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Reliability Estimation of Burr Type III Distribution under Improved Adaptive Progressive Censoring with Application to Surface Coating

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Highlights

- The *SSRe* is proposed to determine the probability of the coating process working properly.
- Based on the Burr III distribution, the *IAPrgCS-II* is applied as a novel censoring scheme.
- The *SSRe* model parameters are attained using frequentist and Bayesian aspects.
- The BEs perform relatively better than ML estimates in the reliability model environment.
- The results show us that the *SSRe* can be used to amend the quality of coatings.

Abstract

The stress-strength reliability (*SSRe*) model is widely investigated in reliability engineering to determine the probability of the strength component overcomes the stress imposed on it. In this paper, we studied the estimation of *SSRe* model based on the Burr III distribution under the improved adaptive progressive type-II censoring scheme (*IAPrgCS-II*). Estimation methods of the *SSRe* parameters are developed using frequentist and Bayesian approaches. The point and interval estimations using the maximum likelihood are considered to estimate the parameters. Two approximations are applied to compute the Bayes estimates. A simulation study is conducted for the comparison of the methods of estimation. Also, parallel to the development of reliability studies, it is necessary to study its application in different sciences such as engineering. Therefore, the droplet splashing (*DrS*) data under two wettabilities are proposed as an application of the considered *SSRe* model and methods. The results show us that the reliability model can be used to amend the quality of coatings.

Keywords

Bayesian and Frequentist Estimators; Burr III; Improved Adaptive Progressive; Stress-Strength Reliability.

1. Introduction

The reliability model has many applications in engineering studies, such as the strength of pressure vessels, fatigue failure of chemical equipment structures, and computer ring network systems. The *SSRe* model is defined as the probability of the strength component (X) overcomes the stress (Y) imposed on it, namely $SSRe = P(X > Y)$. This reliability system is introduced by Church and Harris [9], and since then statistical inference of *SSRe* model has been continuing to be studied under different assumptions. For some recent studies about this topic, one can see, Bhattacharyya [6], Rao et al. [22], Akgul and Senoglu [1], Bai et al. [5], Asadi and Panahi [2], De La Cruz [11] and Demiray and Kizilaslan [12]. Also, one of the important applications of this reliability model is to evaluate the strength of coated microlayers on the surfaces of mechanical devices. In surface coating operations, the fluid breaks up into very fine droplets by the nozzle and then sprayed onto the surface. The adhesion of these droplets depends on the wettability of the surface (Figure 1). The effect of surface wettability (*SWet*) on the coating strength and resistance can be evaluated by the *SSRe* model. Moreover, due to substantial improvement of science and technology, we cannot get the adequate number of failure times of some units which are put on the experiment. Therefore, experimenters deal with the censored sample. The forms of censoring schemes (CS) are varied. The most frequently adopted censoring schemes are type-I, type-II and hybrid (*Hy*) censoring.

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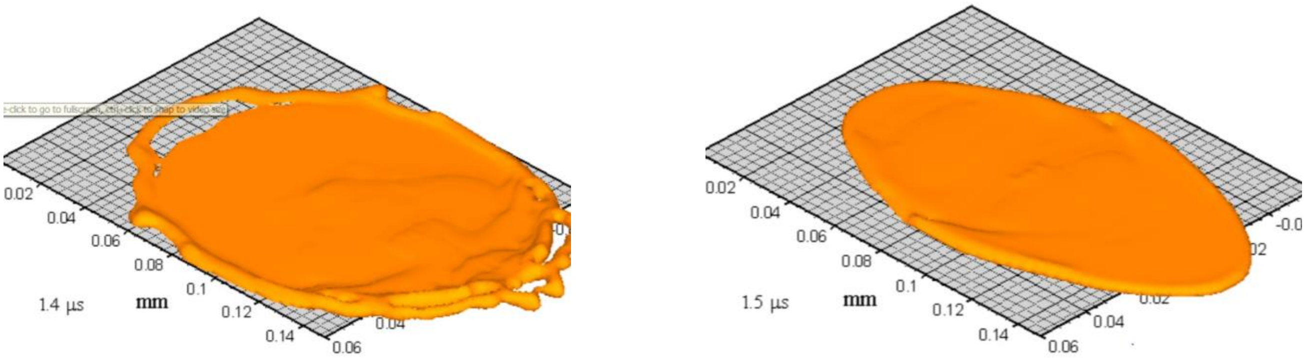

Figure 1. The view of droplet impact onto partial wettable (right) and wettable (left) surfaces

The advent of progressive censoring (*PrC*) schemes ([4]) has greatly improved the two situations above, which means that the random removal of survival units with pre-fixed numbers happens on the basis of type-I or type-II censoring at each failure. Later, generalizations of the *PrC* known as progressive hybrid and adaptive *PrC* schemes were proposed by Kundu and Joarder [16] and Ng et al. [20] respectively. The above censoring schemes have been studied by many researchers based on different distributions. Childs et al. [7] obtained the exact inference for the exponential distribution based on type-I and type-II hybrid CSs. Lee and Seo [17] obtained the estimates for the Gompertz parameters under progressive (*Pr*) censored sample. Similar studies were considered by Ferreira and Silva [14] for Weibull distribution when right *CS* is available, Starling et al. [23] for improving Weibull distribution using generalized type-I *CS*, Chiou and Chen [8] for lifetime performance index under type-II CS. Also, for accelerated life-time reliability model (*ALTM*), Wang et al [24] and Asadi et al. [3] considered estimation methods under *PrC* and adaptive *PrC* respectively. Further, Lone and Panahi [18] studied the *ALTM* based on the Gompertz unified *Hy* censored data. Moreover, the developments and needs in engineering, manufacturing and technology inspire more improved censoring schemes (CS). Recently, Yan et al. [25] introduced a novel CS called the *IAPrgCS-II* which is arose in a reliability studies as follows: An experiment starts with n identical items, prefixed effective number$m$; $(m < n)$ and the prefixed PrC $R = (R_1, R_2, \dots, R_m)$. The R_i at the time of i^{th} failure may change during the test. Let T_1 and T_2, be the two time points, where, $T_1 < T_2$. Based on *IAPrgCS-II*, there are three Cases of observations as follow ([25]):

Case I: $\{X_{1:m:n}, X_{2:m:n}, \dots, X_{m:m:n}\}$ if $X_{m:m:n} < T_1 < T_2$,

Case II: $\{X_{1:m:n}, \dots, X_{k_1:m:n}, \dots, X_{m:m:n}\}$ if $T_1 < X_{m:m:n} < T_2$,

Case III: $\{X_{1:m:n}, \dots, X_{k_1:m:n}, \dots, X_{k_2:m:n}\}$ if $T_1 < T_2 < X_{m:m:n}$.

