PROJECTIVE COLLINEATIONS AS PRODUCTS OF TWO CYCLIC COLLINEATIONS II.

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Abstract. The paper contains one theorem saying that, for arbitrary even *k*, every projective collineation in the three-dimensional projective space is a composition of two *k*-cyclic collineations.

Keywords: collineation, cyclic collneation, composition

The problem of decomposing of a linear transformation into some special transformations was investigated in many papers ([5], [6] for instance). This is also interesting what is the minimal number of such factors required. This task is known as the length problem.

The well known property of the projective geometry says that any projectivity on a projective line P_1 is a composition of two involutions. A generalisation of this fact was given in [2]. Namely, the following theorem was proved: Let $P_1(\mathbf{F})$ be one-dimensional projective space over an algebraically closed field **F** of characteristic 0, let *k* be an arbitrary integer not less than 2, and let *f* be a projective transformation of $P_1(\mathbf{F})$ onto itself. Then there exist exactly *k*-cyclic projective transformations *g*, *h* such that $f=gh$ (a transformation f: X→X is called to be exactly *k*-cyclic if f^k=id, and f^m≠id for $m<k$). In the case of the real projective line P_1(R) the following property [3] holds: Let *f* be a nonsingular projectivity in $P_1(\mathbf{R})$. If detF>0, then for every $2\le k\neq 3$, there exist exactly *k*-cyclic projectivities *g*, *h* such that $f=gh$. If however detF<0, then for every $k\geq 2$ there exist an exactly *k*-cyclic projectivity *g* and an involution *h* such that $f=gh$ (F denotes the matrix of *f*).

In the case of $P_2(\mathbf{R})$ an analogous property [7] was formulated. Namely, the following theorem was proved: If f is a nonsingular projective collineation in $P_2(\mathbf{R})$ and k is an arbitrary integer not less than 3, then f is a composition of two exactly *k*-cyclic collineations.

In this paper we shall deal with the complex projective space $P_3(\mathbf{C})$. E. W. Ellers [6] investigated a much more general situation. Namely, he considered a projective space of an arbitrary dimension (even infinite) over an arbitrary field (not necessarily commutative). However, in the particular case of $P_3(\mathbf{C})$, as it often occurs, the results from [6] can be improved.

First of all, notice that there are 14 types of non-singular projective collineations in $P_3(\mathbf{C})$. They can be described [1], with the help of the Segre's symbols, as follows: [1,1,1,1], [(1,1),1,1], [(1,1),(1,1)], [(1,1,1),1], [(1,1,1,1)], [1,1,2], [(1,2),1], [(1,1),2], [(1,1,2)], [2,2], [(2,2)], [1,3], [(1,3)], [4].

Notice that if a collineation *f* is of type [1,1,1,1], then its matrix, in an allowable coordinate system, is of the form:

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$A_1=\begin{bmatrix} a^2 & 0 & 0 & 0 \\ 0 & b^2 & 0 & 0 \\ 0 & 0 & c^2 & 0 \\ 0 & 0 & 0 & d^2 \end{bmatrix}$. Similarly, for the remaining types of collineations the respective consecutive matrices have the form: $A_2=\begin{bmatrix} a^2 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & b^2 & 0 \\ 0 & 0 & 0 & c^2 \end{bmatrix}$,

$A_3=\begin{bmatrix} a^2 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & b^2 & 0 \\ 0 & 0 & 0 & b^2 \end{bmatrix}$, $A_4=\begin{bmatrix} a^2 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & b^2 \end{bmatrix}$, $A_5=\begin{bmatrix} a^2 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & a^2 \end{bmatrix}$,

$A_6=\begin{bmatrix} a^2 & 0 & 0 & 0 \\ 0 & b^2 & 0 & 0 \\ 0 & 0 & c^2 & 0 \\ 0 & 0 & 0 & c^2 \end{bmatrix}$, $A_7=\begin{bmatrix} a^2 & 1 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & b^2 \end{bmatrix}$, $A_8=\begin{bmatrix} a^2 & 1 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & b^2 & 0 \\ 0 & 0 & 0 & b^2 \end{bmatrix}$,

$A_9=\begin{bmatrix} a^2 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & a^2 & 1 \\ 0 & 0 & 0 & a^2 \end{bmatrix}$, $A_{10}=\begin{bmatrix} a^2 & 1 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & b^2 & 1 \\ 0 & 0 & 0 & b^2 \end{bmatrix}$, $A_{11}=\begin{bmatrix} a^2 & 1 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & a^2 & 1 \\ 0 & 0 & 0 & a^2 \end{bmatrix}$,

$A_{12}=\begin{bmatrix} a^2 & 1 & 0 & 0 \\ 0 & a^2 & 1 & 0 \\ 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & b^2 \end{bmatrix}$, $A_{13}=\begin{bmatrix} a^2 & 1 & 0 & 0 \\ 0 & a^2 & 1 & 0 \\ 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & a^2 \end{bmatrix}$, $A_{14}=\begin{bmatrix} a^2 & 1 & 0 & 0 \\ 0 & a^2 & 1 & 0 \\ 0 & 0 & a^2 & 1 \\ 0 & 0 & 0 & a^2 \end{bmatrix}$.

