Rankings of multi-aspect fuzzy sets in modelling the decision support systems

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The paper presents a method of constructing rankings of fuzzy sets, in particular multi-aspect fuzzy sets. A new class of global membership functions based on the generalized Minkowski norm has been defined. Due to their properties, they can be an alternative to classical product functions. These functions can be used to construct rankings of multi-aspect fuzzy sets, which are an important element of decision support systems. The concepts presented in this paper are illustrated with a numerical example from the area of support of diagnostic decisions based on multi-aspect fuzzy sets.

Keywords: multi-aspect fuzzy set, global membership function, ranking of fuzzy set.

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1. Introduction

In medical diagnosis support systems a medical diagnosis can be defined as a fuzzy set of various diagnoses (disease entities). These diagnoses belong to this set on various degrees of membership depending on whether it relates to disease symptoms identified, risk factors or additional specialized medical examinations. In such cases, fuzzy sets (in the classic version) can be defined as the products [10, 13, 28, 33, 36] of the corresponding (classic "one-aspect") fuzzy sets. The "aggregated" function of membership (usually t-norm) in a set that is the product of multiple fuzzy sets – then appears [10, 13, 33]. In such situations, however, doubts may arise, supported by many analyses and practical studies, as to whether these functions [33, 36] "properly" describe the degree of membership of elements in the fuzzy set. The product of the two fuzzy sets *A* and *B* is the fuzzy set *C* of elements, each of which "to some extent" is simultaneously the member of the both sets. The total "resultant" degree of membership is expressed by the new function of membership of the elements in the set *C*. The way to construct a definition of such function is not obvious [10, 33, 36]. Such a function in decision-making models should guarantee the execution of certain practical postulates [3, 4, 10, 13]. These postulates generally result from intuitive decision expectations in decision support systems. The alternative and more natural approach in such a situation may turn out to be an attempt to use the concept of a multi-aspect fuzzy set.

2. The global Minkowski membership functions and their properties

A global membership function is a certain aggregate of "partial" membership functions when applied to multi-aspect sets [6]. It can also be defined as the function of membership in sets resulting from operations on classical fuzzy sets e.g. on products.

Let μ_A be the multi-aspect function of membership of elements $x \in X$ in the non--empty, multi-aspect fuzzy set *A*. Let $\eta_A : X \to [0,1]$ be a certain scalar function, called the global (aggregate) function of membership of the elements $x \in X$ in the set *A*. From the point of view of practical applications of fuzzy sets in the decision support process, the membership functions should meet a number of postulates:

a) **postulate for monotonicity** – the global membership function $\eta_A(x)$, $x \in X$ meets the postulate for monotonicity, if for $x, y \in X$ such as that $\mu_A(x) \geq \mu_A(y)$ we have $\eta_A(x) \geq \eta_A(y)$.

b) **postulate for differentiation** – the global membership function $\eta_A(x),\ x\in X$ meets the postulate for differentiation if for each ones $x,y\in X$, such as that $\mu_A(x)\ge\mu_A(y)$ and $\mu_A(x)\ne\mu_A(y)$, $\eta_A(x)>\eta_A(y)$ occurs.

c) **postulate for lack of internal contradiction** – the global membership function $\eta_A(x),\ x\in X$ meets the postulate for lack of internal contradiction if for each ones $x,y\in X$, such as that $\eta_A(x)=\eta_A(y)$ and $\mu_A(x)\ne\mu_A(y)$, no $\mu_A(x)\ge\mu_A(y)$ and $\mu_A(y)\ge\mu_A(x)$ occurs.

d) **postulate for continuation** – the global membership function $\eta_A(x),\ x\in X$ meets the postulate for continuation, if for each one $x\in X$, such as that $\mu_A^n(x)\ne 0, n\in\mathcal{N}=\{1,...,N\}$ we have $\eta_A(x)\ne 0$.

The new class of the function of global membership $\eta_A(x),\ x\in X$ in the multi-aspect set $A=\{(x,\mu_A(x))|x\in X\}$, can be defined by using the concept of height (upper pole) of the fuzzy set or the concept of threshold (lower pole) [6].

Let

$$\mathrm{hgt}(A)=\overset{*}{y}(A)=\left(\overset{*}{y}_1(A),...,\overset{*}{y}_n(A),...,\overset{*}{y}_N(A)\right)$$

be the upper pole of the set *A*, where

$$\overset{*}{y}_n(A)=\max_{x\in X}\mu_A^n(x)=\overset{*}{y}_n, n\in\mathcal{N} \tag{1}$$

Let's mark, for simplicity's sake, that $\mathrm{hgt}(A)=\overset{*}{y}=\left(\overset{*}{y}_1,...,\overset{*}{y}_n,...,\overset{*}{y}_N\right)\in\mathcal{R}^N$.