Where $(k_1 \& \ k_1 + 1 < m)$ and k_2 are the number of failures before times T_1 and T_2 respectively. So, the joint likelihood function of the three forms of failure times is defined as:

$$L(parameters|data) = A \prod_{i=1}^{j_2} f(x_{i:m:n}) \prod_{i=1}^{j_1} (1 - F(x_{i:m:n}))^{R_i} (1 - f(T^*))^{R*}, \quad (1)$$

where, (j_1, j_2, R^*, T^*) is $(m, m, 0, 0)$, $(k_1, m, n - m - \sum_{i=1}^{k_1} R_i, x_{m:m:n})$ and $(k_1, k_2, n - k_2 - \sum_{i=1}^{k_1} R_i, T_2)$ for cases I, II and III respectively. This scheme guarantees that the experimental time will not exceed a prefixed time. The Burr III is one of the most widely used distributions in the reliability studies. Let X be a random variable follows the Burr III, then, the PDF, $f(.)$,and CDF $F(.)$, of X are given, respectively, by

$$f(x; \alpha, \beta) = \frac{\alpha\beta x^{-\beta-1}}{(1+x^{-\beta})^{\alpha+1}}; \ x > 0; \ \alpha \text{ and } \beta > 0, \quad (2)$$

and

$$F(x; \alpha, \beta) = (1 + x^{-\beta})^{-\alpha}; \ x > 0; \ \alpha \text{ and } \beta > 0. \quad (3)$$

Recently, more and more scholars have turned attention to Burr III distribution. Cordeiro et al. [10] considered the model estimates under *Pr* censored sample. Panahi [21] used the different approaches to attain the estimations for the parameters of this distribution under unified *Hy* censored sample. Dutta and Kayal [13] obtained the estimations of this model based on unified *Pr* hybrid censoring. Also, the use of different types of data in reliability encourages researchers to study the application of reliability models in different sciences. In this study, we try to expand the applications of the reliability model in engineering field. Hence, this study aims to look into the reliability model through different estimation methods based on *IAPrgCS-II* and consider its application in the coating process. As far as we know, the classical and Bayesian estimates of the reliability model have not been studied under *IAPrgCS-II*. Therefore, we consider the estimation of *SSRe*, in the case of each component has a Burr III distribution with common shape parameter β and both are exposed to *IAPrgCS-II*. The *SSRe* model parameters are acquired using maximum likelihood estimators (MLEs). Using the normal approximation, approximate confidence intervals (*ApCIs*) for the *SSRe's* parameters are obtained. The Lindley's approximation (*LiA*) and Markov Chain Monte Carlo via Metropolis-Hastings (*MeHa*) algorithm are used for obtaining the BEs. The *MeHa* technique is considered to compute the associated credible intervals (*CrI*s). Another importance is the implementation of the obtained methods to the *DrS* data under two *SWets*. For this aim, droplet splashing data under two *SWets* are compared by using the considered *SSRe* model results. The ML and *ApCIs* estimates for the *SSRe* are derived in section 2. In section 3, the *LiA* technique and MCMC method with *MeHa* algorithm are applied to acquire the BEs of *SSRe*. In section 4, we offer a simulation study, under which the above estimation methods are comparatively analyzed. In section 5, an illustration of how the proposed model and methods may be utilized in engineering problems is presented with the analysis of the *DrS* data. A summary and some conclusions are given in section 6.

2. Model Description and Classical Inference

The *SSRe* model is widely utilized in the reliability engineering to determine the probability of the system working properly. In other words, a system will properly work as long as X exceeds Y. For example, consider X as the pressure inside the chemical equipment and Y as the strength of the equipment wall. Then *SSRe* represents the probability of damage to the equipment. Let the model consists of strength variable such that $X \sim BIII(\alpha,\beta)$ and independent stress variable $Y \sim BIII(\lambda,\beta)$. Then, the *SSRe* model can be written as:

$$SSRe = P(X > Y) = \int_0^\infty P(X > Y|Y = y)\, f_Y(y)dy$$
$$= \int_0^\infty f_Y(y)F_Y(x)\, dy = \frac{\lambda}{\alpha+\lambda}, \quad (4)$$

which is a continous function of *SSRe* parameters (Figure 2).

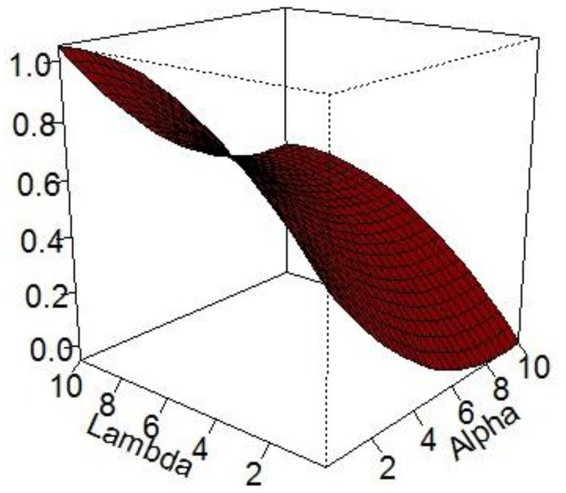

Figure 2. The 3D plot of SSRe model.

Now, let $\underset{\frown}{X} = (X_{1:m_1:n_1},\dots,X_{j_1:m_1:n_1},\dots,X_{j_2:m_1:n_1})$ and $\underset{\frown}{Y} = (Y_{1:m_2:n_2},\dots,Y_{h_1:m_2:n_2},\dots,Y_{h_2:m_2:n_2})$ be *IAPrgCS-II* samples from the $BIII(\alpha,\beta)$ and $BIII(\lambda,\beta)$ with CSs $R_1 = (R_1,\dots,R_{j_1},0,\dots,0,R_1^*)$ and $R_2 = (R_1,\dots,R_{h_1},0,\dots,0,R_2^*)$ respectively. Where, (R_1^*,R_2^*) for different *IAPrgCS-II* cases are (0,0), $(n_1 - m_1 - \sum_{i=1}^{k_{1x}} R_i\,, n_2 - m_2 - \sum_{i=1}^{k_{1y}} R_i)$ and $(n_1 - k_{2_x} - \sum_{i=1}^{k_{1x}} R_i\,, n_2 - k_{2_y} - \sum_{i=1}^{k_{1x}} R_i)$ respectively. The joint PDF of the observed sample, labeled$L(\alpha,\beta,\lambda|data)$, is:

$$L(\alpha,\beta,\lambda|data) \propto \prod_{i=1}^{j_2}\left(\alpha\beta x_{i:m_1:n_1}^{-\beta-1}(1 + x_{i:m_1:n_1}^{-\beta})^{-\alpha-1}\right)\prod_{i=1}^{j_1}\left(1-(1+x_{i:m_1:n_1}^{-\beta})^{-\alpha}\right)^{R_i}\left(1-(1+T_1^{*-\beta})^{-\alpha}\right)^{R_1^*} \times \prod_{i=1}^{h_2}\left(\lambda\beta y_{i:m_2:n_2}^{-\beta-1}(1+y_{i:m_2:n_2}^{-\beta})^{-\lambda-1}\right)\prod_{i=1}^{h_1}\left(1-(1+y_{i:m_2:n_2}^{-\beta})^{-\lambda}\right)^{R_i}\left(1-(1+T_2^{*-\beta})^{-\lambda}\right)^{R_2^*}, \quad (5)$$

where, the values of $j_1, j_2, h_1, h_2, T_1^*$and T_2^* are defined in Table 1

Table 1. Various choices of $j_1, j_2, h_1, h_2, T_1^*$and T_2^*.

