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AXISYMMETRIC THERMAL STRESSES IN A HALF-SPACE IN THE FRAMEWORK OF FRACTIONAL THERMOELASTICITY

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Abstract

A theory of thermal stresses based on the time-fractional heat conduction equation is considered. The Caputo fractional derivative is used. The fundamental solution to the axisymmetric heat conduction equation in a half-space under the Dirichlet boundary condition and the associated thermal stresses are investigated.

1. Introduction

Numerical applications of fractional calculus to problems of mechanics can be found in the literature. We can quote investigations on viscoelasticity [6], creep [20], hereditary mechanics of solids [21], Brownian motion [5], stress and strain localization in solids [1] (see also [4], [22], [23], [26], [27]). The theory of thermal stresses based on the time-fractional heat conduction equation was proposed by the author [11] and was developed in the subsequent studies [12], [13], [15]–[17]. Axisymmetric problems for the time-fractional heat conduction equation in a half-space were investigated in [14]. In this paper we study associated thermal stresses.

2. Formulation of the problem

A thermoelastic state of a solid is governed by the equilibrium equation in terms of displacements

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\mathbf{u} = \beta_T K_T\,\mathrm{grad}\,T, \tag{1}$$

the stress-strain-temperature relation

$$\boldsymbol{\sigma} = 2\mu\mathbf{e} + (\lambda\,\mathrm{tr}\,\mathbf{e} - \beta_T K_T T)\mathbf{I}, \tag{2}$$

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and the time-fractional heat conduction equation

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\,\Delta T, \quad 0 < \alpha \le 2. \tag{3}$$

Here $\mathbf{u}$ is the displacement vector, $\boldsymbol{\sigma}$ the stress tensor, $\mathbf{e}$ the linear strain tensor, T the temperature, λ and μ are Lamé constants, $K_T = \lambda + 2\mu/3$, β_T is the thermal coefficient of volumetric expansion, a denotes the thermal diffusivity, $\mathbf{I}$ stands for the unit tensor, and $\frac{\partial^\alpha}{\partial t^\alpha}$ is the Caputo fractional derivative [2], [3], [10]

$$\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\mathrm{d}^n f(\tau)}{\mathrm{d}\tau^n}\,\mathrm{d}\tau, \quad n-1 < \alpha < n,$$

where $\Gamma(x)$ is the gamma function.

The Caputo derivative has the following Laplace transform rule

$$\mathcal{L}\left\{\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha}\right\} = s^\alpha f^*(s) - \sum_{k=0}^{n-1} f^{(k)}(0^+)s^{\alpha-1-k}, \quad n-1 < \alpha < n,$$

where the asterisk denotes the Laplace transform, s is the transform variable.

In this paper we will consider the axisymmetric fractional heat conduction equation

$$\frac{\partial^\alpha T}{\partial t^\alpha} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right), \quad 0 < \alpha \le 2, \tag{4}$$

in the domain $0 \le r < \infty$, $0 < z < \infty$, $0 < t < \infty$ under zero initial conditions

$$t = 0: \quad T = 0, \quad 0 < \alpha \le 2, \tag{5}$$

$$t = 0: \quad \frac{\partial T}{\partial t} = 0, \quad 1 < \alpha \le 2, \tag{6}$$

and the Dirichlet boundary condition

$$z = 0: \quad T = f(r,t). \tag{7}$$

The zero conditions at infinity

$$\lim_{r\to\infty} T(r,z,t) = 0, \quad \lim_{z\to\infty} T(r,z,t) = 0 \tag{8}$$

are also assumed.

In the quasi-static statement of the thermoelasticity problem, initial values of mechanical quantities are not considered. The boundary of a half-space is load free, hence

$$z = 0: \quad \sigma_{zz} = 0, \quad \sigma_{rz} = 0. \tag{9}$$

3. Representation of stresses

Just as in the classical theory of thermal stresses [8], [9], we can introduce the displacement potential Φ

$$\mathbf{u} = \operatorname{grad} \Phi. \tag{10}$$

In the quasi-static case, from the equilibrium equation (1) we get

$$\Delta\Phi = mT, \quad m = \frac{1+\nu}{1-\nu}\,\frac{\beta_T}{3}, \tag{11}$$

with ν being the Poisson ratio. The part of stresses due to the displacement potential Φ describes the influence of the temperature field and is given as

$$\boldsymbol{\sigma}^{(1)} = 2\mu\left(\operatorname{grad}\operatorname{grad}\Phi - \mathbf{I}\,\Delta\Phi\right). \tag{12}$$