The global (total) degree of the membership of $\eta_A(x)$ the element $x\in X$ in the multi-aspect fuzzy set *A* can be determined by using the distance of its multi-aspect $\eta_A(x)\in\mathcal{R}^N$ image from the upper pole $\mathrm{hgt}(A)=\overset{*}{y}\in\mathcal{R}^N$. This distance can be determined by means of the so-called Minkowski standard (metric) [3, 5, 32]. It shall be the number:

$$\left\|y-\overset{*}{\mu}_A(x)\right\|_p,\ p\ge 1 \text{ for } x\in X.$$

The value of the global membership function $\eta_A^p(x)$ (after standardization) shall be written as follows:

$$\eta_A^p(x)=\frac{\left\|\overset{*}{y}\right\|_p-\left\|\overset{*}{y}-\mu_A(x)\right\|_p}{\left\|\overset{*}{y}\right\|_p}=1-\gamma_p\left\|\overset{*}{y}-\mu_A(x)\right\|_p,$$

where

$$\gamma_p=\frac{1}{\left\|\overset{*}{y}\right\|_p},\ p\ge 1 \tag{2}$$

Let's notice that $0\le\eta_A^p(x)\le 1,\ x\in X,\ p\ge 1$.

Definition 1

The function $\eta_A^p(x)=1-\gamma_p\left\|\overset{*}{y}-\mu_A(x)\right\|_p,\quad p\ge 1$ shall be called the Minkowski global function of membership in the multi-aspect fuzzy set *A*.

Theorem 1

The function $\eta_A^p(x)=1-\gamma_p\left\|\overset{*}{y}-\mu_A(x)\right\|_p,\ p\ge 1$ meets the monotonicity postulate.

The evidence:

The entry "$\mu_A(x)\ge\mu_A(y)$" can be replaced by the notation: $\mu_A^n(x)\ge\mu_A^n(y),\ n\in\mathcal{N},\ x,y\in X$ notation $\eta_A(x)\ge\eta_A(y)$ we can replace with notation

$$1-\gamma_p\left\|\overset{*}{y}-\mu_A(x)\right\|_p\ge 1-\gamma_p\left\|\overset{*}{y}-\mu_A(y)\right\|_p$$

You have to show that this notation is true. By subtracting the number 1 from both sides of the inequality and multiplying both sides by the number $(-\gamma_p)$ we shall obtain:

$$\left\|\overset{*}{y}-\mu_A(x)\right\|_p\le\left\|\overset{*}{y}-\mu_A(y)\right\|_p,$$

which according to the Minkowski definition [3] shall be written as follows:

$$\left(\sum_{n\in\mathcal{N}}\left|\overset{*}{y}_n-\mu_A^n(x)\right|^p\right)^{1/p}\le\left(\sum_{n\in\mathcal{N}}\left|\overset{*}{y}_n-\mu_A^n(y)\right|^p\right)^{1/p}$$

It follows from (1) that $\overset{*}{y}_n\ge\mu_A^n(x)$ and $\overset{*}{y}_n\ge\mu_A^n(y)$ for $n\in\mathcal{N}$.

So we can write:

$$\left(\sum_{n\in\mathcal{N}}\left(\overset{*}{y}_n-\mu_A^n(x)\right)^p\right)^{1/p}\leq\left(\sum_{n\in\mathcal{N}}\left(\overset{*}{y}_n-\mu_A^n(y)\right)^p\right)^{1/p}$$

This inequality is true because $\overset{*}{y}_n-\mu_A^n(x)\leq\overset{*}{y}_n-\mu_A^n(y)$ occurs for each $n\in\mathcal{N}$, because after subtracting $\overset{*}{y}^n$ from both sides and multiplying by (–1) we obtain: $\mu_A^n(x)\geq\mu_A^n(y)$ and this is true by assumption. ■

The functions $\eta_A^p(x)=1-\gamma_p\left\|\overset{*}{y}-\mu_A(x)\right\|_p$, for $1\leq p<\infty$ are also met by the other postulates. The selection of an appropriate membership function, especially in the usage of fuzzy sets being the result of various operations (for example, the product of fuzzy sets or multi--faceted sets) is very essential. In case of the classic product of two fuzzy sets *A* and *B*, the so-called *t-norms* (see [10, 13, 33, 36]), and in particular the following membership functions, are used most often:

1) $\mu_\cap^1(x)=\min\left\{\mu_A(x),\mu_B(x)\right\},\ x\in X$
2) $\mu_\cap^2(x)=\mu_A(x)\mu_B(x),\ x\in X$ (3)
3) $\mu_\cap^3(x)=\max\left\{0,\mu_A(x)+\mu_B(x)-1\right\},\ x\in X$

The Minkowski membership global functions presented in Definition 1 have a number of interesting properties [3, 5, 9, 13, 32]. Including in relation to the above-mentioned product-derived membership functions $\mu_\cap^n(x),\ x\in X$, $n=1,\ 2,\ 3$. One of these functions is referred to by the following statement.

Theorem 2

Let $A=\left\{\left(x,\mu_A(x)\right)\middle|x\in X\right\}$, be the multi--aspect, standardised [33] fuzzy set with the membership function

$$\mu_A(x)=\left(\mu_A^1(x),...,\mu_A^n(x),...,\mu_A^N(x)\right),\ x\in X$$

Let also:

$\mu_A^{\min}(x)=\min\left\{\mu_A^n(x),\ n\in\mathcal{N}\right\},\ x\in X$ – be the "min-type" membership function [33, 36],

$\eta_A^p(x)=1-\gamma_p\left\|\overset{*}{y}-\mu_A(x)\right\|_p$, $p\geq1$ (*)– be the Minkowski membership function with the parameter $p\geq1$.

The thesis: for $p\to\infty$ there is: the $\eta_A^\infty(x)=\mu_A^{\min}(x),\ x\in X$.

The evidence: for standardized (normal) sets we have $\overset{*}{y}_n=\max_{x\in X}\mu_A^n(x)=1,\ n\in\mathcal{N}$, so we have:

$$\operatorname{hgt}(A)=\overset{*}{y}(A)=\overset{*}{y}=\left(\overset{*}{y}_1,...,\overset{*}{y}_n,...,\overset{*}{y}_N\right)=$$
$$=(1,...,1)\in\mathcal{R}^N$$

furthermore $\gamma_\infty=\dfrac{1}{\left\|\overset{*}{y}\right\|_\infty}=1$,

because

$$\left\|\overset{*}{y}\right\|_\infty=\left\|(1,...,1)\right\|_\infty=\max\{1,...,1\}=1 .$$

The formula (*) for $p\to\infty$ shall take the form of: $\eta_A^\infty(x)=1-\max\left\{\left(1-\mu_A^n(x)\right)\middle|n\in\mathcal{N}\right\}$ (**)

Let $k\in\mathcal{N}$ be that $\mu_A^k(x)=\min_{n\in\mathcal{N}}\mu_A^n(x),\ x\in X$ (there always exists such $k\in\mathcal{N}$). We have then:

$$\mu_A^{\min}(x)=\min\left\{\mu_A^n(x),\ n\in\mathcal{N}\right\}=\mu_A^k(x),\ x\in X.$$

If that's the case, then based on (**) we also have

$$\eta_A^\infty(x)=1-\max\left\{\left(1-\mu_A^n(x)\right)\middle|n\in\mathcal{N}\right\}=1-\left(1-\mu_A^k(x)\right)=$$
$$=1-1+\mu_A^k(x)=\mu_A^k(x),\ x\in X.$$

Thus: $\eta_A^\infty(x)=\mu_A^{\min}(x),\ x\in X$, what should have been shown. ■

Many interesting additional properties of the Minkowski functions of type (15) can be found in his papers [3, 5, 6, 32].

3. The rankings of fuzzy sets

The membership functions play an important role when the fuzzy sets they define are used in decision support algorithms (for example in diagnostic decision support). Their typical application is to rank the fuzzy set elements [5, 13]. The ranking of fuzzy set elements is usually associated with the ranking of elements of its carrier supp (*A*) [6].

$$\operatorname{supp}(A)=$$
$$=\left\{x\in X\middle|\text{exist } n\in\mathcal{N},\text{ that } \mu_A^n(x)>0\right\}\subset X \quad (4)$$

Definition 2

Any sequence supp (A) of the r (A) sets forming its division [3, 5], shall be called the ranking of the elements of the carrier $A_k \subset \operatorname{supp}(A)$.

$$r(A) = (A_1, ..., A_k, ..., A_K)$$

It is therefore such a sequence of sub-sets A_k for $k \in \mathcal{K} = \{1, ..., k, ..., K\}$
that

$$1)\quad A_k \cap A_m = \varnothing \text{ dla } k \neq m$$

$$2)\quad \bigcup_{k \in \mathcal{K}} A_k = \operatorname{supp}(A) \qquad (5)$$

The number of possible divisions of the set shall be determined by the so-called Bell number [5].