$\boldsymbol{Cases}$	$\boldsymbol{j_1}$	$\boldsymbol{j_2}$	$\boldsymbol{h_1}$	$\boldsymbol{h_2}$	$\boldsymbol{T_1^*}$	$\boldsymbol{T_2^*}$
I	m_1	m_1	m_2	m_2	0	0
II	k_{1_x}	m_1	k_{1_y}	m_2	$x_{m_1:m_1:n_1}$	$y_{m_2:m_2:n_2}$
III	k_{1_x}	k_{2_x}	k_{1_y}	k_{2_y}	T_{2_x}	T_{2_y}

Therefore, $ln\, L\,(\alpha,\beta,\lambda|data)$is obtained as:

$$ln\, L\,(\alpha,\beta,\lambda|data) = l(\alpha,\beta,\lambda|data) = j_2\, ln\,\alpha + h_2\, ln\,\lambda + (j_2 + h_2)\, ln\,\beta - (\beta+1)\left(\sum_{i=1}^{j_2} ln\, x_{i:m_1:n_1} + \sum_{i=1}^{h_2} ln\, y_{i:m_2:n_2}\right) - (\alpha+1)\sum_{i=1}^{j_2} ln\,\aleph_{i:m_1:n_1} + \sum_{i=1}^{j_1} R_i\, ln(1 - (\aleph_{i:m_1:n_1})^{-\alpha}) + R_1^* \ln(1 - (\wp_1)^{-\alpha}) - (\lambda + 1)\sum_{i=1}^{h_2} ln\, P_{i:m_2:n_2} + \sum_{i=1}^{h_1} R_i\, ln(1 - (P_{i:m_2:n_2})^{-\lambda}) + R_2^* \ln(1 - (\wp_2)^{-\lambda}) \quad (6)$$

Here, $\aleph_{i:m_1:n_1} = 1 + x_{i:m_1:n_1}^{-\beta}, P_{i:m_2:n_2} = 1 + y_{i:m_2:n_2}^{-\beta}, \wp_1 = 1 + T_1^{*-\beta}$ and $\wp_{2_2} = 1 + T_2^{*-\beta}$

The MLEs can be obtained by taking derivatives of α, $\beta$and λ and make them equal to 0 . Using the invariance property of the MLE helps us to obtain MLE of *SSRe*, denoted by $\widehat{SSRe}$ as replacing the parameters in Equation (4) with their estimates to be found by using the $\hat{\alpha}$ and $\hat{\lambda}$. That is the $\widehat{SSRe}$ can be written as:

$$\widehat{SSRe} = \frac{\hat{\lambda}}{\hat{\alpha}+\hat{\lambda}}. \quad (7)$$

2.1. *ApCI* for *SSRe* Based on MLE

In this subsection, we construct *ApCI* for the *SSRe* parameter as an approximate confidence interval. Theoretically speaking, the variance-covariance of $\eta = (\alpha,\beta,\lambda)$ can be derived as follows:

$$I(\alpha,\beta,\lambda) = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}, I_{ij} = \frac{-\partial^2 l(\alpha,\beta,\lambda|data)}{\partial\theta_i\partial\theta_j}; i,j = 1,2,3,$$

where, $I_{ij};\ i,j = 1,2,3$ are given in the Appendix. It is worth mentioning that the expected values in the Fisher information matrix cannot be obtained explicitly since the distribution of the MLEs under the *IAPrgCS-II* cannot be obtained explicitly. Therefore, the observed information matrix $I(\alpha,\beta,\lambda)$ is applied in the asymptotic normality of the MLE. Based on regularity conditions, the estimator $\widehat{SSRe}$ asymptotically normal with mean $SSRe$ and variance ξ respectively, where (Rao et al. (2013)),

$$\xi = \frac{Q^*}{|I(\alpha,\beta,\lambda)|}; Q^* = \left(-\frac{\lambda}{(\alpha+\lambda)^2}\right)^2 d_{11} + \left(\frac{\alpha}{(\alpha+\lambda)^2}\right)^2 d_{22} - 2\left(\frac{\lambda}{(\alpha+\lambda)^2}\right)\left(\frac{\alpha}{(\alpha+\lambda)^2}\right) d_{12},$$

and

$$I^{-1}(\alpha,\beta,\lambda) = \frac{1}{|I(\alpha,\beta,\lambda)|}\begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix} = \frac{1}{|I(\alpha,\beta,\lambda)|}\begin{bmatrix} I_{22}I_{33} - I_{23}^2 & I_{13}I_{23} & -I_{13}I_{22} \\ I_{13}I_{23} & I_{11}I_{33} - I_{13}^2 & -I_{11}I_{23} \\ -I_{13}I_{22} & -I_{11}I_{23} & I_{11}I_{22} \end{bmatrix}. \quad (10)$$

2.2. Testing Problem

In this article, the shape parameters (β) of the stress and strength (S-S) variables are considered equal, which is an important issue of practical opinion. Therefore, in this section, this issue has been considered using the likelihood ratio test

(*LIRT*). Suppose that the S-S variables follow the Burr III distributions with shape parameters β_1, β_2 and the following hypothesis is considered as:

$H_0: \beta_1 = \beta_2 = \beta$ versus $H_1: \beta_1 \neq \beta_2$.

Based on large sample size, the *LIR* statistic is:

$$\eta = -2\{l_1(\hat{\beta}_1, \hat{\beta}_2|data) - l_2(\breve{\beta}|data)\} \sim \chi_1^2. \quad (11)$$

Where, $l_1(\hat{\beta}_1, \hat{\beta}_2|data)$ and $l_2(\breve{\beta}|data)$ are the log-likelihood functions based on $(\hat{\beta}_1, \hat{\beta}_2)$ and common parameter $(\breve{\beta})$ cases, respectively. The *LIRT* can be obtained using the asymptotic distribution of η.

3. Bayesian estimation

Different from frequency estimation, Bayesian estimation treats parameters as unknown random variables rather than unknown deterministic variables. Results from the Bayesian method are always better than the results from classical approaches for the reason of considering more known information. In this section, approximate BEs of *SSRe* model are attained when all the parameters α, β and λ have independent gamma (InG) distributions with parameters $(a_i, b_i); i = 1,2,3$ respectively, as prior distributions. Note that the gamma distribution is versatile for adjusting different shapes of the density function. Jeffery's prior can be obtained as a special case of the gamma prior. The squared error (*SqE*) loss function is proposed. The *SqE* is a balance type loss function. Based on this loss function, we can provide equal over and underestimates to the actual value of parameter. So, we prefer the *SqE* to compute the BEs. Then, the joint posterior PDF of the three various parameters is indicated as below:

$$\pi(\alpha,\beta,\lambda|data) = \frac{L(\alpha,\beta,\lambda|data)\pi(\alpha,\beta,\lambda)}{\int_0^\infty\int_0^\infty\int_0^\infty L(\alpha,\beta,\lambda|data)\pi(\alpha,\beta,\lambda)d\alpha d\beta d\lambda}$$
$$\propto \alpha^{j_2+a_1-1}e^{-\alpha(b_1+\sum_{i=1}^{j_2}ln(\aleph_{i:m_1:n_1}))}\lambda^{h_2+a_2-1}e^{-\lambda(b_2+\sum_{i=1}^{h2}ln(P_{i:m_2:n_2}))}$$
$$\times \beta^{j_2+h_2+a_3-1}e^{-\beta(b_3+\sum_{i=1}^{j_2}ln\,x_{i:m_1:n_1}+\sum_{i=1}^{h_2}ln\,y_{i:m_2:n_2})}$$
$$\prod_{i=1}^{j_2}\left(x_{i:m_1:n_1}(\aleph_{i:m_1:n_1})\right)^{-1}\times\prod_{i=1}^{j_1}\left((1-(\aleph_{i:m_1:n_1})^{-\alpha})^{R_i}\right)\times (1-(\wp_1)^{-\alpha})^{R_1^*}\prod_{i=1}^{h_2}\left(y_{i:m_2:n_2}(P_{i:m_2:n_2})\right)^{-1}\prod_{i=1}^{h_1}\left(1-(P_{i:m_2:n_2})^{-\lambda}\right)^{R_i}\left(1-(\wp_2)^{-\lambda}\right)^{R_2^*}. \quad (12)$$

Using the *SqE* loss function, the BEs of the *SSRe*, can be written as

$$SSR\hat{e}_{Bayes_{SE}}\int_0^\infty\int_0^\infty\int_0^\infty \widehat{SSRe}\times\pi(\alpha,\beta,\lambda|data) \quad (13)$$

Two approximation methods, *LiA* technique and *MeHa* algorithm are applied to obtain BE of *SSRe* due to the multiple integrals in Equation (13) are not obtained analytically and difficulties in numerical computations of these integrals.