In cylindrical coordinates in the case of axial symmetry

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\,\frac{\partial\Phi}{\partial r} + \frac{\partial^2\Phi}{\partial z^2} = mT, \tag{13}$$

$$\sigma_{rr}^{(1)} = 2\mu\left[\frac{\partial^2\Phi}{\partial r^2} - \Delta\Phi\right], \tag{14}$$

$$\sigma_{\theta\theta}^{(1)} = 2\mu\left[\frac{1}{r}\,\frac{\partial\Phi}{\partial r} - \Delta\Phi\right], \tag{15}$$

$$\sigma_{zz}^{(1)} = 2\mu\left[\frac{\partial^2\Phi}{\partial z^2} - \Delta\Phi\right], \tag{16}$$

$$\sigma_{rz}^{(1)} = 2\mu\,\frac{\partial^2\Phi}{\partial r\partial z}. \tag{17}$$

The Hankel transform of order n with respect to the radial coordinate r

$$\mathcal{H}_{(n)}\left\{f(r)\right\} = \int_0^\infty f(r)\,J_n(r\xi)\,r\,\mathrm{d}r,$$

$$f(r) = \int_0^\infty \mathcal{H}_{(n)}\left\{f(r)\right\}\,J_n(r\xi)\,\xi\,\mathrm{d}\xi$$

is often used for solving problems in cylindrical coordinates. The following formulae are helpful in applications [25]

$$\mathcal{H}_{(n)}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + \frac{1}{r}\,\frac{\mathrm{d}f(r)}{\mathrm{d}r} - \frac{n^2}{r^2}\,f(r)\right\} = -\xi^2\mathcal{H}_{(n)}\left\{f(r)\right\},$$

$$\mathcal{H}_{(1)}\left\{\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right\} = -\xi\,\mathcal{H}_{(0)}\left\{f(r)\right\},$$

$$\mathcal{H}_{(2)}\left\{\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} - \frac{1}{r}\,\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right\} = \xi^2\mathcal{H}_{(0)}\left\{f(r)\right\}.$$

In the case $n=0$, simultaneously with the notation $\mathcal{H}_{(0)}\left\{f(r)\right\}$, we will use the notation $\mathcal{H}_{(0)}\left\{f(r)\right\}=\widehat{f}(\xi)$.

From (14)–(17) we have

$$\mathcal{H}_{(0)}\left\{\sigma_{rr}^{(1)}+\sigma_{\theta\theta}^{(1)}\right\} = 2\mu\,\xi^2\widehat{\Phi}-4\mu\frac{\partial^2\widehat{\Phi}}{\partial z^2}, \tag{18}$$

$$\mathcal{H}_{(2)}\left\{\sigma_{rr}^{(1)}-\sigma_{\theta\theta}^{(1)}\right\} = 2\mu\,\xi^2\widehat{\Phi}, \tag{19}$$

$$\mathcal{H}_{(0)}\left\{\sigma_{zz}^{(1)}\right\} = 2\mu\,\xi^2\widehat{\Phi}, \tag{20}$$

$$\mathcal{H}_{(1)}\left\{\sigma_{rz}^{(1)}\right\} = -2\mu\,\xi\frac{\partial\widehat{\Phi}}{\partial z}. \tag{21}$$

It follows from (11) that

$$\frac{\partial^2\widehat{\Phi}}{\partial z^2}-\xi^2\widehat{\Phi}=m\,\widehat{T}. \tag{22}$$

