Jan Długosz University in Częstochowa
Scientific Issues, Mathematics XVI, Częstochowa 2011

M/M/n/m QUEUEING SYSTEMS WITH NON-IDENTICAL SERVERS

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Abstract. $M/M/n/m$ queueing systems with identical servers are well known in queueing theory and its applications. The analysis of these systems is very simple thanks to the fact that the number of customers $\eta(t)$ in the system at arbitrary time instant t forms a Markov chain. The main purpose of this paper is to analyse the $M/M/n/m$ system under assumption that its servers are different, i.e. they have different parameters of service time.

1. The analysis of classical $M/M/n/m$ queueing system

Let us consider the classical $M/M/n/m$ queueing system ($m < \infty$). Let a be an arrival rate of customers, μ be the parameter of service time distribution. This system is illustrated in Fig. 1.

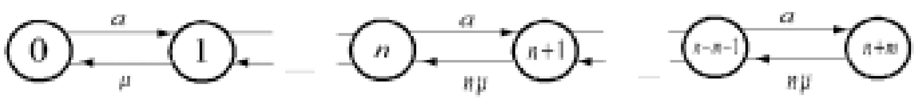

Figure 1: A graph of the classical $M/M/n/m$ queueing system ($m < \infty$)

Denote $\rho = a/(n\mu)$. In the case $\rho < \infty$ the steady state exists, and we have the following equations for steady state probabilities $p_k = P\{\eta = k\}$:

$$\begin{cases} ap_{k-1} = k\mu p_k, & k = \overline{1, n}, \\ ap_{k-1} = n\mu p_k, & k = \overline{n+1, n+m}, \end{cases} \tag{1}$$

where η is the stationary number of customers present in the system, $k = 0, 1, \ldots$.

It can be easily shown [1] that the solution of this system has the form:

$$p_k = \begin{cases} \dfrac{(n\rho)^k}{k!} p_0, & k = \overline{1, n}, \\ \dfrac{n^n \rho^k}{n!} p_0, & k = \overline{n+1, n+m}. \end{cases} \tag{2}$$

The value $p_0 = P\{\eta = 0\}$ is obtained from the normalization condition $\sum_{k=0}^{n+m} p_k = 1$. If $m = \infty$, the graph of the system is similar. In this case $\eta(t)$ is a birth-death process and we can also obtain p_k probabilities (if $\rho < 1$) from analogous equations. The final formulas are also very similar.

Assume now that μ_i is a service time distribution parameter of the ith server in the system, $i = \overline{1, n}$ and these parameters may be different. The problem appeared for the first time in [2],[3] and [4]. In this case it is possible that arriving customer has to choose one of free different servers. Hence we must define the discipline of such a choice. We shall analyze the following disciplines:

1. RANDOM CHOICE.

   If there are k free servers, an arriving customer chooses every free server with the same probability $\frac{1}{k}$. Such systems will be denoted as $M/M/n/m$-$NHRC$ (non-homogeneous servers with classical random choice).

2. FASTEST SERVER CHOICE.

   An arriving customer chooses the fastest server. Such systems will be denoted as $M/M/n/m - NHFC$ (non-homogeneous servers with the fastest server choice).

We shall describe the behaviour of the system by the following generalized Markovian process:

$$(\eta(t), i_1(t), i_2(t), \ldots, i_l(t)), \tag{3}$$

where $l = \min(\eta(t), n)$ and $i_1(t), i_2(t), \ldots, i_l(t)$ is the sequence of the numbers of busy servers written increasingly. If $\eta(t) = 0$, the process reduces to $\eta(t)$.

The process (3) is characterized by the following functions:

$$P_0(t) = P\{\eta(t) = 0\}; \tag{4}$$

$$P_{kf_1f_2\ldots f_l}(t) = P\{\eta(t) = k,\ i_1(t) = f_1, i_2(t) = f_2, \ldots, i_l(t) = f_l\},$$
$$k = \overline{1, n+m}. \tag{5}$$

It is clear that if $k \geq n$, then the process $\eta(t)$ can be understood as a birth-death process and we can rewrite functions from (5) as $P_k(t) = P\{\eta(t) = k\}$. If $k < n$, we have

$$P_k(t) = P\{\eta(t) = k\} = \sum_{\{F_k^n\}} P_{kf_1f_2\dots f_k}(t), \qquad (6)$$

where $\{F_k^n\}$ is the set of all combinations of the set $\{f_1, f_2, \dots, f_k\}$.

Now we shall investigate these systems in some special cases and try to find formulas for p_k in the general case.