3.1. Lindley's Approximation (*LiA*)

In this subsection, we discuss the BEs of *SSRe* using *LiA* technique. This approximation method saves computation time for the Bayesian method and we can also obtain mean and variance of *SSRe* explicitly. The detailed derivations are omitted to maintain brevity. With respect to *SqE* function, the BE of *SSRe* leads to:

$$SSRe^{LIN} = \widehat{SSRe} + \frac{1}{2}\sum_{i=1}^{3}\sum_{i=1}^{3}(u_{ij}+2u_i\rho_j)\,\sigma_{ij} + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{h=1}^{3} l_{ijk}\,\sigma_{ij}\sigma_{kh}\,u_h\,|\xi=\hat{\xi}$$
$$= \frac{\hat{\lambda}}{\hat{\alpha}+\hat{\lambda}} + \frac{1}{2}[\Psi(u_1\sigma_{11}+u_2\sigma_{12}) + Z(u_1\sigma_{21}+u_2\sigma_{22}) + T(u_1\sigma_{31}+u_2\sigma_{32})]$$
$$+u_1(\rho_1\sigma_{11}+\rho_2\sigma_{12}+\rho_3\sigma_{13}) + u_2(\rho_1\sigma_{21}+\rho_2\sigma_{22}+\rho_3\sigma_{21}) + u_{12}\sigma_{12} + \frac{1}{2}(u_{11}\sigma_{11}+u_{22}\sigma_{22}),$$

obtained at $\hat{\xi} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})$. Some elements of $SSRe^{LIN}$ are:

$\Psi = \sigma_{11}l_{111} + \sigma_{33}l_{331}$, $Z = \sigma_{22}l_{222} + \sigma_{33}l_{332}$ and $T = 2\sigma_{13}l_{133} + 2\sigma_{23}l_{233} + \sigma_{33}l_{333}$,

$$u_1 = -\frac{\lambda}{(\alpha+\lambda)^2}, u_2 = \frac{\alpha}{(\alpha+\lambda)^2}, u_{11} = \frac{2\lambda}{(\alpha+\lambda)^3}, u_{12} = u_{21} = \frac{\lambda-\alpha}{(\alpha+\lambda)^3}, u_{22} = -\frac{2\alpha}{(\alpha+\lambda)^3},$$

$$u_3 = 0, u_{13} = u_{23} = u_{31} = u_{32} = 0, \rho_1 = \frac{a_1-1}{\alpha} - b_1, \rho_2 = \frac{a_2-1}{\lambda} - b_2, \rho_3 = \frac{a_3-1}{\beta} - b_3, l_{111} = \frac{\partial l(\alpha,\beta,\lambda|data)}{\partial\alpha^3},$$

$$l_{122} = \frac{\partial l(\alpha,\beta,\lambda|data)}{\partial\alpha\partial\lambda^2}, l_{222} = \frac{\partial l(\alpha,\beta,\lambda|data)}{\partial\lambda^3}, l_{333} = \frac{\partial l(\alpha,\beta,\lambda|data)}{\partial\beta^3}, l_{123} = \frac{\partial l(\alpha,\beta,\lambda|data)}{\partial\alpha\partial\lambda\partial\beta},$$

Also, σ_{ij} is $(i,j)^{th}$ element of the inverse of the matrix $[-l_{ij}]$. One disadvantage of *LiA* technique is that it requires higher order partial (*HOP*) derivatives of the $l(\alpha,\beta,\lambda|data)$. Further, the *LiA* cannot be applied to create the *CrI*s due to lack of an explicit form of the PDF for *SSRe*. So, the MCMC via *MeHa* algorithm is used to derive another approximate BE and constructed the *CrI*s. The *MeHa* algorithm is free from *HOP* derivatives.

3.2. MCMC Method

As mentioned earlier, the *LiA* has some disadvantages. In many cases, due to the complexity of the form of the PDF and the applied CS, the calculation of the *HOP* derivatives is complicated. That is why it motivates us to consider the more flexible method in the BEs for *SSRe*. The important sampling (*ImS*) and *MeHa* are two approaches of the MCMC. It is easy to calculate the BEs based on the *ImS* method, which is important in practice. But if the form of the PDF becomes complicated, the conditional posterior distribution of the parameters cannot be easily attained, so the *MeHa* algorithm is a more appropriate choice for calculating the BEs. This algorithm is an efficient method to calculate the BEs which can be easily conducted and help to reduce the operation complexity of high dimensionel distribution. It is an attractive approach to set up the Markov chain from the conditional distribution of each parameter. So, the BEs and the corresponding *CrI*s of the *SSRe* based on *IAPrgCS-II* are computed by using the *MeHa* ([15,19]) algorithm. To apply this method, the marginal posterior PDFs of the α, β and λ can be re-expressed as:

$$\pi_1(\alpha|\beta,\lambda,data) = \alpha^{j_2+a_1-1}e^{-\alpha(b_1+\sum_{i=1}^{j_2}ln(\aleph_{i:m_1:n_1}))}\times\prod_{i=1}^{j_1}\left((1-(\aleph_{i:m_1:n_1})^{-\alpha})^{R_i}\right)\times(1-(\wp_1)^{-\alpha})^{R_1^*}, \quad (14)$$

and

$$\pi_2(\lambda|\alpha,\beta,data) = \lambda^{h_2+a_2-1}e^{-\lambda(b_2+\sum_{i=1}^{h2}ln(P_{i:m_2:n_2}))}\times$$

$$\prod_{i=1}^{h_1}\left(1-(\mathrm{P}_{i:m_2:n_2})^{-\lambda}\right)^{R_i}\left(1-(\wp_2)^{-\lambda}\right)^{R_2^*} \quad (15)$$

$$\pi_3(\beta|\alpha,\lambda,data)$$
$$=\beta^{j_2+h_2+a_3-1}e^{-\beta(b_3+\sum_{i=1}^{j_2}\ln x_{i:m_1:n_1}+\sum_{i=1}^{h_2}\ln y_{i:m_2:n_2})}$$
$$\times\prod_{i=1}^{j_2}(\aleph_{i:m_1:n_1})^{-1}\prod_{i=1}^{j_1}\left((1-(\aleph_{i:m_1:n_1})^{-\alpha})^{R_i}\right)$$
$$\times(1-(\wp_1)^{-\alpha})^{R_1^*}\prod_{i=1}^{h_2}\left(\mathrm{P}_{i:m_2:n_2}\right)^{-1}\times$$
$$\prod_{i=1}^{h_1}\left(1-(\mathrm{P}_{i:m_2:n_2})^{-\lambda}\right)^{R_i}\left(1-(\wp_2)^{-\lambda}\right)^{R_2^*}. \quad (16)$$

It is clearly seen that the Equations (14)-(16) do not show standard forms and therefore it is not possible to obtain sample directly by standard process. In that case, if the posterior density function is roughly symmetric, a normal distribution can be used to approximate it. So, we will use the *MeHa* technique to calculate the BE of the *SSRe*.

