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The practical stability of the discrete, fractional order, state space model of the heat transfer process

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In the paper the practical stability problem for the discrete, non-integer order model of one dimmensional heat transfer process is discussed. The conditions associating the practical stability to sample time and maximal size of finite-dimensional approximation of heat transfer model are proposed. These conditions are formulated with the use of spectrum decoposition property and practical stability conditions for scalar, positive, fractional order systems. Results are illustrated by a numerical example.

Key words: discrete non-integer order system, heat transfer, diffusion equation, Grünvald-Letnikov operator, practical stability

1. Introduction

Mathematical models of distributed parameter systems obtained on the basis of partial differential equations can be described in an infinite-dimensional state space, usually Hilbert space, but Sobolev space can also be applied. This problem has been analyzed by many researchers. Fundamentals are given for example in [12]. Analysis of a hyperbolic system in Hilbert space is presented in [2]. An overview of literatute is presented also in this paper.

Non-integer order calculus serves as main area of aplication if modeling of processes that are hardly described by the standard tools are concerned. Non-integer models of physical phenomena were presented by many Authors, for example in [4, 5, 7, 15, 22, 25]. Analysis of anomalous diffusion problem with the use of fractional order (FO) approach and semigroup theory was presented for example by [23].

It is well known, that heat transfer processes can be modeled with the use of non-integer order approach. This problem has been investigated for example in [6, 13, 15]. It is important to notice that all known models have a form

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of transfer function or partial differential equation. The time-continuous, non-integer order state-space model for heat transfer process was presented in [18] and [20]. However number of real implementations require discrete-time model to be used. Such a situation appears each case, when model needs to be real-time implemented in a PLC or a microcontroller.

This paper presents a stability problem for a new, discrete, non-integer order, state-space model describing heat transfer in one dimensional metallic rod. The model is obtained via discretization of time-continuous, state space model with the use of Power Series Expansion (PSE) approximation. The stability analysis is curried out with the use of approach presented in [3]. It bases additionally on results proposed in [24].

The paper is organized as follows: in section 2 elementary ideas and definitions are recalled. Section 3 describes the non integer order, state space model of the plant and its discretization with the use of PSE approximation. In section 4 practical stability conditions for the model are proposed and proved. Numerical verification employing real experimental data is is given in section 5.

2. Preliminaries

Elementary ideas and definitions are started with recalling Gamma Euler function (see for example [11]):

Definition 1 *(The Gamma function)*

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t. \tag{1}$$

Mittag-Leffler function is a non-integer order generalization of exponential function $e^{\lambda t}$ and it plays crucial role in solution of FO state equation. The one parameter Mittag-Leffler function is defined as follows:

Definition 2 *(The one parameter Mittag-Leffler function)*

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha+1)}. \tag{2}$$

The two parameter Mittag-Leffler function is defined as follows:

Definition 3 *(The two parameters Mittag-Leffler function)*

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k\alpha+\beta)}. \tag{3}$$

For $\beta = 1$ the two parameter function (3) turns to one parameter function (2).

The fractional-order, integro-differential operator can be described by different definitions, given by Grünvald and Letnikov (GL defintion), Riemann and Liouville (RL definition) and Caputo (C definition). In the further consideration GL and C definitions are used. They are given below [4, 21]).

Definition 4 *(The Grünvald-Letnikov definition of the FO operator)*

$$ {}_0^{GL}D_t^{\alpha} f(t) = \lim_{h\to 0} h^{-\alpha} \sum_{l=0}^{\left[\frac{t}{h}\right]} (-1)^l \binom{\alpha}{j} f(t-lh). \tag{4}$$

In (4) $\binom{\alpha}{l}$ is a binomial coefficient into real numbers:

$$ \binom{\alpha}{l} = \left\{ \begin{array}{cc} 1, & l=0 \\ \dfrac{\alpha(\alpha-1)\ldots(\alpha-l+1)}{l!}, & l>0 \end{array} \right\}. \tag{5}$$

The Caputo definition is described as follows:

Definition 5 *(The Caputo definition of the FO operator)*

$$ {}_0^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(N-\alpha)} \int_0^{\infty} \frac{f^{(N)}(\tau)}{(t-\tau)^{\alpha+1-N}} \mathrm{d}\,\tau, \tag{6}$$

where $N-1<\alpha<N$ denotes the non-integer order of operation and $\Gamma(..)$ is the complete Gamma function expressed by (1).