The general solution of the homogeneous equation (22) has the form

$$\widehat{\Phi}=C_1\mathrm{e}^{-\xi z}+C_2\mathrm{e}^{\xi z}, \tag{23}$$

where the integration constant C_2 should be equal to zero according to the condition at infinity (8). To find the particular solution of the non-homogeneous equation (22) we consider the following equation

$$\frac{\partial^2\widehat{\Phi}}{\partial z^2}-\xi^2\widehat{\Phi}=\delta(z), \tag{24}$$

where $\delta(z)$ is the Dirac delta function. The solution of (24) is written as

$$\widehat{\Phi}=-\frac{1}{2\xi}\mathrm{e}^{-\xi|z|}. \tag{25}$$

Hence, the particular solution of (22) is represented in the convolution form

$$\widehat{\Phi}(\xi,z,t)=-\frac{1}{2\xi}\int_0^\infty\widehat{T}(\xi,\eta,t)\,\mathrm{e}^{-\xi|z-\eta|}\,\mathrm{d}\eta. \tag{26}$$

Assuming in (23) $C_1=0$, we get

$$\mathcal{H}_{(0)}\left\{\sigma_{zz}^{(1)}\right\}=-\mu m\xi\int_0^\infty\widehat{T}(\xi,\eta,t)\,\mathrm{e}^{-\xi|z-\eta|}\,\mathrm{d}\eta, \tag{27}$$

$$\mathcal{H}_{(1)}\left\{\sigma_{rz}^{(1)}\right\}=-\mu m\xi\int_0^\infty\widehat{T}(\xi,\eta,t)\,\mathrm{e}^{-\xi|z-\eta|}\,\mathrm{sign}\,(z-\eta)\,\mathrm{d}\eta. \tag{28}$$

The part of stress field expressed in terms of the biharmonic Love function

$$\sigma_{rr}^{(2)} = 2\mu \frac{\partial}{\partial z}\left[\nu \Delta L - \frac{\partial^2 L}{\partial r^2}\right], \tag{29}$$

$$\sigma_{\theta\theta}^{(2)} = 2\mu \frac{\partial}{\partial z}\left[\nu \Delta L - \frac{1}{r}\frac{\partial L}{\partial r}\right], \tag{30}$$

$$\sigma_{zz}^{(2)} = 2\mu \frac{\partial}{\partial z}\left[(2-\nu) \Delta L - \frac{\partial^2 L}{\partial z^2}\right], \tag{31}$$

$$\sigma_{rz}^{(2)} = 2\mu \frac{\partial}{\partial r}\left[(1-\nu) \Delta L - \frac{\partial^2 L}{\partial z^2}\right] \tag{32}$$

with

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)^2 L = 0 \tag{33}$$

allows us to satisfy the prescribed boundary conditions for the components of the total stress tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)}$.

In the Hankel transform domain the biharmonic equation for the Love function

$$\left(\frac{\partial^2}{\partial z^2} - \xi^2\right)^2 \widehat{L} = 0 \tag{34}$$

has the solution bounded at $z \to \infty$:

$$\widetilde{L} = (A + B\xi z)\, \mathrm{e}^{-\xi z}, \tag{35}$$

where A and B are constants which should be found from the boundary conditions.

In the Hankel transform domain we have

$$\mathcal{H}_{(0)}\left\{\sigma_{rr}^{(2)} + \sigma_{\theta\theta}^{(2)}\right\} = 2\mu\xi^3\left[-A + (1+4\nu)B - B\xi z\right] \mathrm{e}^{-\xi z}, \tag{36}$$

$$\mathcal{H}_{(2)}\left\{\sigma_{rr}^{(2)} - \sigma_{\theta\theta}^{(2)}\right\} = 2\mu\xi^3\left(A - B + B\xi z\right) \mathrm{e}^{-\xi z}, \tag{37}$$

$$\mathcal{H}_{(0)}\left\{\sigma_{zz}^{(2)}\right\} = 2\mu\xi^3\left[A + (1-2\nu)B + B\xi z\right] \mathrm{e}^{-\xi z}, \tag{38}$$

$$\mathcal{H}_{(1)}\left\{\sigma_{rz}^{(2)}\right\} = 2\mu\xi^3\left(A - 2\nu B + B\xi z\right) \mathrm{e}^{-\xi z}. \tag{39}$$