Definition 3

The set A_k is called the k-th element of the ranking r (A) (the k-th cluster or the k-th category of the ranking r (A)) of the *fuzzy* set A.
A very interesting approach (according to Definition 2) to multi-criteria ranking creation is the concept of using the concept of minimal and maximal elements of a set (set clustering) [3, 5].
Each fuzzy set is naturally accompanied by *a specific ranking* resulting from its global membership function η_A [5]. The sets A_k for $k \in \mathcal{K} = \{1, ..., k, ..., K\}$, in this case, is defined by the following recursive formula:

$$A_k = \arg \max_{x \in X - \bigcup_{l=0}^{k-1} A_l} \eta_A(x),$$
$$k \in \mathcal{K} = \{1, ..., k, ..., K\},\ A_0 = \varnothing \qquad (6)$$

The ranking of elements of a fuzzy set, determined according to the formula (6), is very effective and comfortable decision support tool when solving optimization problems with the use of fuzzy sets of acceptable decisions. The numerical example, in the form of the ranking, shall be presented below to illustrate the importance (impact) of formulas defining global membership functions on the result of decision optimization. The results obtained by using various functions of membership in multi-aspect fuzzy sets as well as to sets that are the product of classic fuzzy sets shall be analysed. The important concepts in the context of rankings are those derived from the definition of the minimal and maximal elements of a set [3, 32]. These are the lower and upper fronts of a fuzzy set.

Definition 4

The below set shall be called *the upper front of the fuzzy set A* (the ceiling of the fuzzy set):

$$\operatorname{roof}(A) = \left\{ \begin{array}{l} x \in \operatorname{supp}(A) \,|\, \text{not exist } y \in \operatorname{supp}(A), \\ \text{that } \mu_A(y) \geq \mu_A(x) \text{ and } \mu_A(y) \neq \mu_A(x) \end{array} \right\}$$

an element belonging to the upper front (ceiling) of a fuzzy set is such an element belonging to the carrier of the set that among the other elements of the carrier there is no element that has "larger membership" in this set in the sense of all aspects.

Definition 5

The below set shall be called *the bottom front of the fuzzy set* (the floor of the fuzzy set):

$$\operatorname{floor}(A) = \left\{ \begin{array}{l} x \in \operatorname{supp}(A) \,|\, \text{not exist } y \in \operatorname{supp}(A), \\ \text{that } \mu_A(y) \leq \mu_A(x) \text{ and } \mu_A(y) \neq \mu_A(x) \end{array} \right\}$$

an element belonging to the lower front (floor) of the fuzzy set is such that among other elements there is no element that would have an even smaller membership in this set. The elements being members of the fronts of the fuzzy set are called *the front elements* (upper or lower respectively). They play an important role in decision support (optimization) processes.

4. The two-aspect fuzzy set as a medical diagnosis and its ranking

In this part of the paper, a certain example of a two-aspect fuzzy set will be considered, for which selected characteristics and global membership functions, based on different concepts including global Minkowski functions, shall be determined. Due to the possibility of graphical interpretation, the number of considered features (facets) of the diagnosis shall be reduced to two facets. In the medical diagnostic process [1, 4, 8, 10, 16, 29] of determining a patient's health status, three facets are generally taken into account: the occurrence of disease symptoms and their severity, the intensity of occurrence of disease risk factors, and the "clarity" of the results of the laboratory tests performed (if any). Based on this data, a patient's health status can be defined as a triad of fuzzy sets. Similarly, the pattern of a disease entity (illness) can be defined in terms of analogous three fuzzy sets. The degree of similarity of the patient's health condition (fuzzy set) in terms of symptoms of a specific disease entity (fuzzy set) defines the value of the function of membership of this disease entity in the determined (fuzzy) diagnosis in terms of disease symptoms. The same occurs in the aspect of identified risk factors and derived results of additional laboratory tests. Hence, each disease entity can be said to be in a potential (fuzzy) diagnosis with a defined value of the membership function separately in terms of three characteristics: identified disease symptoms, existing risk factors and laboratory test results. The typical initial diagnostic process carried out by a doctor involves only two aspects in the first stage: the disease symptoms and the risk factors present. The possible need for laboratory tests represents the next iteration in the diagnostic process. In this example, only two facets (leading to the initial diagnosis) will be considered: the identified symptoms and the risk factors. Thus, the initial diagnosis in this case will be understood as a two-aspect fuzzy set in the form of:

$$A = \{(x, \mu_A(x)) | x \in X\}$$

where $\mu_A(x) = (\mu_A^1(x), \mu_A^2(x)),\ x \in X$ is the vector (two-faceted) function of the membership of individual disease entities in the set that constitutes the fuzzy diagnosis. The example: let the set $X = \{1, 2, ..., 30\}$ be the set of possible disease entities (disease repository). The number $\mu_A^1(x)$ is the value of the function of membership of the disease entity x in diagnosis A in terms of the symptoms found in the patient and the number $\mu_A^2(x)$ is the value of the function of membership of the disease entity x in diagnosis A in terms of the risk factors found [4, 6, 8, 10].

We will say about the disease entity $x \in X$ that it fits more (has larger membership) to the set A (to the fuzzy diagnosis A) than the disease $y \in X$ if $\mu_A(x) \geq \mu_A(y)$ and $\mu_A(x) \neq \mu_A(y)$ occurs. The image of the fuzzy set A is the set $O_A(X) = \{\mu_A(x) | x \in \text{supp}(A)\}$. It is "a cloud of thirty dots" in two-dimensional space. Data on the values of the membership function $\mu_A(x) = (\mu_A^1(x), \mu_A^2(x)),\ x \in X$ are presented in the table next to Figure 1. This figure shows an image of a fuzzy set $O_A(X) = \{\mu_A(x) | x \in \text{supp}(A)\}$. Then we have the situation that there is $\mu_A(3) = \mu_A(5) = (0.8,\ 0.9)$ for $x \in \{3, 5\}$ (see Fig.1). Hence $|O_A(X)| = 29$. The further characteristics of the fuzzy set: $\text{hgt}(A) = \overset{*}{y}(A) = (1,1) \in \mathcal{R}^2$ the *ceiling* of the fuzzy set A (*the upper front of the fuzzy set*) is the set $\text{roof}(A) = \{3,\ 5,\ 7,\ 9,\ 16\}$, the *floor* of the fuzzy set A (*the lower front of the fuzzy set*) is the set $\text{floor}(A) = \{11,\ 13,\ 20\}$.

The sets $\text{roof}(A) = \{3,\ 5,\ 7,\ 9,\ 16\}$ and $\text{floor}(A) = \{11,\ 13,\ 20\}$ are marked in Figure1. Table 1 is structured as follows. The elements of the set X are listed in the first column. The values of the selected global membership function are presented in the next five columns:

1) $\mu_\cap^1(x) = \min\{\mu_A^1(x), \mu_A^2(x)\},\ x \in X$
2) $\mu_\cap^2(x) = \mu_A^1(x)\mu_A^2(x),\ x \in X$
3) $\mu_A^L(x) = \eta_A^{p=1}(x),\ x \in X,$ (see definition 1)
4) $\mu_A^E(x) = \eta_A^{p=2}(x),\ x \in X,$ (see definition 1)
5) $\mu_\cap^3(x) = \max\{0, \mu_A^1(x) + \mu_A^2(x) - 1\},\ x \in X$

X	$\mu_A^1(x)$	$\mu_A^2(x)$
1	**0,50**	**0,20**
2	**1,00**	**0,50**
3	**0,80**	**0,90**
4	**0,50**	**0,50**
5	**0,80**	**0,90**
6	**0,30**	**0,40**
7	**0,90**	**0,80**
8	**0,70**	**0,10**
9	**1,00**	**0,60**
10	**0,20**	**1,00**
11	**0,30**	**0,10**
12	**0,60**	**0,40**
13	**0,10**	**0,30**
14	**0,30**	**0,80**
15	**0,90**	**0,30**
16	**0,40**	**1,00**
17	**0,90**	**0,60**
18	**0,40**	**0,30**
19	**0,70**	**0,70**
20	**0,00**	**0,50**
21	**0,50**	**0,10**
22	**0,70**	**0,30**
23	**0,50**	**0,80**
24	**0,20**	**0,60**
25	**0,80**	**0,50**
26	**0,60**	**0,70**
27	**0,10**	**0,70**
28	**0,80**	**0,20**
29	**0,40**	**0,70**
30	**0,70**	**0,60**