The MeHa Algorithm:

Step 1: Set $i=1$ and Start by using the initial values of $(\hat{\alpha},\hat{\beta},\hat{\lambda})$.

Step 2: Use *MeHa* steps to generate $\alpha^{(i)}$, $\beta^{(i)}$ and $\lambda^{(i)}$ from the distributions$N(\alpha^{(i-1)},Variance(\alpha))$,$N(\beta^{(i-1)},Variance(\beta))$ and $N(\lambda^{(i-1)},Variance(\lambda))$ respectively, for $i=1,\dots,M$.

Step 3: Calculate the $SSRe^{(i)}$ as, $\frac{\lambda^{(i)}}{\alpha^{(i)}+\lambda^{(i)}}$.

Step 4: Set $i=i+1$.

Step 5: Repeat steps 1-4, *M* times and obtain $SSRe^{(1)},\dots,SSRe^{(M)}$.

Step 6,7: The BE of $SSRe$is shown as:

$$\widehat{SSRe}_{MH_{SE}}=\frac{1}{M-NB}\sum_{i=NB+1}^{M}SSRe^{(i)};\quad \text{where}, NB\ is\ burn-in\ period.$$

Step 8: Order $SSRe^{(i)};\ i=1,..,M$, then the $100(1-\gamma)\%$ *CrI*s of $SSRe$is given as follows:$\left[\widehat{SSRe}_{(M\gamma/2)},\widehat{SSRe}_{(M(1-\gamma/2))}\right]$.

4. Simulation Study

This section is devoted to the comparative study of the proposed estimates under different *IAPrgCS-II* censored schemes. Different estimation methods of *SSRe* parameters are applied using frequency and Bayesian estimations. For comparson and analysis, 10000 cycles of simulations are repeated for the whole process. We consider values (α,β,λ) as (3.5, 2, 4) and (2, 1, 6) to provide values of *SSRe* as 0.533 ($SSRe_1$) and 0.750 ($SSRe_2$), respectively. Two different $T_i;\ i=1,2$ values: $T_1=0.7,T_2=1.2$ and $T_1=1.2,T_2=2.5$are taken for both S-S variables. Based on *IAPrgCS-II*, the observed data (*OD*) and the removed items during the test (*RIDT*) are respectively as the following Cases (*I, II, III*):

I: The *OD* and *RIDT* are $X_{l:m_1:n_1}$ (or $Y_{l:m_2:n_2};\ l=1,..,m_2$) and $n_1-m_1-\sum_{l=1}^{m_1-1}R_l$ (or $n_2-m_2-\sum_{l=1}^{m_2-1}R_l$).

II: The *OD* and *RIDT* are $X_{l:m_1:n_1};l=1,..,m_1$ (or $Y_{l:m_2:n_2};l=1,..,m_2$) and $n_1-m_1-\sum_{l=1}^{k_{1x}}R_l$ (or $n_2-m_2-\sum_{l=1}^{k_{1y}}R_l$).

III: The *OD* and *RIDT* are$X_{l:m_1:n_1};\ l=1,..,k_{2_x}$ (or $Y_{l:m_2:n_2};\ l=1,..,k_{2_y}$) and $n_1-k_{2_x}-\sum_{l=1}^{k_{2x}}R_l$ (or $n_2-k_{2_y}-\sum_{l=1}^{k_{2y}}R_l$).

We have formulated three systematic CSs including,

- ***CSI :*** $R=\left(0^{\times(m/2-1)},n-m,0^{\times(m/2)}\right)$,
- ***CSII :*** $\begin{cases}R=\left(3^{\times(m/2)},0^{\times(m/2)}\right); & m/n=50\%,\\ R=\left(1^{\times(n-m)},0^{\times(2m-n)}\right); & m/n=80\%.\end{cases}$
- ***CSIII :***$R=\left(n-m,0^{\times(m-1)}\right)$

In the Bayesian estimations, very small values of the hyper-parameters , *i.e.* $a_i = b_i = 0.0001$ for i = 1, 2, 3 can be considered. But due to the closeness of the BEs under this prior distribution, we prefer informative priors such as $SSRe_1: a_1=7, b_1=2, a_3=2, b_3=1, a_2=8, b_2=2$and $SSRe_2: a_1=2, b_1=1, a_2=12, b_2=2,\ a_3=b_3=1$. We propose four different sets of (*m*,*n*) as (30,15), (30,24), (50, 25) and (50,40). For the *MeHa* algorithm, we take *M* = 10000 and *NB* = 1000. The MSEs and average lengths (ALs) of the *SSRe* for $SSRe_1$ and $SSRe_2$ are presented in Tables 23, 4 and 5. According to the results shown in Tables 1, 2, 3 and 4, we can get some conclusions. The simulation research revealed that as the sample size increased, the MSE values of the various estimates generally decrease which shows the consistancy of estimators. Moreover, average lengths of the *ApCIs* and *CrIs* are decreasing with larger sample sizes. The *CrI*s have also smaller AL than *ApCIs*. The ML estimates and BEs are deserving of comparison. The BEs using *MeHa* algorithm outperforms *LiA* technique in terms of MSEs. Moreover, the BEs using the MCMC method are better than other estimates based on MSEs values. Choosing different parameter values for the S-S components can affect the performance of the estimation, because different parameter values provide different ranges of samples. The values of $T_i;\ i=1,2$ for the S-S variables are assumed to be the same for both components to account for the simultaneous testing time. We can be seen that the increasing of T_1 and T_2 for other fixed values cause smaller MSE values in all cases since it provides more testing-time to let more failure observe. The performances of the estimators are better when *m* increases. Also from the censoring schemes, the CSII seems to be superior to other CSs. From the above analysis of the results, we present the conclusion that the results of the BEs perform better than ML estimates in the proposed model environment.

5. Real Data Analysis

Usage and modeling of *DrS* data have been paying attention to the engineering researchers when we consider the problems in the reducing the coating strength and so damaging of the surfaces. Therefore, a series of the splashing data of silicon oil drops is analyzed here to elucidate the practical application of the model we study in the proceeding sections. This data originaly reported by Asadi and Panahi [2]. The glass surface (GS) and Teflon surface (TS) with sample sizes $n_1=n_2=40$ are considered to acquire the *SSRe* estimates. The K-S statistic and associated *p*-value for GS are 0.106 and 0.716, respectively. The same quantities for the TS data set are 0.109 and 0.688 respectively. Thus, the Burr III is a reasonably good fit for these data. The histograms and PP plots also support this results (see Figures 3 and 4).

If the *SSRe* is greater than 0.50, we can say that the TS will increase the splash of drops. Moreover, if the system reliability is less than 0.50, we will think of the reverse result of the aforementioned scenario. Now, we fit these random samples to Burr III distribution and their MLEs are attained as $\hat{\alpha} = 156.7151, \hat{\beta}_1 = 6.1391, \quad \hat{\lambda} = 267.3969, \hat{\beta}_2 = 7.1174$. We assumed that this distribution has the same shape parameter to illustrate the proposed method. For testing that $H_0 = \beta_1 = \beta_2 = \beta$, we perform a *LIRT* and the respective value is 1.5154. Thus, both stress and strength variables can be modelled by using the Burr III with equal β parameter. The non-information prior is selected to estimate the unknown parameters because the information is relatively limited. The hyper-parameters under the non-informative prior are taken as $a_1 = b_1 = a_3 = b_3 = a_2 = b_2 = 0.00001$, which are close to zero.