Non-integer order spatial derivative is given by Riesz and it has the following form (see for example [26]):

Definition 6 *(The Riesz definition of FO spatial derivative)*

$$ \frac{\partial^{\gamma}\Theta(x,t)}{\partial x^{\gamma}} = -r_{\gamma}\left({}_0D_x^{\gamma} + {}_xD_1^{\gamma}\right)\Theta(x,t), \tag{7}$$

where:

$$ r_{\gamma} = \frac{1}{2\cos\left(\dfrac{\pi\gamma}{2}\right)}. \tag{8}$$

In (7) ${}_0D_x^{\gamma}$ and xD_1^{γ} denote left- and right-side Riemann-Liouville spatial derivatives expressed as follows:

$$ {}_0D_x^{\gamma} = \frac{1}{\Gamma(2-\alpha)} \frac{\partial}{\partial x} \int_0^{x} \frac{\Theta(\xi,t)\mathrm{d}\,\xi}{(x-\xi)^{\gamma-1}}, \tag{9}$$

$$ {}_xD_1^{\gamma} = \frac{1}{\Gamma(2-\alpha)} \frac{\partial}{\partial x} \int\limits_x^1 \frac{\Theta(\xi,t)\mathrm{d}\xi}{(\xi-x)^{\gamma-1}}. \tag{10} $$

In (9) and (10) $\Gamma(..)$ denotes the Gamma function.

Next a linear, fractional order state equation can be defined. It has the following form (see for example [1, 9, 10]):

$$ \begin{aligned} \frac{\mathrm{d}^{\alpha}x(t)}{\mathrm{d}t^{\alpha}} &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{11} $$

where $x(t) \in \mathbb{R}^N$ is a state vector, $u(t) \in \mathbb{R}^P$ is a control vector and $y(t) \in \mathbb{R}^M$ is an output vector, $0 < \alpha < 1$ is a fractional order of the equation.

The GL definition is limit case for $h \to 0$ of fractional order backward difference, commonly employed to discrete FO calculations:

Definition 7 *(The fractional order backward difference)*

$$ (\Delta^{\alpha}x)(t) = \frac{1}{h^{\alpha}} \sum_{l=0}^{L} (-1)^l \binom{\alpha}{l} x(t-lh). \tag{12} $$

Let us denote coefficients $(-1)^l \binom{\alpha}{l}$ by d_l:

$$ d_l = (-1)^l \binom{\alpha}{l} \tag{13} $$

The coefficients (13) can be also calculated with the use of the following, equivalent recursive formula (see for example [4], p. 12), useful in numerical calculations:

$$ \begin{aligned} d_0 &= 1, \\ d_l &= \left(1 - \frac{1+\alpha}{l}\right) d_{l-1}, \qquad l = 1, \ldots, L. \end{aligned} \tag{14} $$

It is proven in [3] that:

$$ \sum_{l=1}^{\infty} d_l = 1 - \alpha. \tag{15} $$

In (12) L denotes a memory length necessary to correct approximation of a non integer order operator. Unfortunately good accuacy of PSE approximation requires long memory L what can make difficulties in implementation.

The discrete, fractional order state equation using definition (12) is written as follows (see for example [14]):

$$\begin{cases} (\Delta_L^{\alpha} x)(t+h) = A^{+}x(t) + B^{+}u(t), \\ y(t) = C^{+}x(t), \end{cases} \tag{16}$$

where $x(t) \in \mathbb{R}^N$ is the state vector, $u(t) \in \mathbb{R}^P$ is the control, $y(t) \in \mathbb{R}^M$ is the output. A^+, B^+ and C^+ are state, control and output matrices respectively. If we shortly denote k-th time instant: hk by k, then equation (16) turns to:

$$\begin{cases} (\Delta_L^{\alpha} x)(k+1) = A^{+}x(k) + B^{+}u(k), \\ y(k) = C^{+}x(k), \end{cases} \tag{17}$$

where:

$$A^{+} = h^{\alpha} A, \tag{18}$$

$$B^{+} = h^{\alpha} B, \tag{19}$$

$$C^{+} = C. \tag{20}$$

The solution of state equation (17) takes the form:

$$x(k+1) = G^{+}x(k) - \sum_{l=2}^{L} A_l^{+} x(k-l) + h^{\alpha} B^{+} u(k), \tag{21}$$

where:

$$G^{+} = A^{+} + \alpha I, \tag{22}$$

$$A_l^{+} = d_l I_{N\times N}\,. \tag{23}$$

At this moment the idea of practical stability needs to be introduced. It was proposed by Kaczorek in [8] and it was considered also in [3, 24]. It associates the stability of discrete FO system described by state equation (17) to the asymptotic stability of its approximated solution given by (21).

