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A NUMERICAL SOLUTION FOR A CLASS OF TIME FRACTIONAL DIFFUSION EQUATIONS WITH DELAY

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This paper describes a numerical scheme for a class of fractional diffusion equations with fixed time delay. The study focuses on the uniqueness, convergence and stability of the resulting numerical solution by means of the discrete energy method. The derivation of a linearized difference scheme with convergence order $O(\tau^{2-\alpha} + h^4)$ in L_∞-norm is the main purpose of this study. Numerical experiments are carried out to support the obtained theoretical results.

Keywords: fractional diffusion equation with delay, difference scheme, convergence analysis.

1. Introduction

Recently, significantly increased attention regarding partial differential equations which contain fractional derivatives (FPDEs) and integrals has been observed. Due to their ability to model some phenomena more efficiently than partial differential equations with integer derivatives, FPDEs are utilized in many areas of science. Nowadays, the interest of scientists in FPDEs in fields of science and engineering involves anomalous diffusion mechanisms, such as fluid flow in porous materials (Benson *et al.*, 2001), underground environmental problems (Hatano and Hatano, 1998), anomalous transport in biology (Höfling and Franosch, 2013), finance (Raberto *et al.*, 2002; Scalas *et al.*, 2000), viscoelasticity (Bagley and Torvik, 1983), etc., and many other scientific areas. Time delay has been considered in numerous mathematical models, e.g., physiological systems (Batzel and Kappel, 2011), population dynamics (Liu, 2015; Tumwiine *et al.*, 2008) and HIV-infection modeling (Culshaw *et al.*, 2003; Yan and Kou, 2012). Relative controllability and relative constrained controllability of linear fractional systems with delays in the state were discussed by Sikora (2016).

Sufficient conditions for the controllability of linear and nonlinear fractional dynamical systems in finite dimensional spaces were obtained by Balachandran and Kokila (2012). The authors used Schauder fixed point theorem and the controllability Grammian matrix defined by the Mittag-Leffler matrix function. Some theoretical analysis of fractional differential equations with time delay was introduced by Lakshmikantham (2008). Alternative results concerning the existence and attractivity dependence of solutions for a class of non-linear fractional functional differential equations were presented by Chen and Zhou (2011). Some numerical solutions for time delay differential equations were proposed in the literature by means of finite difference methods and others (Bellen and Zennaro, 2003; Jackiewicz *et al.*, 2014; Rihan, 2009; Solodushkin *et al.*, 2017).

Ferreira (2008) studied energy estimates for delay diffusion-reaction. A backward Euler scheme with L_2-convergence order $O\left(\tau + h^2\right)$ was proposed. Zhang

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and Sun (2013) introduced a linearized compact difference scheme for a class of nonlinear delay partial differential equations with initial and Dirichlet boundary conditions. Karatay *et al*. (2013) predicted an approximation for the time Caputo fractional derivative at time $t_{k+1/2}$ with fractional order $0 < \alpha < 1$. They extended the idea of the Cranck–Nicholson method to time fractional heat equations with convergence order $O\left(\tau^{2-\alpha}+h^2\right)$. Some numerical contributions in fractional functional differential equations with delay based on BDF-type shifted Chebyshev polynomials were discussed by Pimenov and Hendy (2015). A numerical solution of a heat conduction equation with delay for the case of a variable coefficient of heat conductivity was proposed by Lekomtsev and Pimenov (2015).

The time fractional reaction diffusion wave equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = K\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t,u(x,t)) \tag{1}$$

has appeared in a broad variety of engineering, biological and physics processes where anomalous diffusion occurs (Wyss, 1986; Schneider and Wyss, 1989), such as those in sub-diffusive or super-diffusive processes.

When $0 < \alpha < 1$, Eqn. (1) is a time-fractional diffusion equation, while if $1 < \alpha < 2$, it is a time fractional wave-diffusion equation. In the case where $\alpha = 1$, we obtain the classical diffusion equation, and when $\alpha = 2$, we obtain the classical wave equation.

Some numerical methods are introduced in the literature for different forms of (1) (Meerschaert and Tadjeran, 2004; Ren and Sun, 2015). Recently, we proposed a difference method for class of non-linear delay distributed order fractional diffusion equations (Pimenov *et al*., 2017). In this approach, a theoretical analysis of the proposed linear difference scheme is made. Also, a finite difference scheme for semi-linear space-fractional diffusion equations with time delay is given by Hao *et al*. (2016).

Based on the ideas of Zhang and Sun (2013) and Karatay *et al*. (2013), we are interested in constructing a linearized difference scheme for (1) which is induced with fixed time delay in the source function as in the simulation of dynamical systems. Specifically, we consider

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = K\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t,u(x,t),u(x,t-s)), \tag{2a}$$

with the following initial and boundary conditions:

$$u(x,t) = \psi(x,t), \quad 0 \le x \le L, \quad t \in [-s,0), \tag{2b}$$

$$u(0,t) = \phi_0(t), \quad u(L,t) = \phi_L(t), \quad t > 0, \tag{2c}$$

where $s > 0$ is the delay parameter and K is a positive constant. The fractional derivative is introduced in the Caputo sense (Miller and Ross, 1993), that is,

$$\begin{aligned} {}_0^C D_t^\alpha u(x,t) &\equiv \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} \\ &:= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\zeta)^{-\alpha}\frac{\partial u(x,\zeta)}{\partial \zeta}\,\mathrm{d}\zeta, \\ &\qquad 0<\alpha<1. \end{aligned} \tag{3}$$

In this paper, we propose a high-order linearized difference scheme for the time fractional diffusion equation with delay. The degree of complexity is how to approximate the time fractional derivative and the non-linear delay source function. Throughout this work, like Zhang and Sun (2013), we suppose that the function $f(x,t,\mu,\nu)$ and the solution $u(x,t)$ of (2) are sufficiently smooth in the following sense:

- Let m be an integer satisfying $ms \le T < (m+1)s$. Define $I_r = (rs,(r+1)s)$, $r = -1,0,\dots,m-1$, $I_m = (ms,T)$, $I = \bigcup_{q=-1}^m I_q$, and assume that $u(x,t) \in C^{(6,2)}([0,L]\times(0,T])$.
- The partial derivatives $f_\mu(x,t,\mu,\nu)$ and $f_\nu(x,t,\mu,\nu)$ are continuous in the ϵ_0-neighborhood of the solution. Define

$$c_1 = \sup_{\substack{0<x<L,\,0<t\le T\\ |\epsilon_1|\le\epsilon_0,|\epsilon_2|\le\epsilon_0}} |f_\mu + \epsilon_1,\ u(x,t-s)+\epsilon_2)|,$$

$$c_2 = \max_{\substack{0<x<L,\,0<t\le T\\ |\epsilon_1|\le\epsilon_0,|\epsilon_2|\le\epsilon_0}} |f_\nu + \epsilon_1,\ u(x,t-s)+\epsilon_2)|.$$

The rest of this paper is arranged in the following way. We present the derivation of the difference scheme in the following section. Next, in Section 3, the solvability, convergence and stability for the difference scheme are discussed. In Section 4, numerical examples are given to illustrate the accuracy of the presented scheme and to support our theoretical results. Finally, the paper ends with conclusion and some remarks.