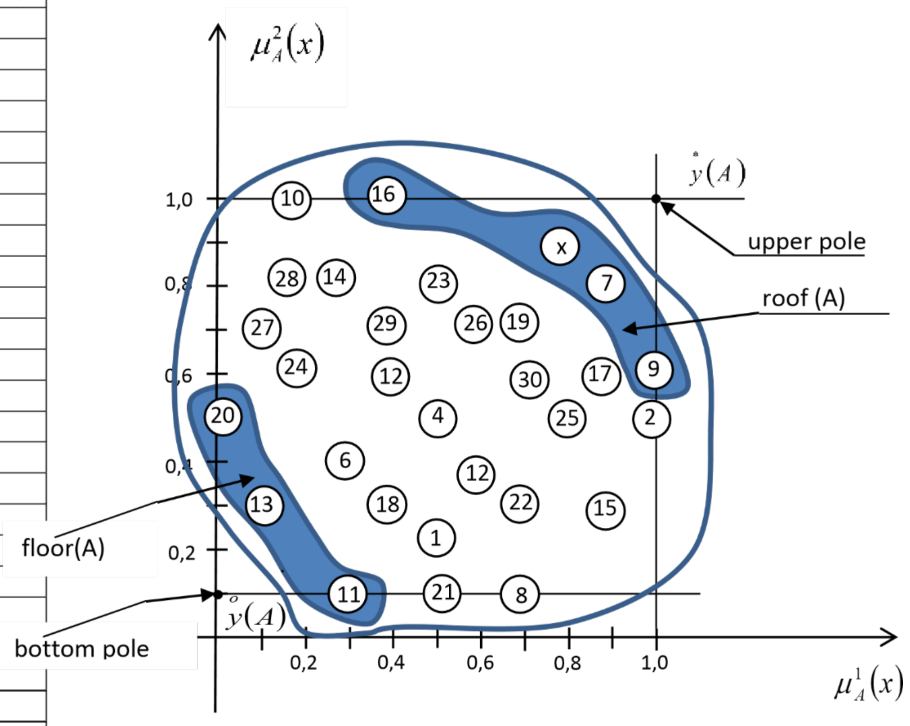

Fig. 1. Image of multi aspects fuzzy set

Because of $\mu_\cap^1(x) = \eta_A^{p=\infty}(x),\ x \in X$, (see Theorem 2), the table does not include a column with the values of the $\eta_A^{p=\infty}(x),\ x \in X$ function. The next (last) five columns of the table are the rankings of the fuzzy set *A*, derived respectively by the values of the individual global membership functions. In accordance with Definition 2, the rankings of the elements of the fuzzy set *A*, derived from the values of the above global membership functions, are as follows:

1) $$r^1(\min) = \begin{pmatrix} \{3,5,7\},\{19\},\{9,17,30\}, \\ \{2,4,23,25\},\{12,16,29\}, \\ \{6,14,15,18,22\}, \\ \{1,10,24,28\}, \\ \{8,11,13,21,27\},\{20\} \end{pmatrix},$$

(9 clusters0

2) $$r^2(\text{product}) = \begin{pmatrix} \{3,5,7\},\{9\},\{17\},\{2\}, \\ \{19\},\{26,30\},\{16,23,25\}, \\ \{29\},\{15\},\{4\},\{12,14\}, \\ \{22\},\{10\},\{28\}, \\ \{6,18,24\},\{1\},\{8,27\}, \\ \{21\},\{11\},\{13\},\{20\} \end{pmatrix},$$

(21 clusters)

3) $$r^L = \begin{pmatrix} \{3,5,7\},\{9\},\{2,17\},\{16,19\}, \\ \{23,25,26,30\},\{10,15\},\{14,29\}, \\ \{4,12,22,28\},\{8,24,27\},\{1,6,18\}, \\ \{21\},\{20\},\{11,13\} \end{pmatrix},$$

(13 clusters)

4) $$r^E = \begin{pmatrix} \{3,5,7\},\{9,19\},\{17\},\{2,26,30\}, \\ \{23,25\},\{16\},\{29\},\{4\},\{12,14,15\}, \\ \{22\},\{10\},\{28\},\{24\},\{6,18\},\{1,8\}, \\ \{27\},\{21\},\{11,13\},\{20\} \end{pmatrix},$$