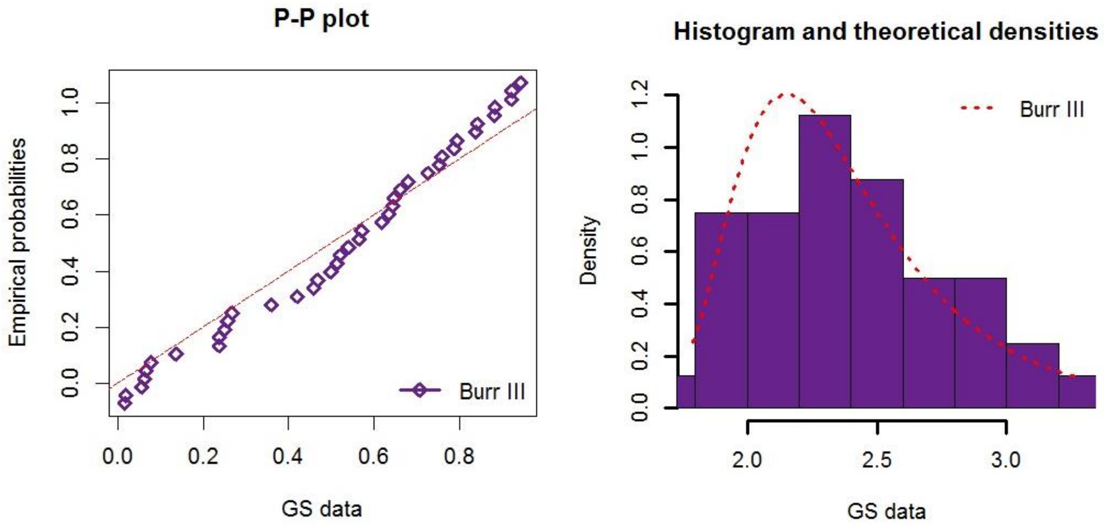

Figure 3. The PP plot and histogram for GS data.

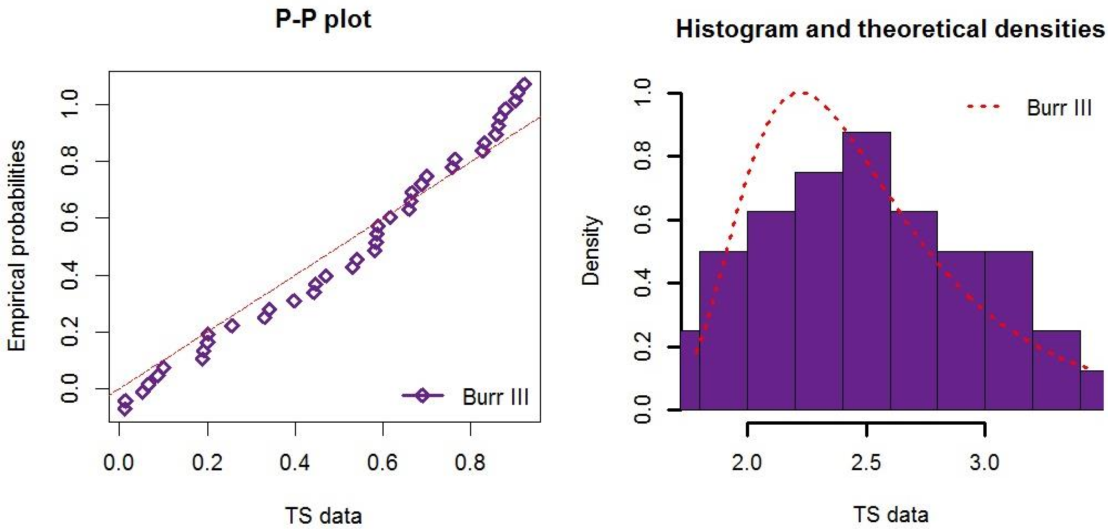

Figure 4. The PP plot and histogram for TS data.

It is observed that since all the estimates of *SSRe* values are greater than 0.50, the GS should be used for decreasing the splashing phenomenon on the considered scenario. We assume three *IAPrgCS-II* censoring plan by considering$R_1 = (0 * 8,5,5,5,5,0 * 8), R_2 = (0 * 10,5,5,5,5,0 * 6)$, as:

SCI: $n_1 = n_2 = 40, \quad m_1 = m_2 = 20, R_1^* = R_2^* = 0$, $T_{1_{TS}}(k_{1_{TS}}) = 2.6(24), T_{2_{TS}}(k_{2_{TS}}) = 2.9(31), T_{1_{GS}}(k_{1_{GS}}) = 2.4(22), T_{2_{GS}}(k_{2_{GS}}) = 2.7(33), T_1^* = T_2^* = 0$.

SCII: $n_1 = n_2 = 40, \quad m_1 = m_2 = 20, R_1^* = R_2^* = 5$, $T_{1_{TS}}(k_{1_{TS}}) = 2.2(11), T_{2_{TS}}(k_{2_{TS}}) = 2.9(31), T_{1_{GS}}(k_{1_{GS}}) = 2.2(13), T_{2_{GS}}(k_{2_{GS}}) = 2.7(33), T_1^* = 2.51604, T_2^* = 2.34780$.

SCIII: $n_1 = n_2 = 40, \quad m_1 = m_2 = 20, \quad R_1^* = R_2^* = 8$, $T_{1_{TS}}(k_{1_{TS}}) = 2.2(11), T_{2_{TS}}(k_{2_{TS}}) = 2.4(17), T_{1_{GS}}(k_{1_{GS}}) = 2.2(13), T_{2_{GS}}(k_{2_{GS}}) = 2.31(17), T_1^* = 2.4, T_2^* = 2.31$.

We also checked the convergnce of the Markov chain by using trace (Tr) and density (De) plots (Figures 6and 7) under SCIII. Figure 5 shows that the Markov Chain fluctuates around its center with similar variations. Moreover, we observe from the Figure 6, the density plot has a symmetric and unimodal shape. The Tr and De plots for other schemes are similar.

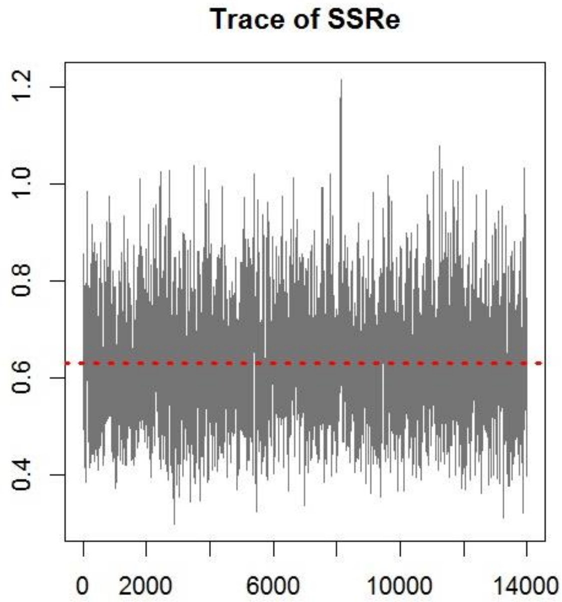

Figure 5. The *Tr* plot of *SSRe*.