From the load free boundary condition (9) we obtain

$$\mathcal{H}_0\left\{\sigma_{zz}^{(1)} + \sigma_{zz}^{(2)}\right\} = 0, \quad \mathcal{H}_1\left\{\sigma_{rz}^{(1)} + \sigma_{rz}^{(2)}\right\} = 0, \tag{40}$$

$$A = -\frac{(1-4\nu)m}{2\xi^2}\int_0^\infty \widetilde{T}(\xi,\eta,t)\, \mathrm{e}^{-\xi\eta}\, \mathrm{d}\eta, \qquad B = \frac{m}{\xi^2}\int_0^\infty \widetilde{T}(\xi,\eta,t)\, \mathrm{e}^{-\xi\eta}\, \mathrm{d}\eta,$$

$$\mathcal{H}_{(0)}\left\{\sigma_{zz}\right\} = \mu m \int_0^\infty \xi\, \widehat{T}(\xi,\eta,t) \left[(2\xi z+1)\,\mathrm{e}^{-\xi(z+\eta)} - \mathrm{e}^{-\xi|z-\eta|}\right] \mathrm{d}\eta,$$

$$\mathcal{H}_{(1)}\{\sigma_{rz}\} = \mu m \int_0^\infty \xi \widehat{T}(\xi,\eta,t)\Big[(2\xi z-1)\,\mathrm{e}^{-\xi(z+\eta)} - \mathrm{e}^{-\xi|z-\eta|}\,\mathrm{sign}\,(z-\eta)\Big]\,\mathrm{d}\eta.$$

4. Solution to the fractional heat conduction equation

We start from the fundamental solution to the Dirichlet problem for the time-fractional heat conduction equation (4) with zero initial conditions (5) and (6) and the prescribed boundary value of temperature

$$z = 0: \quad T = \frac{U_0}{2\pi r}\,\delta(r)\,\delta(t). \tag{41}$$

In the case of the Dirichlet boundary condition at a surface $z = 0$ the sin-Fourier transform is used:

$$\mathcal{F}\{f(z)\} = \widetilde{f}(\eta) = \int_0^\infty f(z)\,\sin(z\eta)\,\mathrm{d}z,$$

$$\mathcal{F}^{-1}\left\{\widetilde{f}(\eta)\right\} = f(z) = \frac{2}{\pi}\int_0^\infty \widetilde{f}(\eta)\,\sin(z\eta)\,\mathrm{d}\eta,$$

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(z)}{\mathrm{d}z^2}\right\} = -\eta^2\widetilde{f}(\eta) + \eta f(z)\Big|_{z=0},$$

where the tilde denotes the Fourier transform, η is the transform variable.

Applying the Laplace transform with respect to time t, the Hankel transform with respect to the radial coordinate r, and the sin-Fourier transform with respect to the spatial coordinate z gives

$$\widetilde{\widehat{T}}^{*} = \frac{aU_0\eta}{2\pi}\,\frac{1}{s^\alpha + a(\xi^2+\eta^2)} \tag{42}$$

or after inversion of integral transforms [14]

$$T = \frac{aU_0 t^{\alpha-1}}{\pi^2}\int_0^\infty\int_0^\infty E_{\alpha,\alpha}\left[-a(\xi^2+\eta^2)t^\alpha\right] J_0(r\xi)\,\sin(z\eta)\,\xi\eta\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{43}$$

Here the following formula [10]

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-\beta}}{s^\alpha+b}\right\} = t^{\beta-1}\,E_{\alpha,\beta}(-bt^\alpha)$$

has been used, where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function in two parameters α and β [3].

It should be emphasized that partition of the total stress tensor $\boldsymbol{\sigma}$ into the stress tensors $\boldsymbol{\sigma}^{(1)}$ and $\boldsymbol{\sigma}^{(2)}$ is not unique. Sometimes, it is helpful to suppose that $\Phi\big|_{z=0} = 0$ [8]. In this case, the sin-Fourier transform with respect to the spatial coordinate z (as well as the Laplace transform with respect to time t) can be applied to (22) resulting in

$$\Phi = -\frac{aU_0 m t^{\alpha-1}}{\pi^2} \int_0^\infty \int_0^\infty E_{\alpha,\alpha}\left[-a(\xi^2+\eta^2)t^\alpha\right] \times \frac{\xi\,\eta}{\xi^2+\eta^2} J_0(r\xi)\,\sin(z\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{44}$$