Definition 8 *(Practical stability)*
The fractional order system described by (17) *is practically stable if its finite dimensional solution* (21) *is asymptotically stable.*

If we additionally assume that the considered FO discrete system (17) is positive, then simple practical stability conditions can be applied. These conditions are given in [3, 24]. In this paper the following is used (Theorems 3 and 5 in [3]):

Theorem 1 *(Necessary and sufficient pracical stability condition of positive system* (17) *for fixed memory length L)*
The positive,FO system (17) *with order* $0 < \alpha < 1$ *is practically stable if and only if the standard positive system:*

$$x(k+1) = \left(G^{+} + \sum_{l=2}^{L} A_l\right) x(k) \tag{24}$$

is asymptotically stable.

Theorem 2 *(Necessary and sufficient practical stability condition of positive system* (17)*) independently on memory length)*
The positive FO system (17) *with order* $0 < \alpha < 1$ *is practically stable for each memory length L if and only if the standard positive system:*

$$x(k+1) = (A^{+} + I)x(k) \tag{25}$$

is asymptotically stable.

Both of the above theorems will are to stability analysis for discrete model of heat plant considered in this paper. This is presented in the next section.

3. The plant and its non-integer order, state-space model

3.1. The time-continuous model

Let us consider an experimental heat plant shown in figure 1. It has the form of a thin copper rod 260 [mm] long. It is heated by an electric heater of the length Δx_u installed at the end. The input signal of the system is the standard voltage signal 0–10 [V]. It is transformed to the current of 0–1.5 [A], which supplies the heater. The temperature of the rod is measured with the use of a Pt-100 sensors Δx long located at the points: 0.29, 0.50 and 0.73 of rod length.

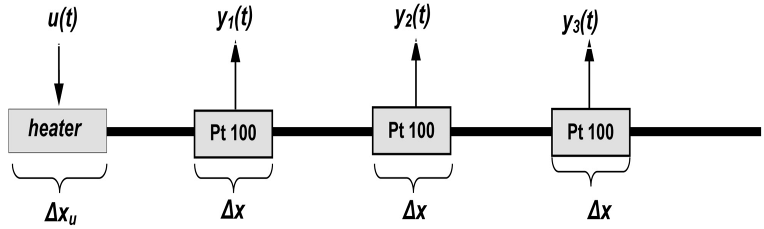

Figure 1: An experimental heat plant

Basic mathematical model describing the heat conduction of the plant is the partial differential equation of the parabolic type with the homogeneous Neumann boundary conditions at the ends, the homogeneous initial condition, the heat exchange along the rod and distributed control and observation. It is given with details for example in [16, 17]. The presented non integer order model with respect to both time and space coordinates is motivated by the fact that the non integer order differentiation better describes the heat exchange and diffussion in the plant that integer order model (see [19]). Assume that non integer order difference with respect to time is described by Caputo definition (6) and non integer order difference with respect to length is described by the Riesz definition (7). Then the non integer order heat transfer equation takes the following form:

$$\begin{cases} {}^{C}D_t^{\alpha}Q(x,t) = a\dfrac{\partial^{\beta}Q(x,t)}{\partial x^{\beta}} - R_aQ(x,t) + b(x)u(t), \\ \dfrac{\partial Q(0,t)}{dx} = 0, \qquad t \geqslant 0, \\ \dfrac{\partial Q(1,t)}{dx} = 0, \qquad t \geqslant 0, \\ Q(x,0) = 0, \qquad 0 \leqslant x \leqslant 1, \\ y(t) = y_0\displaystyle\int_0^1 Q(x,t)c(x)\mathrm{d}x, \end{cases} \tag{26}$$

where $\alpha, \beta > 0$ denote non integer orders of the system, a, R_a denote coefficients of heat conduction and heat exchange. The equation (26) can be expressed as an inifinte dimensional state equation in the Hilbert space, analogically, as it is presented in [20]:

$$\begin{cases} {}^{C}D_t^{\alpha}Q(t) = AQ(t) + Bu(t), \\ Q(0) = 0, \\ y(t) = y_0CQ(t), \end{cases} \tag{27}$$

where:

$$\begin{cases} AQ = a\dfrac{\partial^{\beta}Q(x)}{\partial x^{\beta}} - R_aQ, \qquad a, R_a > 0, \\ D(A) = \left\{Q \in H^2(0,1) : Q'(0) = 0,\ Q'(1) = 0\right\}, \\ H^2(0,1) = \left\{u \in L^2(0,1) : u', u'' \in L^2(0,1)\right\}, \\ CQ(t) = \langle c, Q(t)\rangle, \quad Bu(t) = bu(t). \end{cases} \tag{28}$$