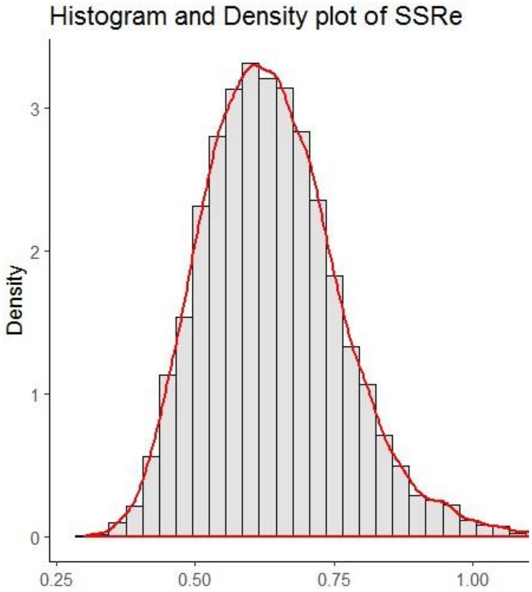

Figure 6. The *De* plot of *SSRe*.

The point estimation results along with 95% *ApCI* and *CrI* are reported in Table 5 for this real data study. We observed close estimates to each other for both estimation methods. Moreover, both approximate Bayes estimates of *SSRe* are similar. The *CrI* lengths (*CrILs*) are obtained smaller than *ApCI* lengths (*ApCILs*). It is observed that since all the estimates of *SSRe* values are greater than 0.50, the TS increases the splashing phenomenon.

6. Conclusions

Based on the potential effectiveness of the *SSRe* in engineering, the reliability modelling and assessment of the system are important. Also, due to advances and needs in engineering, manufacturing and technology, the use of improved censoring schemes is inevitable. Therefore, in this paper, we studied the *SSRe* estimation for Burr III distribution based on *IAPrgCS-II*. The main reason for selecting this censoring plan is that it can guarantee that the test time will not exceed a prefixed time. Plenty of estimation approaches from both frequency and Bayesian schools are applied under *IAPrgCS-II*. In the Bayesian procedure, *LiA* technique and *MeHa* algorithm are used to acquire the *SSRe* parameters. The *ApCI* and *CrI* are evaluated based on the Fisher matrix and the same procedure of the Bayes point estimates. In addition to theoretical derivation, a simulation study is conducted and the *DrS* datasets under two wettabilities are analyzed to implement of the proposed model. We observed consistent and expected results. This study has novelty concerning the reliability model under *IAPrgCS-II* as well as engineering application of the obtained results. More efforts will be made on this distribution by considering the multicomponent systems under *IAPrgCS-II*.

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Appendix

$$I_{11} = \frac{\partial^2 l(\alpha,\beta,\lambda|data)}{\partial\alpha^2} = -\frac{j_2}{\alpha^2} - \sum_{i=1}^{j_1} R_i \frac{(\aleph_{i:m_1:n_1})^{-\alpha}(ln(\aleph_{i:m_1:n_1}))^2}{(1-(\aleph_{i:m_1:n_1})^{-\alpha})^2} - R_1^* \frac{(\wp_1)^{-\alpha}(ln(\wp_1))^2}{(1-(\wp_1)^{-\alpha})^2}$$

$$I_{22} = \frac{\partial^2 l(\alpha,\beta,\lambda|data)}{\partial\beta^2} = -\frac{j_2 + h_2}{\beta^2} - (\alpha+1)\sum_{i=1}^{j_2} \frac{x_{i:m_1:n_1}^{-\beta}(ln\, x_{i:m_1:n_1})^2}{(\aleph_{i:m_1:n_1})^2} - \alpha \sum_{i=1}^{j_1} R_i \frac{(\aleph_{i:m_1:n_1})^{-2\alpha+1} x_{i:m_1:n_1}^{-2\beta}(ln\, x_{i:m_1:n_1})^2 \text{ƛ}_{i:m_1:n_1}}{(1-(\aleph_{i:m_1:n_1})^{-\alpha})^2}$$

$$-\frac{\alpha R_1^*(\wp_1)^{-2\alpha+1} T_1^{*-2\beta}(ln\, T_1^*)^2 \text{ƛ}_{T_1^*}}{(1-(\wp_1)^{-\alpha})^2} - (\lambda+1)\sum_{i=1}^{h_2} \frac{y_{i:m_2:n_2}^{-\beta}(ln\, y_{i:m_2:n_2})^2}{(P_{i:m_2:n_2})^2}$$

$$-\lambda \sum_{i=1}^{h_1} R_i \frac{(P_{i:m_2:n_2})^{-2\lambda+1} y_{i:m_2:n_2}^{-2\beta}(ln\, y_{i:m_2:n_2})^2 \nu_{i:m_2:n_2}}{(1-(P_{i:m_2:n_2})^{-\lambda})^2} - \frac{\lambda R_2^*(\wp_2)^{-2\lambda+1} T_2^{*-2\beta}(ln\, T_2^*)^2 \nu_{T_2^*}}{(1-(\wp_2)^{-\lambda})^2},$$

$$I_{33} = \frac{\partial^2 l(\alpha,\beta,\lambda|data)}{\partial\lambda^2} = -\frac{h_2}{\lambda^2} - \sum_{i=1}^{h_1} R_i \frac{(P_{i:m_2:n_2})^{-\lambda}(ln(P_{i:m_2:n_2}))^2}{(1-(P_{i:m_2:n_2})^{-\lambda})^2} - R_2^* \frac{(\wp_2)^{-\lambda}(ln(\wp_2))^2}{(1-(\wp_2)^{-\lambda})^2},$$

$$I_{12} = I_{21} = \frac{\partial^2 l(\alpha,\beta,\lambda|data)}{\partial\alpha\partial\beta} = \sum_{i=1}^{j_2} \frac{x_{i:m_1:n_1}^{-\beta}(ln\, x_{i:m_1:n_1})}{\aleph_{i:m_1:n_1}} - \sum_{i=1}^{j_2} R_i \frac{(\aleph_{i:m_1:n_1})^{-\alpha-1} x_{i:m_1:n_1}^{-\beta}(ln\, x_{i:m_1:n_1})\varpi_{i:m_1:n_1}}{(1-(\aleph_{i:m_1:n_1})^{-\alpha})^2}$$

$$-\frac{R_1^*(\wp_1)^{-\alpha-1} T^{*-\beta}(ln\, T_1^*)\varpi_{T_1^*}}{(1-(\wp_1)^{-\alpha})^2},$$

$$I_{23} = I_{32} = \sum_{i=1}^{h_2} \frac{y_{i:m_2:n_2}^{-\beta}\, ln\, y_{i:m_2:n_2}}{P_{i:m_2:n_2}} - \sum_{i=1}^{h_1} R_i \frac{(P_{i:m_2:n_2})^{-\lambda-1} y_{i:m_2:n_2}^{-\beta}\, ln\, y_{i:m_2:n_2}}{1-(P_{i:m_2:n_2})^{-\lambda}}$$

$$-\frac{R_2^*(\wp_2)^{-\lambda-1}T_2^{*-\beta}\ln T_2^{*-\beta}}{1-(\wp_2)^{-\lambda}},$$