2. Derivation of the difference scheme

We aim to obtaining a numerical solution based on the Crank–Nicholson method. We need some notation. Take two positive integers M and n, let $h = L/M$, $\tau = s/n$ and write $x_i = i\,h$, $t_k = k\,\tau$ and $t_{k+1/2} = \left(k+\frac{1}{2}\right)\tau = \frac{1}{2}(t_k + t_{k+1})$. Cover the space-time domain by $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$, where $\Omega_h = \{x_i | 0 \le i \le M\}$, $\Omega_\tau = \{t_k | -n \le k \le N\}$, $N = \lfloor T/\tau \rfloor$. Let $\mathcal{W} = \{\nu | \nu = v_i^k, 0 \le i \le M, -n \le k \le N\}$ be a grid function space on $\Omega_{h\tau}$. For $\nu \in \mathcal{W}$, we write $v_i^{k+1/2} = \frac{1}{2}\left(v_i^k + v_i^{k+1}\right)$ and $\delta_x^2 v_i^k = \left(v_{i+1}^k - 2v_i^k + v_{i-1}^k\right)/h^2$.

Lemma 1. (Zhang and Sun, 2013) *Let* $q(x) \in C^6[x_{i-1}, x_{i+1}]$. *Then*

$$\frac{1}{12}\left(q''(x_{i-1}) + 10q''(x_i) + q''(x_{i+1})\right) - \frac{1}{h^2}\left(q(x_{i-1}) - 2q(x_i) + q(x_{i+1})\right) = \frac{h^4}{240} q^{(6)}(\omega_i),$$

where $\omega_i \in (x_{i-1}, x_{i+1})$.

We define the grid function on $\Omega_{h\tau}$: $U(i,k) = u(x_i, t_k)$. In the work of Karatay *et al*. (2013), an approximation to the time Caputo fractional derivative at $t_{k+1/2}$ with $0 < \alpha_l < 1$ was given as

$$\begin{aligned}\frac{\partial^\alpha u(x_i, t_{k+1/2})}{\partial t^\alpha} &= \omega_1 U_i^k + \sum_{m=1}^{k-1}(\omega_{k-m+1} - \omega_{k-m}) u_i^m \\ &\quad - \omega_k U_i^0 + \frac{\sigma}{2^{1-\alpha}}\left(U_i^{k+1} - U_i^k\right) + O\left(\tau^{2-\alpha}\right),\end{aligned} \tag{4}$$

where

$$\omega_i = \sigma\left(\left(i + \frac{1}{2}\right)^{1-\alpha} - \left(i - \frac{1}{2}\right)^{1-\alpha}\right), \tag{5}$$

$$\sigma = \frac{1}{\tau^\alpha \Gamma(2-\alpha)}, \quad 0 \le i \le M, \quad 0 \le k \le N-1. \tag{6}$$

We are now in a position to apply (4) to (2a) at the points $(x_i, t_{k+1/2})$, and arrive at

$$\begin{aligned}&\left[\omega_1 U_i^k + \sum_{m=1}^{k-1}(\omega_{k-m+1} - \omega_{k-m}) U_i^m - \omega_k U_i^0 + \frac{\sigma}{2^{1-\alpha}}\left(U_i^{k+1} - U_i^k\right) + O\left(\tau^{2-\alpha}\right)\right] + O\left(\Delta\alpha\right)^4 \\ &= K \frac{\partial^2 u(x_i, t_{k+1/2})}{\partial x^2} + f(x_i, t_{k+1/2}, u(x_i, t_{k+1/2}), u(x_i, t_{k+1/2} - s)).\end{aligned} \tag{7}$$

Lemma 2. *For* $g = (g_0, g_1, \ldots, g_M)$, *let the linear operator* $\mathfrak{A}$ *be defined as*

$$\mathfrak{A} g_i = \frac{1}{12}(g_{i-1} + 10 g_i + g_{i+1}), \quad 1 \le i \le M-1.$$

Then we have

$$\begin{aligned}&\mathfrak{A}\left[\omega_1 U_i^k + \sum_{m=1}^{k-1}(\omega_{k-m+1} - \omega_{k-m}) U_i^m - \omega_k U_i^0 + \frac{\sigma}{2^{1-\alpha}}\left(U_i^{k+1} - U_i^k\right)\right] \\ &= K\delta_x^2 U_i^{k+1/2} + \mathfrak{A} f\left(x_i, t_{k+1/2}, \frac{3}{2}U_i^k - \frac{1}{2}U_i^{k-1}, \frac{1}{2}U_i^{k+1-n} + \frac{1}{2}U_i^{k-n}\right) + R_i^k,\end{aligned} \tag{8}$$

where

$$\left|R_i^k\right| = O\left(h^4 + \tau^{2-\alpha}\right), \tag{9}$$

$$1 \le i \le M-1, \quad 0 \le k \le N-1.$$

Proof. We use Taylor expansions

$$\frac{\partial^2 u(x_i, t_{k+1/2})}{\partial x^2} = \left(\frac{\partial^2 u(x_i, t_k)}{\partial x^2} + \frac{\partial^2 u(x_i, t_{k+1})}{\partial x^2}\right) + O\left(\tau^2\right),$$

$$u(x_i, t_{k+1/2}) = U_i^{k+1/2} = \frac{3}{2}U_i^k - \frac{1}{2}U_i^{k-1} + O\left(\tau^2\right),$$

$$u(x_i, t_{k+1/2} - s) = U_i^{k-n+\frac{1}{2}} = \frac{1}{2}U_i^{k+1-n} + \frac{1}{2}U_i^{k-n} + O\left(\tau^2\right),$$

in (7) and obtain

$$\begin{aligned}&\left[\omega_1 U_i^k + \sum_{m=1}^{k-1}(\omega_{k-m+1} - \omega_{k-m}) U_i^m - \omega_k U_i^0 + \frac{\sigma}{2^{1-\alpha}}\left(U_i^{k+1} - U_i^k\right) + O\left(\tau^{2-\alpha}\right)\right] \\ &= \frac{K}{2}\left(\frac{\partial^2 u(x_i, t_k)}{\partial x^2} + \frac{\partial^2 u(x_i, t_{k+1})}{\partial x^2}\right) + O\left(\tau^2\right) \\ &\quad + f\left(x_i, t_{k+1/2}, \frac{3}{2}U_i^k - \frac{1}{2}U_i^{k-1}, \frac{1}{2}U_i^{k+1-n} + \frac{1}{2}U_i^{k-n}\right),\end{aligned}$$

where we use the continuity of the derivatives of f in its third and fourth components. We rewrite this as

$$\begin{aligned}&\Bigg[\omega_1 U_i^k + \sum_{m=1}^{k-1}(\omega_{k-m+1}-\omega_{k-m})U_i^m - \omega_k U_i^0\\&+\frac{\sigma}{2^{1-\alpha}}\left(U_i^{k+1}-U_i^k\right)\Bigg] + O(\tau^{2-\alpha})\\&=\frac{K}{2}\left(\frac{\partial^2 u(x_i,t_k)}{\partial x^2}+\frac{\partial^2 u(x_i,t_{k+1})}{\partial x^2}\right)\\&\quad+f\bigg(x_i,t_{k+1/2},\frac{3}{2}U_i^k-\frac{1}{2}U_i^{k-1},\\&\quad\frac{1}{2}U_i^{k+1-n}+\frac{1}{2}U_i^{k-n}\bigg)+O\left(\tau^2\right),\end{aligned}\tag{10}$$