The stress tensor $\boldsymbol{\sigma}^{(1)}$ has the following components:

$$\sigma_{rr}^{(1)} = -\frac{2\mu m a U_0 t^{\alpha-1}}{\pi^2} \int_0^\infty \int_0^\infty E_{\alpha,\alpha}\left[-a(\xi^2+\eta^2)t^\alpha\right] \times \frac{\xi\,\eta}{\xi^2+\eta^2} \left[\frac{\xi}{r} J_1(r\xi) + \eta^2 J_0(r\xi)\right] \sin(z\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{45}$$

$$\sigma_{\theta\theta}^{(1)} = -\frac{2\mu m a U_0 t^{\alpha-1}}{\pi^2} \int_0^\infty \int_0^\infty E_{\alpha,\alpha}\left[-a(\xi^2+\eta^2)t^\alpha\right] \times \frac{\xi\,\eta}{\xi^2+\eta^2} \left[\left(\xi^2+\eta^2\right) J_0(r\xi) - \frac{\xi}{r} J_1(r\xi)\right] \sin(z\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{46}$$

$$\sigma_{zz}^{(1)} = -\frac{2\mu m a U_0 t^{\alpha-1}}{\pi^2} \int_0^\infty \int_0^\infty E_{\alpha,\alpha}\left[-a(\xi^2+\eta^2)t^\alpha\right] \times \frac{\xi^3\eta}{\xi^2+\eta^2} J_0(r\xi)\,\sin(z\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{47}$$

$$\sigma_{rz}^{(1)} = \frac{2\mu m a U_0 t^{\alpha-1}}{\pi^2} \int_0^\infty \int_0^\infty E_{\alpha,\alpha}\left[-a(\xi^2+\eta^2)t^\alpha\right] \times \frac{\xi^2\,\eta^2}{\xi^2+\eta^2} J_0(r\xi)\,\cos(z\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta. \tag{48}$$

It follows from the load free condition (41) that

$$A = -(1-2\nu)B, \tag{49}$$

$$B = \frac{m a U_0 t^{\alpha-1}}{\pi^2} \int_0^\infty E_{\alpha,\alpha}\left[-a\left(\xi^2+\eta^2\right)t^\alpha\right] \frac{\eta^2}{\xi^2(\xi^2+\eta^2)}\,\mathrm{d}\eta. \tag{50}$$

To investigate several particular cases of the obtained solution it is convenient to pass to polar coordinates in the (ξ,η)-domain:

$$\xi = \varrho\cos\vartheta, \qquad \eta = \varrho\sin\vartheta.$$

Equation (43) for temperature is rewritten as

$$(51)\quad \begin{aligned} T \;=\; & \frac{aU_0 t^{\alpha-1}}{\pi^2} \int_0^\infty \varrho^3 E_{\alpha,\alpha}(-a\varrho^2 t^\alpha)\,\mathrm{d}\varrho \\ & \times \int_0^{\pi/2} J_0(r\varrho\cos\vartheta)\,\sin(z\varrho\sin\vartheta)\,\sin\vartheta\,\cos\vartheta\,\mathrm{d}\vartheta. \end{aligned}$$

Substitution $v = \cos\vartheta$ and evaluation of the arising integral [19]

$$\int_0^1 x\,\sin\left(b\sqrt{1-x^2}\right)\,J_0(cx)\,\mathrm{d}x = \frac{b\sin\left(\sqrt{b^2+c^2}\right)}{(b^2+c^2)^{3/2}} - \frac{b\cos\left(\sqrt{b^2+c^2}\right)}{b^2+c^2}$$

allows us to obtain

$$(52)\quad \begin{aligned} T \;=\; & \frac{aU_0 t^{\alpha-1} z}{\pi^2 (r^2+z^2)^{3/2}} \int_0^\infty \varrho\, E_{\alpha,\alpha}(-a\varrho^2 t^\alpha) \\ & \times \left[\sin\left(\varrho\sqrt{r^2+z^2}\right) - \varrho\sqrt{r^2+z^2}\cos\left(\varrho\sqrt{r^2+z^2}\right)\right]\mathrm{d}\varrho. \end{aligned}$$

