1)$ order knowledge operators $K_i^{n+1}A = K_i K_j K_i \ldots A$, where on the right hand side there are $(n + 1)$ K_m operators ($m = i, j$). Let us consider two agents i and j and suppose that the reflexive game takes place at the level n. This means that we have $K_i^{n+1}A$ and/or $K_j^{n+1}A$ which are understood as the perlocutionary effects of illocutionary acts and satisfy requirements (12)–(17). We know that $A \subseteq \ldots \subseteq K_j^n A \subseteq K_i^{n+1}A$ and $A \subseteq \ldots \subseteq K_i^n A \subseteq K_j^{n+1}A.$ Therefore, $K_i^{n+1}A \cap K_j^{n+1}A \neq \varnothing.$

The payoff of a reflexive game at the nth level in accordance with $K_i^{n+1}A$ or $K_j^{n+1}A$ is called a performative equilibrium of this game.

We have the following possibilities:

- both $K_i^{n+1}A$ and $K_j^{n+1}A$ are a performative equilibrium. This means that both agents i and j are on the same nth level of reflexion, simultaneously,
- only $K_i^{n+1}A$ is a performative equilibrium (then we can take $K_j^{n+1}A = K_j^n A$) – this means that agent i is at the nth level of reflexion, but agent j is at the $(n - 1)$th level of reflexion,
- only $K_j^{n+1}A$ is a performative equilibrium (then we can take $K_i^{n+1}A = K_i^n A$). This means that agent j is at the nth level of reflexion, but agent i is at the $(n - 1)$th level of reflexion.

In a reflexive game at level n for agent i, it is important that $K_i^n A \subseteq K_j^{n+1}A$, i.e. that agent i really is at the level n. Choosing an appropriate level of reflexion n may mean victory in a game.

Now let us define $K_i^{n+1}A$ on p-adic probabilities. Assume we have $p \in \mathbf{N}$ reflexive players i, j. Then all possible combinations $K_{\alpha_n} \ldots K_{\alpha_1} K_{\alpha_0} A$, where $\alpha_k \in \{i, j, \ldots\}$, can be presented by finite p-adic integers

$$\ldots 00\beta_n \ldots \beta_2\beta_1\beta_0 = \sum_{k=0}^{n} \beta_k p^k$$

where $\beta_k \in \{0, \ldots, p-1\}$ for each $k = 0, \ldots, n$ and there is a bijection between the sets $\{0, \ldots, p-1\}$ and $\{i, j, \ldots\}$.

Let Ω be a finite set of possible states of the world and $A \subseteq \Omega$. Then a finite p-adic probability measure P_i^{n+1} is defined on the sets $A, B \subseteq \Omega$ as follows:

$$P_i^{n+1}(\varnothing) = 0 \quad \text{and} \quad P_i^{n+1}(\Omega) = 1$$

$$\text{if} \quad P_j^{n}(A) > 0, \quad \text{then} \quad P_i^{n+1}(A) > 0$$

$$P_i^{n}(A) > 0 \quad \text{iff} \quad P_i^{n+1}(A) > 0$$

$$\text{if} \quad P_j^{n+1}(A) = 1, \quad \text{then} \quad P_i^{n}(A) = 1$$

$$P_i^{n+1}(A) = \frac{\sum_{k=0}^{n} \alpha_k p^k}{\sum_{k=0}^{n} (p-1) p^k} \quad \text{and} \quad P_i^{n+1}(B) = \frac{\sum_{k=0}^{n} \beta_k p^k}{\sum_{k=0}^{n} (p-1) p^k}$$

where $\alpha_k,\ \beta_k \in \{0,\ \ldots,\ p-1\}$ for each $k = 0, \ldots, n$,

$$P_i^{n+1}(A \cup B) = P_i^{n+1}(A) + P_i^{n+1}(B)\ \mathit{if}\ A \cap B = \varnothing, \qquad P_i^{n+1}(\neg A) = 1 - P_i^{n+1}(A)$$

$$P_i^{n+1}(A|B) = \frac{\inf\left(\sum_{k=0}^{n} \alpha_k p^k,\ \sum_{k=0}^{n} \beta_k p^k\right)}{\sum_{k=0}^{n} \beta_k p^k}$$

where inf is defined digit by digit. For instance, if we have just two agents, then at the zero level of reflexion we have only two probability values: either 0 or 1 (meaning, e.g. that an agent either does not follow the content $A \subseteq \Omega$ or does). At the first level of reflexion we already have the following four probability values: 0, 1/3, 2/3, 1 (meaning, e.g. that neither agent follows the content $A \subseteq \Omega$, one of them does not follow, another does, and both of them follow), etc.

Now we can define $K_i^{n+1}A$ in the following way:

$$K_i^{n+1}A = \left\{\omega : A \subseteq \mathbf{Q}_i(\omega) = \left\{a : P_i^{n+1}(a \mid \omega) > 0\right\}\right\}$$

Note that according to this definition, taking into account our assumption that if $P_j^n(a \mid \omega) > 0$, then $P_i^{n+1}(a \mid \omega) > 0$, we have $K_j^n A \subseteq K_i^{n+1}A$ for each agent *j* participating in the reflexive game.

Let us suppose that there are just three reflexive players *k*, *l*, *m* at the reflexion level *n* = 2. Then $P_i^2(A \subseteq \Omega) \in \{0, 1/8, 2/8, 3/8, 4/8, 5/8, 6/8, 7/8, 1\}$ for each $i \in \{k, l, m\}$. At the infinite level of reflexion, we have the following *p*-adic probabilities:

$$P_i^\infty(A) = \lim_{n\to\infty} P_i^n(A)$$

The knowledge operators $K_i^{n+1}A$ satisfy the following relations:

$$(K_i^{n+1}A \cap K_i^{n+1}B) \Rightarrow K_i^{n+1}(A \cap B)$$

$$K_i^{n+1}(A \cup B) \Rightarrow (K_i^{n+1}A \cup K_i^{n+1}B)$$

$$K_i^{n+1}(A \cup B) = (K_i^{n+1}A \cap K_i^{n+1}B)$$

$$A \subseteq B \Rightarrow K_i^{n+1}A \subseteq K_i^{n+1}B$$

$$A \subseteq K_i^{n+1}A$$

$$K_i^{n+1}K_i^n A = K_i^n A$$

Using (finite) *p*-adic probabilities, we understand reflexion levels discretely. Therefore, between *n* and *n* + 1 there are no other reflexive levels. For any finite number of agents we can always define a reflexive level *n* such that probabilities are distributed on an appropriate finite set of *p*-adic numbers. The larger *n* (or the larger the number of reflexive agents), the more finite *p*-adic probabilities.

5. Conclusion

The reflexion disagreement theorem opens the door to new mathematics in game theory and decision theory, in particular it shows that it has sense to use stream calculus, non-Archimedean mathematics, and *p*-adic analysis. Within this mathematics, we can formalize reflexive games of different reflexive levels (up to the infinite reflexive level). These results can be implemented in new mathematical tools of behavioral finance.

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Received 12 June 2013
Accepted 7 January 2014