According to Lemma 1 we have

$$\mathfrak{A}\frac{\partial^2 u(x_i,t_k)}{\partial x^2} = \delta_x^2 U_i^k + \frac{h^4}{240}\frac{\partial^6 u}{\partial x^6}(\theta_i^k,t_k),\quad \theta_i^k\in(x_{i-1},x_{i+1}).$$

Thus, applying $\mathfrak{A}$ to (10), we arrive at

$$\begin{aligned}&\mathfrak{A}\Bigg[\omega_1 U_i^k + \sum_{m=1}^{k-1}(\omega_{k-m+1}-\omega_{k-m})u_i^m - \omega_k U_i^0\\&+\frac{\sigma}{2^{1-\alpha}}\left(U_i^{k+1}-U_i^k\right)\Bigg]\\&=K\delta_x^2 U_i^{k+1/2}\\&\quad+\mathfrak{A}f\bigg(x_i,t_{k+1/2},\frac{3}{2}U_i^k-\frac{1}{2}U_i^{k-1},\\&\quad\frac{1}{2}U_i^{k+1-n}+\frac{1}{2}U_i^{k-n}\bigg)+O\left(\tau^{2-\alpha}+h^4\right)\end{aligned}$$

as $u(x,t)\in C^{(6,2)}(I\times(0,T])$. ∎

The final form of our difference scheme is obtained by neglecting R_i^k and replacing U_i^k with u_i^k in (8):

$$\begin{aligned}&\mathfrak{A}\Bigg[\omega_1 u_i^k + \sum_{m=1}^{k-1}(\omega_{k-m+1}-\omega_{k-m})u_i^m - \omega_k u_i^0\\&+\frac{\sigma^l}{2^{1-\alpha_l}}\left(u_i^{k+1}-u_i^k\right)\Bigg]\\&=K\delta_x^2 u_i^{k+1/2}+\mathfrak{A}f\bigg(x_i,t_{k+1/2},\frac{3}{2}u_i^k-\frac{1}{2}u_i^{k-1},\\&\quad\frac{1}{2}u_i^{k+1-n}+\frac{1}{2}u_i^{k-n}\bigg),\end{aligned}\tag{11a}$$

and supplying appropriate initial and boundary conditions:

$$u_0^k=\phi_0(t_k),\quad u_M^k=\phi_L(t_k),\quad 1\le k\le N,\tag{11b}$$

$$u_i^k=\psi(x_i,t_k),\quad 0\le i\le M,\quad -n\le k\le 0.\tag{11c}$$

3. Analysis of the difference scheme

Before introducing the uniqueness, convergence and stability theorems in L_∞ norm for the proposed difference scheme using a discrete energy method, we introduce some notation.

If the spatial domain $[0,L]$ is covered by $\Omega_h=\{x_i|0\le i\le M,\}$, let $V_h=\{v|v=(v_0,\dots,v_M),\quad v_0=v_M=0\}$ be a grid function space on Ω_h. For any $u,v\in V_h$, introduce the discrete inner products and corresponding norms as

$$\begin{aligned}\langle u,v\rangle &= -h\sum_{i=1}^{M-1}(\mathfrak{A}u_i)(\delta_x^2 v_i)\\&=h\sum_{i=1}^{M-1}(\delta_x u_{i+1/2})(\delta_x v_{i+1/2})\\&\quad-\frac{h^2}{12}h\sum_{i=1}^{M-1}(\delta_x^2 u_i)(\delta_x^2 v_i),\end{aligned}$$

$$|u|_1^2=h\sum_{i=1}^{M}(\delta_x u_{i-1/2})^2,$$

$$\|u\|^2=h\sum_{i=1}^{M-1}(u_i)^2,\quad \|u\|_\infty=\max_{1\le i\le M-1}|u_i|.$$

According to Samarskii and Andreev (1976) or Zhang and Sun (2013), for any $u\in V_h$ the following inequalities are fulfilled:

$$\frac{2}{3}|u|_1^2\le\langle u,u\rangle\le|u|_1^2,$$

$$\|u\|_\infty\le\frac{\sqrt{L}}{2}|u|_1,\quad \|u\|^2\le\frac{L}{6}|u|_1^2.\tag{12}$$

It is directly observed from (12) that

$$\|u\|^2\le\frac{L^2}{4}\langle u,u\rangle,\quad \|u\|_\infty^2\le\frac{3L}{8}\langle u,u\rangle.\tag{13}$$

Lemma 3. *For any $v\in V_h$, we have $\|\mathfrak{A}v\|^2\le\|v\|^2$.*

Lemma 4. *For any $u,v\in V_h$, we have*

$$-h\sum_{i=1}^{M-1}(\delta_x^2 u_i)v_i = h\sum_{i=1}^{M}(\delta_x u_{i-1/2})(\delta_x v_{i-1/2}).$$

For the ease of further analysis, Eqn. (4) can be rewritten as

$$\begin{aligned}&\frac{\partial^\alpha u(x_i,t_{k+1/2})}{\partial t^\alpha}\\&=\frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\left[a_{k-m+1}^\alpha u_{t,m-1}+a_0^\alpha u_{t,k}\right]\\&\quad+O\left(\tau^{2-\alpha}\right),\end{aligned}\tag{14}$$

such that

$$a_0^\alpha=\left(\frac{1}{2}\right)^{1-\alpha},$$

$$a_l^\alpha = (l+1/2)^{1-\alpha} - (l-1/2)^{1-\alpha}, \quad l \geq 1.$$

Then

$$\frac{\partial^\alpha u(x_i, t_{k+1/2})}{\partial t^\alpha} = \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{m=0}^{k} C_{k-m}^{(k+1)} u_{t,m}, \quad (15)$$

such that

$$u_{t,m} = \frac{u_{m+1} - u_m}{\tau},$$

where $c_0^{(k+1)} = a_0^\alpha$ for $j = 0$ and for $j \geq 1$ we have

$$C_m^{(k+1)} = \begin{cases} a_0^\alpha, & m = 0, \\ a_m^\alpha, & 1 \leq m \leq k-1, \\ a_k^\alpha, & m = k. \end{cases}$$

Then at $0 < \alpha \leq 1$ and for $u(x,t) \in C^2[0,T]$, we have

$$\begin{aligned} &\frac{\partial^\alpha u(x_i, t_{k+1/2})}{\partial t^\alpha} \\ &= \sum_{n=0}^{k} g_n^{(k+1)} \Big[u(x_i, t_{n+1}) - u(x_i, t_n) \Big] + O\left(\tau^{2-\alpha}\right) \\ &:= \Delta_{t_{k+1/2}}^\alpha u + O\left(\tau^{2-\alpha}\right), \end{aligned} \quad (16)$$

such that

$$g_n^{(k+1)} = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} C_{k-n}^{(k+1)}.$$