2. Analysis of $M/M/2/m$–NHRC and $M/M/3/m$–NHRC queueing systems

Let us consider the generalization of $M/M/n/m$ queueing system with two non-identical servers and random choice of a free server. Denote as μ_1, μ_2 the service time parameters for the servers. We will also use the notation: $\rho = \frac{a}{\mu_1+\mu_2}$. A graph of such a system is presented in Fig. 2.

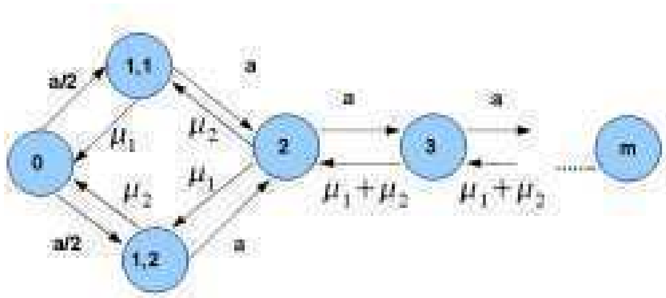

Figure 2: A graph of the $M/M/2/m - NHRC$ queueing system ($m < \infty$)

To describe a behavior of the system we introduce the following Markovian process:

$$(\eta(t), i_1(t), \dots, i_l(t)), \qquad (7)$$

where $l = \min(\eta(t), 2)$.

The process (7) is characterized by the following functions:

$$P_0(t) = P\{\eta(t) = 0\}; \qquad (8)$$

$$P_{11}(t) = P\{\eta(t) = 1,\ i_1(t) = 1\}; \qquad (9)$$

$$P_{12}(t) = P\{\eta(t) = 1,\ i_1(t) = 2\}; \qquad (10)$$

$$P_k(t) = P\{\eta(t) = k\},\ k = \overline{2, m+2}. \qquad (11)$$

The Kolmogorov equations for these functions have the form:

$$P_0'(t) = -aP_0(t) + \mu_1 P_{11}(t) + \mu_2 P_{12}(t); \tag{12}$$

$$P_{11}'(t) = -(a+\mu_1)P_{11}(t) + \frac{a}{2}P_0(t) + \mu_2 P_2(t); \tag{13}$$

$$P_{12}'(t) = -(a+\mu_2)P_{12}(t) + \frac{a}{2}P_0(t) + \mu_1 P_2(t); \tag{14}$$

$$P_2'(t) = -(a+\mu_1+\mu_2)P_2(t) + a(P_{11}(t) + P_{12}(t)) + (\mu_1+\mu_2)P_3(t); \tag{15}$$

$$P_k'(t) = -(a+\mu_1+\mu_2)P_k(t) + aP_{k-1}(t) + (\mu_1+\mu_2)P_{k+1}(t),\ k = \overline{3, m+1}; \tag{16}$$

$$P_{m+2}'(t) = -(\mu_1+\mu_2)P_{m+2}(t) + aP_{m+1}(t); \tag{17}$$

$$P_0(t) + P_{11}(t) + P_{12}(t) + \sum_{k=2}^{m+2} P_k(t) = 1. \tag{18}$$

In the steady state (if $\rho < \infty$) we obtain the following equations for probabilities p_0, p_{11}, p_{12} and p_k that are limits of functions (8)–(11) for $t \to \infty$:

$$ap_0 = \mu_1 p_{11} + \mu_2 p_{12}; \tag{19}$$

$$(a+\mu_1)p_{11} = \frac{a}{2}p_0 + \mu_2 p_2; \tag{20}$$

$$(a+\mu_2)p_{12} = \frac{a}{2}p_0 + \mu_1 p_2; \tag{21}$$

$$(a+\mu_1+\mu_2)p_2 = a(p_{11}+p_{12}) + (\mu_1+\mu_2)p_3; \tag{22}$$

$$(a+\mu_1+\mu_2)p_k = ap_{k-1} + (\mu_1+\mu_2)p_{k+1},\ k = \overline{3, m+1}; \tag{23}$$

$$(\mu_1+\mu_2)p_{m+2} = ap_{m+1}; \tag{24}$$

$$p_0 + p_{11} + p_{12} + \sum_{k=2}^{m+2} p_k = 1. \tag{25}$$

The solution has the form:

$$p_{11} = \frac{a}{2\mu_1}p_0; \tag{26}$$

$$p_{12} = \frac{a}{2\mu_2}p_0; \tag{27}$$

$$p_1 = p_{11} + p_{12} = \frac{ap_0}{2}(\frac{1}{\mu_1} + \frac{1}{\mu_2}); \tag{28}$$

$$p_k = \frac{ap_0}{2}\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right)\rho^{k-1},\ k = \overline{2, m+2}. \tag{29}$$