$$I_{31}=I_{13}=0.$$

$$\lambda_{i:m_1:n_1}=\left(\frac{(\aleph_{i:m_1:n_1})-(\aleph_{i:m_1:n_1})^{\alpha+1}}{x^{\beta}_{i:m_1:n_1}}\right)+(\alpha+1)(\aleph_{i:m_1:n_1})^{\alpha}-1,\ \lambda_{T_1^*}=(\alpha+1)(\wp_1)^{\alpha}+T_1^{*\beta}((\wp_1)-(\wp_1)^{\alpha+1})-1,$$

$$\nu_{i:m_2:n_2}=\left(\frac{(P_{i:m_2:n_2})-(P_{i:m_2:n_2})^{\lambda+1}}{y^{\beta}_{i:m_2:n_2}}\right)+(\lambda+1)(P_{i:m_2:n_2})^{\lambda}-1,\ \nu_{T_2^*}=(\lambda+1)(\wp_2)^{\lambda}+T_2^{*\beta}((\wp_2)-(\wp_2)^{\lambda+1})-1,$$

$$\varpi_{i:m_1:n_1}=1-\alpha\ln(\aleph_{i:m_1:n_1})-(\aleph_{i:m_1:n_1})^{-\alpha},$$

$$\varpi_{T_1^*}=1-\alpha\ln(\wp_1)-(\wp_1)^{-\alpha}.$$

Table 2. The MSEs and CI lengths of the $SSRe_1=0.533$under different CSs and $T_1=0.7, T_2=1.2$.

n	***m***	T_1	T_2	***CS***	***MLE***	$Bayes_{Lindley}$	$Bayes_{MH}$	***ApCILs***	***CrILs***
30	15	0.7	1.2	I	0.10754	0.08854	0.08325	0.49863	0.27944
30	15	0.7	1.2	II	0.10065	0.08184	0.07989	0.47551	0.25002
30	15	0.7	1.2	III	0.10908	0.08897	0.08476	0.49926	0.27979
30	24	0.7	1.2	I	0.10211	0.08365	0.08117	0.47780	0.25498
30	24	0.7	1.2	II	0.09754	0.07859	0.07576	0.46311	0.23765
30	24	0.7	1.2	III	0.10231	0.08343	0.08195	0.47788	0.25513
50	25	0.7	1.2	I	0.10327	0.08654	0.08132	0.47234	0.25779
50	25	0.7	1.2	II	0.09855	0.07740	0.07586	0.46082	0.25030
50	25	0.7	1.2	III	0.10287	0.08612	0.07999	0.47188	0.25762
50	40	0.7	1.2	I	0.09327	0.07336	0.07155	0.43996	0.22564
50	40	0.7	1.2	II	0.08966	0.06953	0.06732	0.43222	0.21978
50	40	0.7	1.2	III	0.09532	0.07387	0.06897	0.43998	0.22420

Table 3. The MSEs and CI lengths of the $SSRe_1=0.533$ under different CSs and $T_1=1.2, T_2=2.5$.

n	***m***	T_1	T_2	***CS***	***MLE***	$Bayes_{Lindley}$	$Bayes_{MH}$	***ApCILs***	***CrILs***
30	15	1.2	2.5	I	0.10207	0.08532	0.08101	0.48334	0.27552
30	15	1.2	2.5	II	0.09864	0.08100	0.07744	0.47246	0.24996
30	15	1.2	2.5	III	0.10213	0.08539	0.08265	0.48375	0.27569
30	24	1.2	2.5	I	0.09779	0.07574	0.07315	0.47042	0.25211
30	24	1.2	2.5	II	0.09643	0.07365	0.07095	0.46009	0.23645
30	24	1.2	2.5	III	0.09772	0.07583	0.07318	0.47125	0.25318
50	25	1.2	2.5	I	0.09953	0.07864	0.07665	0.46809	0.25527
50	25	1.2	2.5	II	0.09758	0.07219	0.06890	0.46001	0.24888
50	25	1.2	2.5	III	0.09968	0.07859	0.07678	0.46799	0.25526
50	40	1.2	2.5	I	0.08856	0.06808	0.06672	0.43686	0.21974
50	40	1.2	2.5	II	0.08231	0.06200	0.06111	0.42705	0.21305
50	40	1.2	2.5	III	0.08866	0.06829	0.06669	0.43689	0.21979

Table 4. The MSEs and CI lengths of the $SSRe_1 = 0.750$under different CSs and $T_1 = 0.7, T_2 = 1.2$.

n	m	T_1	T_2	CS	MLE	$Bayes_{Lindley}$	$Bayes_{MH}$	$ApCILs$	$CrILs$
30	15	0.7	1.2	I	0.08432	0.06756	0.06443	0.35632	0.17853
30	15	0.7	1,2	II	0.08105	0.06291	0.05842	0.35211	0.17688
30	15	0.7	1.2	III	0.08511	0.06763	0.06452	0.35664	0.17859
30	24	0.7	1.2	I	0.08216	0.06550	0.06220	0.35106	0.17676
30	24	0.7	1.2	II	0.07764	0.06009	0.05508	0.34978	0.17523
30	24	0.7	1.2	III	0.08212	0.06549	0.06217	0.35102	0.17650
50	25	0.7	1.2	I	0.08121	0.06442	0.06237	0.35377	0.17642
50	25	0.7	1.2	II	0.07872	0.05889	0.05604	0.35206	0.17613
50	25	0.7	1.2	III	0.08127	0.06440	0.06240	0.35379	0.17650
50	40	0.7	1.2	I	0.07574	0.06019	0.05739	0.33778	0.16884
50	40	0.7	1.2	II	0.07043	0.05514	0.05110	0.33542	0.16607
50	40	0.7	1.2	III	0.07569	0.06014	0.05735	0.33767	0.16881

Table 5. The MSEs and CI lengths of the $SSRe_1 = 0.750$under different CSs and $T_1 = 1.2, T_2 = 2.5$.

n	m	T_1	T_2	CS	MLE	$Bayes_{Lindley}$	$Bayes_{MH}$	$ApCILs$	$CrILs$
30	15	0.7	1.2	I	0.07985	0.06012	0.05964	0.32743	0.17127
30	15	0.7	1.2	II	0.07810	0.05886	0.05677	0.30622	0.16709
30	15	0.7	1.2	III	0.07987	0.06010	0.06021	0.32843	0.17132
30	24	0.7	1.2	I	0.07979	0.05881	0.05895	0.32356	0.17005
30	24	0.7	1.2	II	0.07093	0.05800	0.05401	0.30216	0.16526
30	24	0.7	1.2	III	0.07975	0.05884	0.05918	0.32489	0.17116
50	25	0.7	1.2	I	0.07832	0.05831	0.05805	0.32105	0.16749
50	25	0.7	1.2	II	0.07776	0.05726	0.05600	0.29978	0.16314
50	25	0.7	1.2	III	0.07845	0.05836	0.05811	0.32102	0.16754
50	40	0.7	1.2	I	0.07419	0.05733	0.05647	0.31807	0.16312
50	40	0.7	1.2	II	0.07015	0.05189	0.05085	0.29465	0.15849
50	40	0.7	1.2	III	0.07425	0.05742	0.05687	0.31820	0.16322

Table 6. Estimations of *SSRe* for the real data example under different CSs.

SCs	*MLE*	*MeHa*	*LiA*	*ApCILs*	*CrILs*
SCI	0.62809	0.62722	0.62314	0.3672	0.2588
SCII	0.62189	0.61988	0.61528	0.3517	0.2503
SCIII	0.63763	0.63543	0.63107	0.3698	0.2625