Lemma 5. *For any* $0 < \alpha < 1$, $C_m^{(k+1)} (0 \leq m \leq k, k \geq 1)$*, and if* $3^\alpha \geq 3/2$*, we have*

$$C_k^{k+1} > \frac{1-\alpha}{2}(k+1/2)^{-\alpha}, \quad (17)$$

$$C_0^{(k+1)} > C_1^{(k+1)} > \cdots > C_{k-1}^{(k+1)} > C_k^{k+1}. \quad (18)$$

Proof. For $k \geq 1$, we get

$$\begin{aligned} C_k^{(k+1)} &= (1/2)^{1-\alpha} \Big[(2k+1)^{1-\alpha} - (2k-1)^{1-\alpha} \Big] \\ &> \frac{1-\alpha}{2} \left(\frac{1}{2}\right)^{-\alpha} \int_0^1 \frac{\mathrm{d}\eta}{(2k+1-\eta)^\alpha} \\ &> \frac{1-\alpha}{2} \left(\frac{1}{2}\right)^{-\alpha} (2k+1)^{-\alpha} \\ &> \frac{1-\alpha}{2} \left(k + \frac{1}{2}\right)^{-\alpha}. \end{aligned}$$

Moreover

$$C_0^{(k+1)} = (1/2)^{1-\alpha} > \frac{1-\alpha}{2} \left(\frac{1}{2}\right)^{-\alpha},$$

so that (17) is achieved. Observe that $C_1^{(k+1)} > \cdots > C_{k-1}^{(k+1)} > C_k^{k+1}$, because we have $a_l^\alpha > a_{l+1}^\alpha, \quad l \geq 1$. Accordingly, the inequality (18) is achieved if $a_0^\alpha \geq a_1^\alpha$, which is equivalent to $3^\alpha \geq 3/2$. ∎

Lemma 6. *From Lemma 5 it follows that*

$$g_k^{(k+1)} > g_{k-1}^{(k+1)} > \cdots > g_1^{(k+1)} > g_0^{(k+1)}$$

and

$$\begin{aligned} g_0^{(k+1)} &= \frac{\tau^{-\alpha} C_k^{(k+1)}}{\Gamma(2-\alpha)} \\ &\geq \frac{\frac{1-\alpha}{2}(k+1/2)^{-\alpha}}{\tau^\alpha \Gamma(2-\alpha)} \\ &\geq \frac{1-\alpha}{2T^\alpha \Gamma(2-\alpha)} = k_0. \end{aligned}$$

Lemma 7. (Alikhanov, 2015) *If*

$$\{g_k^{(k+1)} > g_{k-1}^{(k+1)} > \cdots > g_0^{(k+1)} > 0, \quad k = 0, 1, \ldots, M-1\},$$

then for any function $\nu(t)$ *defined on the mesh* $\{t_k : t_0 < t_1 < t_2 < \ldots t_{M-1} < t_M = T\}$*, we have*

$$\nu^{k+1} \Delta_{t_{k+1/2}}^\alpha \nu \geq \frac{1}{2} \Delta_{t_{k+1/2}}^\alpha (\nu)^2 + \frac{1}{2g_k^{(k+1)}} (\Delta_{t_{k+1/2}}^\alpha \nu)^2,$$

$$\nu^k \Delta_{t_{k+1/2}}^\alpha \nu \geq \frac{1}{2} \Delta_{t_{k+1/2}}^\alpha (\nu)^2 - \frac{1}{2(g_k^{(k+1)} - g_{k-1}^{(k+1)})} (\Delta_{t_{k+1/2}}^\alpha \nu)^2.$$

Based on Lemma 7, we can deduce the following direct result.

Lemma 8. *If*

$$\{g_k^{(k+1)} > g_{k-1}^{(k+1)} > \cdots > g_0^{(k+1)} > 0, \quad k = 0, 1, \ldots, M-1\},$$

then for any function $\nu(t)$ *defined on the mesh* $\{t_k : t_0 < t_1 < t_2 < \ldots t_{M-1} < t_M = T\}$ *we have the following inequality:*

$$\left(\frac{1}{2}\nu^{k+1} + \frac{1}{2}\nu^k\right) \Delta_{t_{k+1/2}}^\alpha \nu \geq \frac{1}{2} \Delta_{t_{k+1/2}}^\alpha (\nu)^2.$$

Lemma 9. (Special Gronwall inequality) (Holte, 2009; Kruse and Scheutzow, 2016) *Let* z_k *and* g_k *be non-negative sequences and such that* K *is a non-negative constant. If*

$$z_k \leq K \sum_{0 \leq i < k} g_i z_i, \quad k \geq 0,$$

then

$$z_k \leq K \exp\Big(\sum_{0 \leq j < k} g_j \Big), \quad k \geq 0.$$

We start to prove that our difference scheme admits a unique solution. Next we show that the obtained solution solves (2).

Theorem 1. *The difference scheme* (11) *is uniquely solvable.*

Proof. Suppose that $u_i^k, \quad 0 \le i \le M$, is the solution for the obtained difference scheme (11). Using the mathematical induction, the base step is fulfilled from the initial condition (11c) as the solution u_i^k is determined for $-n \le k \le 0$. For the inductive hypothesis, let u_i^k be determined when $k = l$; then from (11a) we obtain a system of linear algebraic equations with respect to u_i^l. The proof ends by the inductive step as the coefficient matrix of this system is strictly diagonally dominant, so there exists a unique solution u_i^{l+1}.

We can arrange the system (11) as follows:

$$\begin{aligned}
&\Bigg(\Big[\frac{\sigma}{2^{1-\alpha}} - \frac{K}{2h^2}\Big]u_{i+1}^{k+1} + \Big[\frac{10}{12}\frac{\sigma}{2^{1-\alpha}} + \frac{K}{h^2}\Big]u_i^{k+1} \\
&+ \Big[\frac{1}{12}\frac{\sigma}{2^{1-\alpha}} - \frac{K}{2h^2}\Big]u_{i-1}^{k+1}\Bigg) \\
&+ \Bigg(\Big[\frac{1}{12}(\omega_1 - \frac{\sigma}{2^{1-\alpha}}) - \frac{K}{2h^2}\Big]u_{i+1}^{k} \\
&+ \Big[\frac{10}{12}(\omega_1 - \frac{\sigma}{2^{1-\alpha}}) + \frac{K}{h^2}\Big]u_i^{k} \\
&+ \Big[\frac{1}{12}(\omega_1 - \frac{\sigma}{2^{1-\alpha}}) - \frac{K}{2h^2}\Big]u_{i-1}^{k}\Bigg) \\
&+ \mathfrak{A}\Bigg(\sum_{m=1}^{k-1}(\omega_{k-m+1} - \omega_{k-m})u_i^m - \omega_k u_i^0\Bigg) \\
&\quad = \mathfrak{A}f\Bigg(x_i, t_{k+1/2}, \frac{3}{2}u_i^k - \frac{1}{2}u_i^{k-1}, \\
&\qquad \frac{1}{2}u_i^{k+1-n} + \frac{1}{2}u_i^{k-n}\Bigg).
\end{aligned}$$