Formulas (28)–(29) can be rewritten as

$$p_k = \frac{ap_0}{2}\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)\rho^{k-1},\ k=\overline{1, m+2}, \tag{30}$$

where the value of p_0 can be obtained from the normalization condition (25):

$$p_0 = \left[1+\frac{a}{2}\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right)\sum_{k=1}^{m+2}\rho^{k-1}\right]^{-1}. \tag{31}$$

If we investigate $M/M/2/\infty - NHRC$ system in the steady state ($\rho < 1$), formulas (30) for probabilities p_k will be the same, but the relation (31) will have a simpler form (in this case we use formula for the sum of geometric series).

If we analyse $M/M/3/m - NHRC$ system ($\rho = \frac{a}{\mu_1+\mu_2+\mu_3}$) and write analogous equations, we will obtain the following results:

$$p_1 = \frac{ap_0}{3}\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}+\frac{1}{\mu_3}\right); \tag{32}$$

$$p_k = \frac{a^2 p_0}{6}\left(\frac{1}{\mu_1\mu_2}+\frac{1}{\mu_1\mu_3}+\frac{1}{\mu_2\mu_3}\right)\rho^{k-2},\ k=\overline{2, m+3}; \tag{33}$$

$$p_0 = \left[1+\frac{a}{3}\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}+\frac{1}{\mu_3}\right)+\frac{a^2}{6}\left(\frac{1}{\mu_1\mu_2}+\frac{1}{\mu_1\mu_3}+\frac{1}{\mu_2\mu_3}\right)\sum_{k=2}^{m+3}\rho^{k-2}\right]^{-1}. \tag{34}$$

Analysis of $M/M/n/m - NHRC$ for $n > 3$ is more complicated because the number of equations is increasing exponentially and it is difficult to find the exact solution. But obtained results let predict the general solution for the arbitrary number of servers in the following form:

$$p_k = \begin{cases} \dfrac{a^k (n-k)! p_0}{n!} \displaystyle\sum_{\{F_k^n\}} \frac{1}{\prod_{x_i \in F_n^k} \mu_{x_i}}, & k=\overline{1, n-1}, \\ \dfrac{a^k (n-k)! p_0}{n!} \displaystyle\sum_{\{F_n^n\}} \frac{1}{\prod_{x_i \in F_n^n} \mu_{x_i}} \rho^{k-n}, & k=\overline{n, n+m}, \end{cases} \tag{35}$$

where F_k^n denotes a k-element subset of an n element set.

3. Analysis of $M/M/2/m - NHFC$ queueing system

In this case, if we assume that $\mu_1 > \mu_2$, then we will not have the transition $0 \to (1,2)$ in the graph presented in Fig. 2, and the transition $0 \to (1,1)$ parameter will be equal to a.

So in this case we have the following equations in the steady state:

$$ap_0 = \mu_1 p_{11} + \mu_2 p_{12}; \tag{36}$$

$$(a+\mu_1)p_{11} = ap_0 + \mu_2 p_2; \tag{37}$$

$$(a+\mu_2)p_{12} = \mu_1 p_2. \tag{38}$$

The rest of equations for the functions p_k will be the same as in (22)–(25).

The solution of this system has the form:

$$p_k = \frac{a(a+\mu_2)(\mu_1+\mu_2)p_0}{\mu_1\mu_2(2a+\mu_1+\mu_2)}\rho^{k-1},\ k = \overline{1, m+2}; \tag{39}$$

$$p_0 = \left[1 + \frac{a(a+\mu_2)(\mu_1+\mu_2)}{\mu_1\mu_2(2a+\mu_1+\mu_2)}\sum_{k=1}^{m+2}\rho^{k-1}\right]^{-1}. \tag{40}$$

The general analysis for $n \geq 3$ is complicated and predicting general formulas for p_k is not easy.

References

[1] O. Tikhonenko. *Metody probabilistyczne analizy systemów informacyjnych.* Akademicka Oficyna Wydawnicza EXIT, Warszawa 2006.

[2] H. Gumbel. Waiting lines with heterogeneous servers. *Operations Research*, **8**(4), 504–511, 1960.

[3] V.P. Singh. Markovian queues with three heterogeneous servers. *AIIE Transactions*, **3**(1), 45–48, 1971.

[4] V.P. Singh. Two-server Markovian queues with balking: Heterogeneous vs. homogeneous servers. *Operations Research*, **18**(1), 145–159, 1970.