The following set of the eigenvectors for the state operator A creates the orthonormal basis of the state space:

$$h_n = \begin{cases} 0, & n = 0, \\ \sqrt{2}\cos(n\pi x), & n = 1,2,\ldots . \end{cases} \tag{29}$$

Eigenvalues of the state operator are expressed as follows:

$$\lambda_{\beta_n} = -a\pi^\beta n^\beta - R_a, \quad n = 0,1,2,... \tag{30}$$

and consequently the state operator has the form:

$$A = diag\{\lambda_{\beta_1}, \lambda_{\beta_2}, \lambda_{\beta_3}, ...\}. \tag{31}$$

Next, the spectrum σ of the state operator A is expressed as follows:

$$\sigma(A) = \{\lambda_{\beta_1}, \lambda_{\beta_2}, \lambda_{\beta_3}, ...\}. \tag{32}$$

The input operator B describing the heater has the following form:

$$B = [b_0, b_1, b_2, ...]^{\mathrm{T}}, \tag{33}$$

where $b_n = \langle b, h_n\rangle$, $b(x)$ denotes the control function:

$$b(x) = \begin{cases} 1, & x \in [0, x_0], \\ 0, & x \notin [0, x_0]. \end{cases} \tag{34}$$

The output operator C associated with the temperature sensors is expressed as follows:

$$C = \begin{bmatrix} C_{s1} \\ C_{s2} \\ C_{s3} \end{bmatrix}. \tag{35}$$

Rows of output operator C are as follows:

$$C_{sj} = \left[c_{sj,0}, c_{sj,1}, c_{sj,2}, ...\right], \qquad j = 1,2,3..., \tag{36}$$

where $c_{sj,n} = \langle c, h_n\rangle$, $c(x)$ denotes the output sensor function:

$$c(x) = \begin{cases} 1, & x \in [x_1, x_2], \\ 0, & x \notin [x_1, x_2]. \end{cases} \tag{37}$$

Coordinates x_1 and x_2 depend on the sensor position on the rod and they are equal:

$$\begin{cases} x=0.29: & x_1=0.26, \ x_2=0.32 \\ x=0.50: & x_1=0.47, \ x_2=0.53 \\ x=0.73: & x_1=0.70, \ x_2=0.76. \end{cases}$$

From (34) and (37) it turns out that the control function $b(x)$ and output function $c(x)$ are the interval constant functions. The solution of state equation (27) can be calculated with the use of Laplace transform for Caputo operator assuming that initial condition is equal zero: $Q(x,0)=0, 0\leqslant x\leqslant 1$ and state and control operators are described by (31)–(34). If we assume that the control signal has the form of the Heaviside function $u(t)=1(t)$ then we obtain the solution as follows:

$$y_j(t)=y_{0_j}\sum_{n=1}^{\infty}\frac{\left(E_\alpha(\lambda_{\beta_n}t^\alpha)-1(t)\right)}{\lambda_{\beta_n}}\langle b,h_n\rangle\langle c,h_n\rangle, \qquad j=1,2,3 \tag{38}$$

and consequently the output of the system is expressed as follows:

$$y(t)=[y_1(t),\ y_2(t),\ y_3(t)]^{\mathrm{T}}. \tag{39}$$

Notice that the above non integer order model described by (26)–(38) for integer orders: $\alpha=1$ and $\beta=2$ turns to known integer order model.

The non integer order model described by (27)–(38) is an infinite dimensional model. Its practical application requires its finite dimensional approximation. This can be obtained by "cutting" further modes in state equation (27) and consequently calculating solution (38) and (39) as a finite sum expressed by (40). Consequently operators: A, B and C can be interpreted as matrices.

$$y_j(t)=y_{0_j}\sum_{n=1}^{N}\frac{\left(E_\alpha(\lambda_{\beta_n}t^\alpha)-1(t)\right)}{\lambda_{\beta_n}}\langle b,h_n\rangle\langle c,h_n\rangle, \qquad j=1,2,3. \tag{40}$$

In (40) N denotes the order of finite approximation. Its correct estimation is a crucial problem in using of presented models. An example of its numerical estimation is given by [20]. It is obvious that the increasing N allows to make the model more accurate. Unfortunately, the increasing of N causes loss of stability of discrete model derived from it. Thus we need to estimate the maximal N assuring both the good accuracy and stability of discrete model. This problem is presented in the next section.