According to the system above, the coefficient matrix $\mathsf{A} = (a_{ij})$ is strictly diagonally dominant because

$$|a_{ii}| \ge \sum_{j\ne i}|a_{ij}|,$$

$$\begin{aligned}
a_{ii} &= \frac{10}{12}\frac{\sigma}{2^{1-\alpha}} + \frac{K}{h^2}, \\
a_{i+1,i} &= \frac{1}{12}\frac{\sigma}{2^{1-\alpha}} - \frac{K}{2h^2} \\
&= a_{i-1,i}, \quad \frac{\sigma^l}{2^{1-\alpha}} > 0.
\end{aligned}$$

Therefore, the coefficient matrix is nonsingular and this proves the theorem. ∎

Theorem 2. (Convergence theorem) *Let* $u(x,t) \in C^{6,2}([0,L] \times (-s,T])$ *be the solution of* (2) *such that* $u(x_i,t_k) = U_i^k$ *and* u_i^k $(0 \le i \le M, -n \le k \le N)$ *is the solution of the difference scheme* (11). *Write* $e_i^k = U_i^k - u_i^k$ *for* $0 \le i \le M$, $-n \le k \le N$. *Then if*

$$\tau \le \tau_0 = \left(\frac{\epsilon_0}{4C}\right)^{\frac{1}{2-\alpha}}, \quad h \le h_0 = \left(\frac{\epsilon_0}{4C}\right)^{\frac{1}{4}}, \tag{19}$$

we have

$$\|e^k\|_\infty \le \bar{C}\left(\tau^{2-\alpha} + h^4\right), \quad 0 \le k \le N, \tag{20}$$

where $\bar{C}$ *is a positive constant independent of* h *and* τ.

Proof. The difference scheme in (8) and (11a) can be rewritten in terms of (16) as follows:

$$\begin{aligned}
&\mathfrak{A}\Bigg[\sum_{n=0}^{k} g_n^{(k+1)}\Big(U_i^{n+1} - U_i^n\Big)\Bigg] \\
&\quad = K\delta_x^2 U_i^{k+1/2} + \mathfrak{A}f\big(x_i, t_{k+1/2}, \frac{3}{2}U_i^k - \frac{1}{2}U_i^{k-1}, \\
&\qquad \frac{1}{2}U_i^{k+1-n} + \frac{1}{2}U_i^{k-n}\big) + R_i^{k+1/2},
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\mathfrak{A}\Bigg[\sum_{n=0}^{k} g_n^{(k+1)}\Big(u_i^{n+1} - u_i^n\Big)\Bigg] \\
&\quad = K\delta_x^2 u_i^{k+1/2} + \mathfrak{A}f\big(x_i, t_{k+1/2}, \frac{3}{2}u_i^k - \frac{1}{2}u_i^{k-1}, \\
&\qquad \frac{1}{2}u_i^{k+1-n} + \frac{1}{2}u_i^{k-n}\big).
\end{aligned} \tag{22}$$

The error difference scheme can be obtained by subtracting (22) from (21), the latter with u replaced by U, as follows:

$$\begin{aligned}
&\mathfrak{A}\Bigg[\sum_{n=0}^{k} g_n^{(k+1)}\Big(e_i^{n+1} - e_i^n\Big)\Bigg] \\
&\quad = K\delta_x^2 e_i^{k+1/2} + R_i^{k+1/2} + \mathfrak{A}\Bigg[f\big(x_i, t_{k+1/2}, \frac{3}{2}U_i^k \\
&\qquad - \frac{1}{2}U_i^{k-1}, \frac{1}{2}U_i^{k+1-n} + \frac{1}{2}U_i^{k-n}\big) \\
&\qquad - f\big(x_i, t_{k+1/2}, \frac{3}{2}u_i^k - \frac{1}{2}u_i^{k-1}, \\
&\qquad \frac{1}{2}u_i^{k+1-n} + \frac{1}{2}u_i^{k-n}\big)\Bigg],
\end{aligned} \tag{23}$$

and

$$e_0^k = 0, \quad e_M^k = 0, \quad 1 \le k \le N, \tag{24}$$

$$e_i^k = 0, \quad 0 \le i \le M, \quad -n \le k \le 0. \tag{25}$$

Multiplying (23) by $-h(\delta_x^2 e_i^{k+1/2})$ and summing up for i from 1 to $M-1$ yields

$$\begin{aligned}
&- h\sum_{i=1}^{M-1}\mathfrak{A}\left[\sum_{n=0}^{k} g_n^{(k+1)}\Big(e_i^{n+1}-e_i^{n}\Big)\right]\delta_x^2 e_i^{k+1/2}\\
&= -K\|\delta_x^2 e_i^{k+1/2}\|^2 - h\sum_{i=1}^{M-1}(R_i^{k+1/2})\delta_x^2 e^{k+1/2}\\
&\quad - h\sum_{i=1}^{M-1}\mathfrak{A}\Bigg[f\Big(x_i, t_{k+1/2}, \frac{3}{2}U_i^k-\frac{1}{2}U_i^{k-1},\\
&\quad \frac{1}{2}U_i^{k+1-n}+\frac{1}{2}U_i^{k-n}\Big)\\
&\quad - f\Big(x_i, t_{k+1/2}, \frac{3}{2}u_i^k-\frac{1}{2}u_i^{k-1},\\
&\quad \frac{1}{2}u_i^{k+1-n}+\frac{1}{2}u_i^{k-n}\Big)\Bigg]\delta_x^2 e_i^{k+1/2}. \qquad (26)
\end{aligned}$$

We will prove (20) by strong mathematical induction. The base case is evident: following (25), it is clear that $\|e^k\|_\infty = 0, \quad -n \le k \le 0$, so in particular we have $\|e^0\|_\infty = 0$.

Next, suppose that (20) is fulfilled for $0 \le k \le \ell$; then we will show that (20) holds for $k = \ell + 1$.

From the inductive hypothesis, and when τ and h satisfy (19), we obtain

$$\|e^k\|_\infty \le C\left(\tau^{2-\alpha}+h^4\right) \le \frac{\epsilon_0}{2}, \quad 0 \le k \le \ell. \qquad (27)$$