Take into account matrices:

$B_1=\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & \dfrac{Bc(bd-ac)}{b(d^2-c^2)} \\ 1 & \dfrac{Bc(bd-ac)}{b(d^2-c^2)} & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, where $B=2\cos\dfrac{4\pi}{n}$,

$B_2=\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & \dfrac{Bb}{b+c} \\ 1 & \dfrac{Bc}{b+c} & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, $B_3=\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & \overline{A} \\ 1 & A & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, where $A=\cos\dfrac{4\pi}{n}+i\sin\dfrac{4\pi}{n}$,

$$B_4=\begin{bmatrix}0&0&0&-1\\0&0&1&\frac{Ba}{a+b}\\1&\frac{Bb}{a+b}&0&0\\0&1&0&0\end{bmatrix},\ B_5=B_6=\begin{bmatrix}0&0&0&\frac{-b}{Bc^2(a-b)}\\0&0&\frac{c^2(a-b)}{b}&1\\1&0&0&0\\0&B&0&0\end{bmatrix},$$

$$B_7=\begin{bmatrix}0&0&0&\frac{-1}{Bb(a-b)}\\0&0&1&B\\Bb(a-b)&0&0&0\\0&1&0&0\end{bmatrix},\ B_8=B_9=\begin{bmatrix}0&0&A&0\\0&0&0&\overline{A}\\1&0&0&0\\0&1&0&0\end{bmatrix},$$

$$B_{10}=\begin{bmatrix}0&0&0&\frac{1}{a^2b^2}\\0&0&1&0\\a^2&B&0&0\\0&-b^2&0&0\end{bmatrix},\ B_{11}=\begin{bmatrix}0&0&0&\frac{1}{a^4}\\0&0&1&0\\a^2&B&0&0\\0&-a^2&0&0\end{bmatrix},$$

$$B_{12}=\begin{bmatrix}0&0&B-\frac{Aa}{b}&1\\0&0&\frac{ab}{A}-Ba^2+\frac{a^3A}{b}&-a^2\\1&0&0&0\\0&-\frac{A}{ab}&0&0\end{bmatrix},\ B_{13}=\begin{bmatrix}0&0&A&-\frac{\overline{A}}{a^2}\\0&0&0&\overline{A}\\1&0&0&0\\0&1&0&0\end{bmatrix},$$

$$B_{14}=\begin{bmatrix}0&0&1&0\\0&0&0&-1\\A&0&0&0\\0&-\overline{A}&0&0\end{bmatrix}.$$

It is easy to see that the polynomial x^4-Bx^2+1 is the characteristic polynomial of each of matrices B_i (i=1,2…,14). Similarly, the polynomial $x^4-B\sqrt{\det A_i}\,x^2+\det A_i$ is the characteristic polynomial of the matrix B_iA_i, for all i. In both cases the roots of such a polynomial are all distinct and are roots of even degree of a polynomial of the form: x^n-k.

According to Theorem I [1] p. 353, all the matrices B_iA_i, B_i represent exactly n-cyclic collineations. Then the following property holds:

For a non-singular projective collineation f in $P_3(\mathbf{C})$ and an arbitrary even integer n not less than 6, there exist f two exactly n-cyclic collineations such that f is a composition of them.

If n=4, the reader will find matrices B'_i having analogous properties as matrices B_i. In such a case the respective characteristic polynomial is x^4+1 instead of x^4-Bx^2+1. Hence we can formulate:

Theorem

If f is a non-singular projective collineation in $P_3(\mathbf{C})$ and n is an arbitrary even integer not less than 4 then f is a composition of two exactly n-cyclic collineations.

As we see the length problem was solved in an optimal way.

Conjecture

The author is convinced that for odd integers an analogous theorem is true.

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KOLINEACJE RZUTOWE JAKO ZŁOŻENIA DWÓCH KOLINEACJI CYKLICZNYCH

W pracy pokazano, że każda kolineacja rzutowa trójwymiarowej zespolonej przestrzeni rzutowej jest złożeniem dwóch kolineacji *n*-cyklicznych. Przy tym ma to miejsce dla dowolnej parzystej liczby naturalnej nie mniejszej niż 4.