(19 clusters)

5)

$$r^3 = \begin{pmatrix} \{3,5,7\},\{9\},\{2,17\},\{16,19\}, \\ \{23,25,26,30\},\{10,15\},\{14,29\}, \\ \{1,4,6,8,11,12,13,18,20,21,22,24,27,28\} \end{pmatrix},$$

(8 clusters)

Tab. 1. Data set

X	$\mu_\cap^1(x)$	$\mu_\cap^2(x)$	$\mu_A^L(x)$	$\mu_A^E(x)$	$\mu_\cap^3(x)$	r^1	r^2	r^L	r^E	r^3
1	0.20	0.10	0.35	0.342	0.00	3	3	3	3	3
2	0.50	0.50	0.75	0.650	0.50	5	5	5	5	5
3	0.80	0.72	0.85	0.846	0.70	7	7	7	7	7
4	0.50	0.25	0.50	0.490	0.00	19	9	9	9	9
5	0.80	0.72	0.85	0.846	0.70	9	17	2	19	2
6	0.30	0.12	0.35	0.348	0.00	17	2	17	17	17
7	0.80	0.72	0.85	0.846	0.70	26	19	16	2	16
8	0.10	0.07	0.40	0.342	0.00	30	26	19	26	19
9	0.60	0.60	0.80	0.715	0.60	2	30	23	30	23
10	0.20	0.20	0.60	0.440	0.20	4	23	25	23	25
11	0.10	0.03	0.20	0.209	0.00	23	25	26	25	26
12	0.40	0.24	0.50	0.496	0.00	25	16	30	16	30
13	0.10	0.03	0.20	0.209	0.00	12	29	10	29	10
14	0.30	0.24	0.55	0.496	0.10	16	15	15	4	15
15	0.30	0.27	0.60	0.496	0.20	29	4	14	12	14
16	0.40	0.40	0.70	0.580	0.40	6	12	29	14	29
17	0.60	0.54	0.75	0.702	0.50	14	14	4	15	1
18	0.30	0.12	0.35	0.348	0.00	15	22	12	22	4
19	0.70	0.49	0.70	0.715	0.40	18	10	22	10	6
20	0.00	0.00	0.25	0.195	0.00	22	28	28	28	8
21	0.10	0.05	0.30	0.279	0.00	1	6	8	24	11
22	0.30	0.21	0.50	0.468	0.00	10	18	24	6	12
23	0.50	0.40	0.65	0.629	0.30	24	24	27	18	13
24	0.20	0.12	0.40	0.384	0.00	28	1	1	1	18
25	0.50	0.40	0.65	0.629	0.30	8	8	6	8	20
26	0.60	0.42	0.65	0.650	0.30	11	27	18	27	21
27	0.10	0.07	0.40	0.335	0.00	13	21	21	21	22
28	0.20	0.16	0.50	0.426	0.00	21	11	20	13	24
29	0.40	0.28	0.55	0.531	0.10	27	13	13	11	27
30	0.60	0.42	0.65	0.650	0.30	20	20	11	20	28

The analysis of the derived rankings leads to a number of interesting conclusions. The most important one shows that in the case of the classic membership function [33]: $\eta_A(x) = \mu_\cap^1(x) = \min\{\mu_A^1(x), \mu_A^2(x)\},\ x \in X$, (see [10, 13]) there were relatively many *internal contradictions* resulting from the degree of global membership and membership resulting directly from the partial membership function $\mu_A^1(x), \mu_A^2(x)$. For example, for the pair of the elements $16,\ 29 \in X$, although $\mu_\cap^1(16) = \mu_\cap^1 29 = 0.4$ exists, (which means the same degree of global membership in the fuzzy set *A*), $\mu_A(16) \geq \mu_A(29)$ and $\mu_A(16) \neq \mu_A(29)$ occurs. Since there is $\mu_A^1(16) = \mu_A^1(29) = 0.4$ but $\mu_A^2(16) = 1.0$ and $\mu_A^2(29) = 0.7$ occurs. The function $\mu_\cap^1(x) = \min\{\mu_A^1(x), \mu_A^2(x)\}$ does not meet the postulate for lack of internal contradiction. The element x=16 should have *larger* membership in the set *A* than the element

$x = 29$ but there is $\mu^1_\cap(16) = \mu^1_\cap 29 = 0.4$. The similar situation applies to the pairs of elements (15,18), (9,30) and many others pairs. The function