From (27), we conclude that $|e^k| \le \epsilon_0/2, \quad 0 \le k \le \ell$, and so $|U_i^k - u_i^k| \le \epsilon_0/2$, $|U_i^{k-1}-u_i^{k-1}| \le \epsilon_0/2, \quad 0 \le k \le \ell$. Then $|\frac{3}{2}(U_i^k - u_i^k) - \frac{1}{2}(U_i^{k-1}-u_i^{k-1})| \le \epsilon_0/2$, and the following inequality is fulfilled $|(\frac{3}{2}U_i^k - \frac{1}{2}U_i^{k-1}) - (\frac{3}{2}u_i^k - \frac{1}{2}u_i^{k-1})| \le \epsilon_0, \quad 0 \le i \le M, \quad 0 \le k \le \ell$. In the same way, we conclude that $|\frac{1}{2}(U_i^{k+1-n} - u_i^{k+1-n}) + \frac{1}{2}(U_i^{k-n} - u_i^{k-n})| \le \epsilon_0/2$. Then the following inequality is obtained: $|(\frac{1}{2}U_i^{k+1-n}+\frac{1}{2}U_i^{k-n}) - (\frac{1}{2}u_i^{k+1-n}+\frac{1}{2}u_i^{k-n})| \le \epsilon_0, \quad 0 \le i \le M, \quad 0 \le k \le \ell$. Consequently,

$$\begin{aligned}
&\Big|f\Big(x_i, t_{k+1/2}, \frac{3}{2}U_i^k-\frac{1}{2}U_i^{k-1}, \frac{1}{2}U_i^{k+1-n}+\frac{1}{2}U_i^{k-n}\Big)\\
&- f\Big(x_i, t_{k+1/2}, \frac{3}{2}u_i^k-\frac{1}{2}u_i^{k-1}, \frac{1}{2}u_i^{k+1-n}+\frac{1}{2}u_i^{k-n}\Big)\Big|\\
&\le c_1\Big|\frac{3}{2}e_i^k-\frac{1}{2}e_i^{k-1}\Big| + c_2\Big|\frac{1}{2}e_i^{k+1-n}+\frac{1}{2}e_i^{k-n}\Big|,
\end{aligned}$$

and then

$$\begin{aligned}
&\Big|\mathfrak{A}\Big[f\Big(x_i, t_{k+1/2}, \frac{3}{2}U_i^k-\frac{1}{2}U_i^{k-1}, \frac{1}{2}U_i^{k+1-n}+\frac{1}{2}U_i^{k-n}\Big)\\
&- f\Big(x_i, t_{k+1/2}, \frac{3}{2}u_i^k-\frac{1}{2}u_i^{k-1}, \frac{1}{2}u_i^{k+1-n}+\frac{1}{2}u_i^{k-n}\Big)\Big]\Big|\\
&\le \mathfrak{A}\Big(c_1\Big|\frac{3}{2}e_i^k-\frac{1}{2}e_i^{k-1}\Big| + c_2\Big|\frac{1}{2}e_i^{k+1-n}+\frac{1}{2}e_i^{k-n}\Big|\Big), \qquad (28)
\end{aligned}$$

where $0 \le i \le M, \quad 0 \le k \le \ell$.

Now, we will deal with each part of (26) individually,

$$\begin{aligned}
\eta_1 &:= -h\sum_{i=1}^{M-1}\mathfrak{A}\left[\sum_{n=0}^{k} g_n^{(k+1)}\Big(e_i^{n+1}-e_i^{n}\Big)\right]\delta_x^2 e_i^{k+1/2}\\
&= \sum_{n=0}^{k} g_n^{(k+1)}\langle e^{n+1}-e^n, e^{k+1/2}\rangle.
\end{aligned} \qquad (29)$$

Using Lemma 8 in (29), we obtain

$$\eta_1 \ge \frac{1}{2}\sum_{n=0}^{k} g_n^{(k+1)}\Big(\langle e^{n+1}, e^{n+1}\rangle - \langle e^n, e^n\rangle\Big). \qquad (30)$$

By the Cauchy–Schwarz inequality, we get

$$\begin{aligned}
\eta_2 &:= -h\sum_{i=1}^{M-1}(R_i^{k+1/2})\delta_x^2 e^{k+1/2}\\
&\le \frac{K}{2}|\delta_x^2 e^{k+1/2}\|^2 + \frac{1}{2K}\|R^{k+1/2}\|^2.
\end{aligned} \qquad (31)$$

Moreover,

$$\eta_3 := -h\sum_{i=1}^{M-1}\mathfrak{A}\varrho_i^{k+1/2}\delta_x^2 e_i^{k+1/2}, \qquad (32)$$

such that

$$\begin{aligned}
&\varrho_i^{k+1/2}\\
&= \Bigg[f\Big(x_i, t_{k+1/2}, \frac{3}{2}U_i^k-\frac{1}{2}U_i^{k-1},\\
&\frac{1}{2}U_i^{k+1-n}+\frac{1}{2}U_i^{k-n}\Big)\\
&- f\Big(x_i, t_{k+1/2}, \frac{3}{2}u_i^k-\frac{1}{2}u_i^{k-1}, \frac{1}{2}u_i^{k+1-n}+\frac{1}{2}u_i^{k-n}\Big)\Bigg]
\end{aligned}$$

and

$$\|R^{k+1/2}\|^2 \le Lc_3^2(\tau^{2-\alpha}+h^4)^2.$$

Using (28), we can predict that

$$\begin{aligned}
\eta_3 \le \Big\langle &\mathfrak{A}\Big(c_1|\frac{3}{2}e_i^k-\frac{1}{2}e_i^{k-1}| + c_2|\frac{1}{2}e_i^{k+1-n}\\
&+\frac{1}{2}e_i^{k-n}|\Big), \delta_x^2 e_i^{k+1/2}\Big\rangle.
\end{aligned} \qquad (33)$$

For simplicity, the inner product in the right-hand side of (33) will be denoted by $\langle \xi_1, \xi_2\rangle$. Then $\langle \xi_1, \xi_2\rangle \le \frac{1}{2\theta}\|\xi_1\|^2 + \frac{\theta}{2}\|\xi_2\|^2$, and setting $\theta = K$, we obtain

$$\begin{aligned}
\eta_3 \le \frac{1}{2\theta}\|&\mathfrak{A}\Big(c_1|\frac{3}{2}e_i^k-\frac{1}{2}e_i^{k-1}| + c_2|\frac{1}{2}e_i^{k+1-n}\\
&+\frac{1}{2}e_i^{k-n}|\Big)\|^2 + \frac{\theta}{2}\|\delta_x^2 e_i^{k+1/2}\|^2.
\end{aligned}$$

Recalling Lemma 3, we get

$$\eta_3 \le \frac{1}{2\theta}\|c_1\|\frac{3}{2}e_i^k - \frac{1}{2}e_i^{k-1}\| + c_2|\frac{1}{2}e_i^{k+1-n} + \frac{1}{2}e_i^{k-n}|\|^2 + \frac{\theta}{2}\|\delta_x^2 e_i^{k+1/2}\|^2,$$

$$\eta_3 \le \frac{1}{2\theta}h\sum_{i=1}^{M-1}\Big(c_1|\frac{3}{2}e_i^k - \frac{1}{2}e_i^{k-1}| + c_2|\frac{1}{2}e_i^{k+1-n} + \frac{1}{2}e_i^{k-n}|\Big)^2 + \frac{\theta}{2}\|\delta_x^2 e_i^{k+1/2}\|^2,$$

$$\eta_3 \le \frac{1}{2\theta}\Big[hc_1^2\sum_{i=1}^{M-1}\Big(\frac{3}{2}e_i^k - \frac{1}{2}e_i^{k-1}\Big)^2 + c_2^2 h\sum_{i=1}^{M-1}\Big(\frac{1}{2}e_i^{k+1-n} + \frac{1}{2}e_i^{k-n}\Big)^2\Big] + \frac{\theta}{2}\|\delta_x^2 e_i^{k+1/2}\|^2,$$

$$\eta_3 \le \frac{1}{2\theta}\Big[\frac{5}{2}hc_1^2\sum_{i=1}^{M-1}\Big((e_i^k)^2 + (e_i^{k-1})^2\Big) + \frac{1}{2}c_2^2 h\sum_{i=1}^{M-1}\Big((e_i^{k+1-n})^2 + (e_i^{k-n})^2\Big)\Big] + \frac{\theta}{2}\|\delta_x^2 e_i^{k+1/2}\|^2, \tag{34}$$

which means that

$$\eta_3 \le \frac{1}{2K}\Big[\frac{5}{2}c_1^2\Big((||e^k||^2 + ||e^{k-1}||^2\Big) + \frac{1}{2}c_2^2\Big((||e^{k+1-n}||^2 + ||e^{k-n}||^2\Big)\Big] + \frac{K}{2}\|\delta_x^2 e_i^{k+1/2}\|^2. \tag{35}$$