Similarly,

$$(53)\quad \begin{aligned} \Phi = & -\frac{maU_0 t^{\alpha-1} z}{\pi^2 (r^2+z^2)^{3/2}} \int_0^\infty \frac{1}{\varrho}\, E_{\alpha,\alpha}(-a\varrho^2 t^\alpha) \\ & \times \left[\sin\left(\varrho\sqrt{r^2+z^2}\right) - \varrho\sqrt{r^2+z^2}\cos\left(\varrho\sqrt{r^2+z^2}\right)\right]\mathrm{d}\varrho. \end{aligned}$$

Now we investigate several particular cases of the obtained solution. For classical thermoelasticity $\alpha = 1$ and $E_{1,1}(-x) = \mathrm{e}^{-x}$. Taking into account that

$$\int_0^\infty x^2\,\mathrm{e}^{-a^2x^2}\cos(bx)\,\mathrm{d}x = \frac{\sqrt{\pi}}{4a^3}\left(1-\frac{b^2}{2a^2}\right)\exp\left(-\frac{b^2}{4a^2}\right), \quad a>0,$$

and [18]

$$\int_0^\infty x\,\mathrm{e}^{-a^2x^2}\,\sin(bx)\,\mathrm{d}x = \frac{\sqrt{\pi}b}{4a^3}\,\exp\left(-\frac{b^2}{4a^2}\right), \quad a>0,$$

$$\int_0^\infty \frac{1}{x}\,\mathrm{e}^{-a^2x^2}\,\sin(bx)\,\mathrm{d}x = \frac{\pi}{2}\,\mathrm{erf}\left(\frac{b}{2a}\right), \quad a>0,$$

$$\int_0^\infty \mathrm{e}^{-a^2x^2}\,\cos(bx)\,\mathrm{d}x = \frac{\sqrt{\pi}}{2a}\,\exp\left(-\frac{b^2}{4a^2}\right), \quad a>0,$$

we obtain

$$(54)\quad T = \frac{U_0 z}{8\pi^{3/2}a^{3/2}t^{5/2}}\,\exp\left(-\frac{r^2+z^2}{4at}\right),$$

(55) $$\Phi = -\frac{maU_0 z}{2\pi R^{3/2}}\left[\operatorname{erf}\left(\frac{R}{2\sqrt{at}}\right) - \frac{R}{\sqrt{\pi a t}}\,\exp\left(-\frac{R^2}{4at}\right)\right], \quad R = \sqrt{r^2+z^2},$$

(56) $$B = \frac{maU_0}{2\pi\xi}\left[\frac{1}{\sqrt{\pi a t}\xi}\,\mathrm{e}^{-at\xi^2} - \operatorname{erfc}\left(\sqrt{at}\xi\right)\right].$$

In the case of heat conduction with $\alpha = 1/2$

(57) $$E_{1/2,\,1/2}(-x) = \frac{2}{\sqrt{\pi}}\int_0^\infty \mathrm{e}^{-u^2-2ux}u\,\mathrm{d}u$$

and

(58) $$T = \frac{U_0 z}{16\sqrt{2}\pi^2 a^{3/2} t^{7/4}}\int_0^\infty \frac{1}{v^{3/2}}\,\exp\left(-v^2 - \frac{r^2+z^2}{8va\sqrt{t}}\right)\mathrm{d}v,$$

(59) $$\begin{aligned}\Phi = &-\frac{maU_0 z}{\pi^{3/2}R^{3/2}\sqrt{t}}\int_0^\infty \left[\operatorname{erf}\left(\frac{R}{2^{3/2}t^{1/4}\sqrt{av}}\right)\right.\\ &\left.-\frac{R}{\sqrt{2\pi a v}t^{1/4}}\,\exp\left(-\frac{R^2}{8a\sqrt{t}v}\right)\right]\mathrm{d}v,\end{aligned}$$

(60) $$B = \frac{maU_0}{\pi^{3/2}\sqrt{t}\,\xi}\int_0^\infty v\mathrm{e}^{-v^2}\left[\frac{\mathrm{e}^{-2a\sqrt{t}v\xi^2}}{\sqrt{2\pi a v}t^{1/4}\xi} - \operatorname{erfc}\left(\sqrt{2av}t^{1/4}\xi\right)\right]\mathrm{d}v.$$

Using the integral transform technique, similar results can be obtained for other types of boundary conditions for the time-fractional heat conduction equation.

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