$\eta_A(x) = \mu^1_\cap(x) = \min\{\mu^1_A(x), \mu^2_A(x)\},\ x \in X$

does not meet, in this example (as you can easily see), either the postulate for differentiation or the postulate for lack of internal contradiction. The ranking $r^1(\min)$ obtained is "very blurry" (it has only 9 clusters). In some "areas" it is contradictory. It would be difficult to recommend it for use in decision support algorithms. This function and the resulting ranking coincides with the Minkowski function for $p=\infty$. Even worse in terms of fulfilling the postulate of absence of internal contradiction, differentiation and continuation is the case of the function

$\mu^3_\cap(x) = \max\{0, \mu^1_A(x) + \mu^2_A(x) - 1\},\ x \in X$

[28, 30]. For example, in case of the pair of elements $4,\ 13 \in X$ although $\mu^1_\cap(4) = \mu^1_\cap(13) = 0$ occurs (which means the identical degree of global membership in the fuzzy set A and in addition it means that the degree is equal to zero) then $\mu^1_A(4) = 0.5 > \mu^1_A(13) = 0.1$

and $\mu^2_A(4) = 0.5 > \mu^2_A(13) = 0.3$ occurs.

The postulates for differentiation, for lack of internal contradiction and for continuation are therefore not fulfilled. Likewise in many other situations. The number of clusters in the ranking is only 8. The remaining functions of global membership in the fuzzy set (and especially $\eta_A(x) = \mu^2_\cap(x) = \mu^1_A(x)\mu^2_A(x)$,

and

$\eta^{p=2}_A(x) = \mu^E_A(x),\ x \in X$ bring, in practical terms, much better results (the number of clusters in the ranking is 21 and 19 respectively). They also meet all four postulates. Figure 1 shows the image of the fuzzy set *A* and such decision making characteristics of the set as the upper and lower front of the set as well as the both poles of the set.

5. Conclusion

This paper proposes new global membership functions for multi-aspect fuzzy sets and sets that are products of fuzzy sets. The ranking of a multi-faceted fuzzy set is defined. Comparisons have been made between the properties of classical membership functions (for the product of fuzzy sets) and those of Minkowski membership function, in the context of satisfying decision postulates. It is clear from the analysis that classical membership functions (some t^{th}-norms [26, 27, 33, 36]) of fuzzy sets such as $\mu^1_\cap(x) = \min\{\mu^1_A(x), \mu^2_A(x)\}$,

$\mu^3_\cap(x) = \max\{0,\ \mu^1_A(x) + \mu^2_A(x) - 1\},\ x \in X$

do not practically meet the most important decision postulates. The rankings derived on their basis are not "precise". Therefore, they should not be recommended for use in modelling decision support systems using fuzzy sets methodology. In turn, the functions $\mu^2_\cap(x) = \mu^1_A(x)\mu^2_A(x)$

and $\eta^{p=2}_A(x) = \mu^E_A(x),\ x \in X$ fully meet these postulates.

The rankings obtained on their basis are "almost linear". They are also very similar to each other (almost identical ones). The function $\eta^{p=1}_A(x) = \mu^L_A(x),\ x \in X$ outputs slightly worse results in terms of meeting the postulates, but much better than classical functions. An interesting observation is the fact that all analysed global membership functions in the above example brought the rankings that on the first three (most important) positions of the ranking have exactly the same elements and in the same order. Besides these elements are the members of the *upper front* (*ceiling*) of the fuzzy set *A* [2, 9, 11, 12].

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Rankingi wieloaspektowych zbiorów rozmytych w modelowaniu systemów wspomagania decyzji

A. AMELJAŃCZYK

W artykule przedstawiono metodę konstruowania rankingów zbiorów rozmytych, w szczególności wieloaspektowych zbiorów rozmytych. Zdefiniowano nową klasę globalnych funkcji przynależności w oparciu o uogólnioną normę Minkowskiego. Ze względu na swoje właściwości mogą one stanowić alternatywę dla klasycznych funkcji iloczynowych. Funkcje te można wykorzystać do konstruowania rankingów wieloaspektowych zbiorów rozmytych, które są ważnym elementem systemów wspomagania decyzji. Koncepcje prezentowane w artykule zilustrowano przykładem numerycznym z obszaru wspomagania decyzji diagnostycznych w oparciu o wieloaspektowe zbiory rozmyte danych medycznych.

Słowa kluczowe: wieloaspektowy zbiór rozmyty, globalna funkcja przynależności, ranking zbioru rozmytego.