Substituting by (29), (31) and (35) into (26), we get

$$\sum_{n=0}^{k} g_n^{(k+1)}\Big(\langle e^{n+1}, e^{n+1}\rangle - \langle e^n, e^n\rangle\Big) \le \frac{1}{K}\Big[\frac{5}{2}c_1^2\Big((||e^k||^2 + ||e^{k-1}||^2\Big) + \frac{1}{2}c_2^2\Big(||e^{k+1-n}||^2 + ||e^{k-n}||^2\Big)\Big] + \frac{1}{K}\|R^{k+1/2}\|^2, \tag{36}$$

which can be written as follows:

$$g_k^{(k+1)}\langle e^{k+1}, e^{k+1}\rangle \le \sum_{n=1}^{k}\Big(g_n^{(k+1)} - g_{n-1}^{(k+1)}\Big)\langle e^n, e^n\rangle + g_0^{(k+1)}\langle e^0, e^0\rangle + \frac{1}{K}\Big[\frac{5}{2}c_1^2\Big((||e^k||^2 + ||e^{k-1}||^2\Big) + \frac{1}{2}c_2^2\Big(||e^{k+1-n}||^2 + ||e^{k-n}||^2\Big)\Big] + \frac{1}{K}\|R^{k+1/2}\|^2. \tag{37}$$

Since

$$||\nu||^2 \le \frac{L^2}{4}\langle \nu, \nu\rangle,$$

we obtain

$$g_k^{(k+1)}\langle e^{k+1}, e^{k+1}\rangle \le \sum_{n=1}^{k}\Big(g_n^{(k+1)} - g_{n-1}^{(k+1)}\Big)\langle e^n, e^n\rangle + g_0^{(k+1)}\langle e^0, e^0\rangle + \frac{L^2}{4K}\Big[\frac{5}{2}c_1^2\Big(\langle e^k, e^k\rangle + \langle e^{k-1}, e^{k-1}\rangle\Big) + \frac{1}{2}c_2^2\Big(\langle e^{k+1-n}, e^{k+1-n}\rangle + \langle e^{k-n}, e^{k-n}\rangle\Big)\Big] + \frac{1}{K}\|R^{k+1/2}\|^2, \tag{38}$$

$$g_k^{(k+1)}\langle e^{k+1}, e^{k+1}\rangle \le \sum_{n=1}^{k}\Big(g_n^{(k+1)} - g_{n-1}^{(k+1)}\Big)\langle e^n, e^n\rangle + g_0^{(k+1)}\langle e^0, e^0\rangle + \bar\eta\Big(\langle e^k, e^k\rangle + \langle e^{k-1}, e^{k-1}\rangle + \langle e^{k+1-n}, e^{k+1-n}\rangle + \langle e^{k-n}, e^{k-n}\rangle\Big) + \frac{1}{K}\|R^{k+1/2}\|^2, \tag{39}$$

$$\bar\eta = \frac{1}{K}\max\Big\{\frac{5c_1L^2}{8}, \frac{c_2L^2}{8}\Big\}.$$

Noting that $g_0^{(k+1)} \ge k_0 > 0$, and defining

$$E_k = \max_{0\le l\le k}\{\langle e^0, e^0\rangle + \frac{\bar\eta}{k_0}\Big(\langle e^l, e^l\rangle + \langle e^{l-1}, e^{l-1}\rangle + \langle e^{l+1-n}, e^{l+1-n}\rangle + \langle e^{l-n}, e^{l-n}\rangle\Big) + \frac{1}{k_0K}\|R^{l+1/2}\|^2\},$$

we can rewrite (38) as follows:

$$g_k^{(k+1)}\langle e^{k+1}, e^{k+1}\rangle \le \sum_{n=1}^{k}\Big(g_n^{(k+1)} - g_{n-1}^{(k+1)}\Big)\langle e^n, e^n\rangle + g_0^{(k+1)}E_k. \tag{40}$$

Using the mathematical induction, we are going to prove that

$$\langle e^{k+1}, e^{k+1}\rangle \le E_k, \quad 0 \le k \le \ell \le N-1. \tag{41}$$

For $k = 0$, it is easy to see that (41) can be obtained from (40). Assume that

$$\langle e^{k+1}, e^{k+1}\rangle \le E_k, \quad 0 \le k+1 \le r.$$

Observing that (40), we can write

$$\begin{aligned} & g_r^{(r+1)}\langle e^{r+1}, e^{r+1}\rangle \\ & \le \sum_{n=1}^{r} \Big(g_n^{(r+1)} - g_{n-1}^{(r+1)}\Big)\langle e^n, e^n\rangle + g_0^{(r+1)} E_r \\ & \le \sum_{n=1}^{r} \Big(g_n^{(r+1)} - g_{n-1}^{(r+1)}\Big) E_r + g_0^{(r+1)} E_r \\ & = g_r^{(r+1)} E_r. \end{aligned} \tag{42}$$

Consequently, (41) is proved. Noting that $\langle e^0, e^0\rangle = 0$, we have

$$\begin{aligned} & \langle e^{\ell+1}, e^{\ell+1}\rangle \\ & \le \frac{\bar\eta}{k_0}\Big(\sum_{r=l_0-n}^{l_0-n+1}\langle e^r, e^r\rangle + \sum_{r=l_0-1}^{l_0}\langle e^r, e^r\rangle\Big) \\ & \quad + \frac{1}{k_0 K}\|R^{l_0+1/2}\|^2, \end{aligned} \tag{43}$$

where l_0 is a number at which the maximum of E_ℓ is achieved. Since (43) fulfills all conditions of applying Lemma 9, we obtain

$$\begin{aligned} \langle e^{\ell+1}, e^{\ell+1}\rangle & \le \frac{1}{k_0 K}\|R^{l_0+1/2}\|^2 \exp\Big(\frac{4\bar\eta}{k_0}\Big) \\ & \le C(\tau^{2-\alpha} + h^4)^2, \\ C & = \frac{L c_3^2}{k_0 K}\exp\Big(\frac{4\bar\eta}{k_0}\Big). \end{aligned} \tag{44}$$

Recalling (44), we get

$$\|e^{\ell+1}\|_\infty \le \sqrt{\frac{3L}{8} C\Big(\tau^{2-\alpha} + h^4\Big)^2} \le \bar C\Big(\tau^{2-\alpha} + h^4\Big).$$

Thus, the inductive step for (20) is achieved and this completes the proof. ■

To discuss the stability of the difference scheme (11a)–(11c), we also use the discrete energy method in the same way like the discussion of the convergence.

Let $\{\nu_i^k | 0 \le i \le M,\ 0 \le k \le N\}$ be the solution of

$$\begin{aligned} & \mathfrak{A}\Big[\omega_1 \nu^k + \sum_{m=1}^{k-1}(\omega_{k-m+1} - \omega_{k-m})\nu^m - \omega_k \nu^0 \\ & \quad + \sigma\frac{(\nu_i^{k+1} - \nu_i^k)}{2^{1-\alpha}}\Big] \\ & = K\delta_x^2 \nu_i^{k+1/2} + \mathfrak{A} f(x_i, t_{k+1/2}, \frac{3}{2}\nu_i^k - \frac{1}{2}\nu_i^{k-1}, \\ & \quad \frac{1}{2}\nu_i^{k+1-n} + \frac{1}{2}\nu_i^{k-n}), \end{aligned} \tag{45}$$

$$\nu_0^k = \phi_0(t_k), \quad \nu_M^k = \phi_L(t_k), \quad 1 \le k \le N, \tag{46}$$

$$\nu_i^k = \psi(x_i, t_k) + \rho_i^k, \quad 0 \le i \le M, \quad -n \le k \le 0, \tag{47}$$

where ρ_i^k is the perturbation of $\psi(x_i, t_k)$.

Following the same steps as in the proof of convergence theorem, the following result is obtained.

Theorem 3. (Stability theorem) *Assume that* $\theta_i^k = \nu_i^k - u_i^k$, $0 \le i \le M$, $-n \le k \le N$. *There exist constants* c_4, c_5, h_0, τ_0 *such that*

$$\|\theta^k\|_\infty \le c_4\sqrt{\tau}\sum_{k=-n}^{0}\|\rho^k\|, \quad 0 \le k \le N,$$

$$\|\rho^k\| = \sqrt{h\sum_{i=1}^{M-1}(\rho_i^k)^2},$$

provided that

$$h \le h_0, \quad \tau \le \tau_0, \quad \max_{\substack{-n\le k\le 0\\ 0\le i\le M}} |\rho_i^k| \le c_5.$$

4. Numerical examples

Let u_i^k be the solution of the constructed difference scheme (11a)–(11c) with the step sizes τ and h. Define the maximum norm error by

$$E(\tau, h) = \max_{\substack{0\le i\le M\\ 0\le k\le N}} \|U_i^k - u_i^k\|_\infty.$$

Define the following error rates:

$$rate_1 = \log_2\left(\frac{E(2\tau, h)}{E(\tau, h)}\right),$$

$$rate_2 = \log_2\left(\frac{E(\tau, 2h)}{E(\tau, h)}\right).$$

Example 1. Consider the following test example:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x, t, u(x,t), u(x, t-s)), \quad t \in (0,1), \quad 0 < x < 2, \tag{48}$$

$$\begin{aligned} & f(x, t, u(x,t), u(x, t-s) \\ & = \frac{\Gamma(3)}{\Gamma(3-\alpha)}(2x - x^2)t^{2-\alpha} \\ & \quad + 2t^2 - u(x, t-s) + x(2-x)(t-s)^2, \end{aligned}$$

with the initial and boundary conditions

$$u(x,t) = t^2(2x - x^2), \quad 0 \le x \le 2, \quad t \in [-s, 0), \tag{49}$$

$$u(0,t) = u(2,t) = 0, \quad t \in [0,1]. \tag{50}$$

The exact solution to this problem is

$$u(x,t) = t^2(2x - x^2). \tag{51}$$

♦

Example 2. Consider the following test example:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t,u(x,t),u(x,t-s)),$$
$$t \in (0,1), \quad 0 < x < 1, \quad (52)$$

$$f(x,t,u(x,t),u(x,t-s)$$
$$= \frac{\Gamma(7/2)}{\Gamma(7/2-\alpha)}(x^2-x)t^{5/2-\alpha} - 2t^{5/2}$$
$$+ u^2(x,t-s) - (x^2-x)^2(t-s)^5,$$

with the initial and boundary conditions

$$u(x,t) = t^{\frac{5}{2}}(x-x^2), \quad 0 \le x \le 1, \quad t \in [-s,0), \quad (53)$$

$$u(0,t) = u(1,t) = 0, \quad t \in [0,1]. \quad (54)$$

The exact solution to this problem is

$$u(x,t) = t^{\frac{5}{2}}(x^2-x). \quad (55)$$

♦

Tables 1, 2 and 3, 4 show the errors in maximum norm and their convergence rates for time fractional models 48–51 and 52–55, respectively. From these tables, it can be seen that the orders of convergence of the proposed numerical method are in good agreement with the theoretical results in the theorem.

Table 1. Errors and convergence orders of the difference scheme (11a)–(11c) in the time variable with $h = 1/300$ and $\alpha = 0.25$ with time delay $s = 1$.

τ	$E(\tau,h)$	$rate_1$
$\frac{1}{10}$	0.00113	
$\frac{1}{20}$	0.00034	1.735
$\frac{1}{40}$	0.0001	1.742
$\frac{1}{80}$	0.00003	1.748
$\frac{1}{160}$	0.000009	1.752

Table 2. Errors and convergence orders of the difference scheme (11a)–(11c) in the space variable with $\tau = 1/1000$ and $\alpha = 0.25$ with time delay $s = 1$.

h	$E(\tau,h)$	$rate_2$
$\frac{1}{4}$	0.0025	
$\frac{1}{8}$	0.00017	3.863
$\frac{1}{16}$	0.00001	3.895
$\frac{1}{32}$	0.0000007	3.942
$\frac{1}{64}$	0.0000004	3.975

Table 3. Errors and convergence orders of the difference scheme (11a)–(11c) in the time variable with $h = 1/500$ and $\alpha = 0.75$ with time delay $s = 0.5$.

τ	$E(\tau,h)$	$rate_1$
$\frac{1}{10}$	0.00112	
$\frac{1}{20}$	0.00047	1.245
$\frac{1}{40}$	0.00019	1.248
$\frac{1}{80}$	0.00008	1.249
$\frac{1}{160}$	0.00003	1.252

Table 4. Errors and convergence orders of the difference scheme (11a)–(11c) in the space variable with $\tau = 1/2000$ and $\alpha = 0.75$ with time delay $s = 0.5$.

h	$E(\tau,h)$	$rate_2$
$\frac{1}{4}$	0.0121	
$\frac{1}{8}$	0.00076	3.980
$\frac{1}{16}$	0.00005	3.995
$\frac{1}{32}$	0.000003	3.998
$\frac{1}{64}$	0.0000001	4.010

5. Conclusion

The main contribution of this work lies in building a linearized difference scheme to solve a class of time fractional diffusion equations with non-linear delay. We proved that our scheme is unconditionally convergent and stable in the sense of the maximum norm. In our future work, we plan to increase the time convergence order to two instead of $2-\alpha, 0 < \alpha \le 1$, by using a suitable approximation for the time Caputo fractional derivative in the problem under consideration. The proposed numerical test examples supported our theoretical results.

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