3.2. The time-discrete model

The discrete time model follows directly from continuous model (27) after use discrete version of GL definition (4). Its solution (21) has the following form:

$$\begin{cases} Q^+(k+1) = G^+Q^+(k) - \sum_{l=1}^{L} A_l^+ Q^+(k-l) + B^+u(k), \\ y^+(k) = C^+Q^+(k). \end{cases} \tag{41}$$

In (41) A_l^+ is expressed by (23), G^+, B^+ and C^+ take the following form:

$$\begin{cases} G^+ = diag\{\lambda_{\beta_1}^+, \lambda_{\beta_2}^+ \ldots \lambda_{\beta_N}^+\}, \\ B^+ = h^\alpha B, \\ C^+ = C, \end{cases} \tag{42}$$

where:

$$\lambda_{\beta_n}^+ = \alpha - h^\alpha \left(a\pi^\beta n^\beta + R_a \right). \tag{43}$$

In [19] is proven that the spectrum of the time-continuous system can be decomposed into single, seperated eigenvalues (analogically as in the integer order case). This property is mapped in the discrete time system also. Particularly the solution (41) can be also decomposed to separated “subsolutions” associated with the single eigenvalues (43). This allows to give the fundamental results presented in the next section.

4. Main results

Presentation of main results is started with a decomposition the discrete model (41). The state vector $Q^+(k)$ can be expressed as:

$$Q^+(k) = \begin{bmatrix} q_1^+(k) \\ \ldots \\ q_N^+(k) \end{bmatrix}. \tag{44}$$

The matrices G^+ and A_l^+ describing the solution of the discrete system (41) are diagonal matrices. Consequently the solution (41) can be decomposed into N independent modes, associated with n-th state variable $q_n^+(k)$ and described by n eigenvalues. The stability analysis can be made using free solution (without controls). For fixed memory length L it takes the form:

$$q_n^+(k+1) = \lambda_{\beta_n}^+ q_n^+(k) - \sum_{l=2}^{L} d_l q_n^+(k-l), \qquad n = 1,..,N. \tag{45}$$

The characteristic polynomial associated with solution $q_n^+(k)$ has the following form:

$$w_n(z^{-1}) = 1 - \lambda_{\beta_n}^+ z^{-1} + \sum_{l=2}^{L} d_l z^{-l}. \tag{46}$$

For each memory length the solution takes a form:

$$q_n^+(k+1) = \lambda_{\beta_n}^+ q_n^+(k) - \sum_{l=2}^{\infty} d_l q_n^+(k-l), \qquad n = 1,..,N \tag{47}$$

and analogically the characteristic polynomial is given as follows:

$$w_n(z^{-1}) = 1 - \lambda_{\beta_n}^+ z^{-1} + \sum_{l=2}^{\infty} d_l z^{-l}, \tag{48}$$

where $\lambda_{\beta_n}^+$ is expressed by (43), d_l are expressed by (13) or by (14). Notice that the practical stability or instability for the whole considered system is determined by the asymptotic stability or instability of its separated modes (45) or (47). This can be expressed as the following remarks:

Remark 1 *(The practical stability of the discrete, decomposed FO system)*

1. *The discrete non integer order system* (41) *is practically stable for fixed memory length L if and only if each mode of its solution* (45) *is asymptotically stable.*

2. *The discrete non integer order system* (16) *is practically stable for each memory length L if and only if each mode of its solution* (47) *is asymptotically stable.*

Remark 2 *(The instability of the discrete, decomposed FO system)*

1. *The discrete non integer order system* (41) *is instable for fixed memory length L if and only if there exists at least one instable mode of its solution* (45).

2. *The discrete non integer order system* (16) *will be instable for each memory length L if and only if there exists at least one instable mode of its solution* (47).

The practical stability of the discrete system we deal with can be tested directly using the both above remarks. This requires testing of the localisation of roots of each characteristic polynomial (46) for $n = 1,...,N$. The degree of each polynomial is equal $L+1$. It can be done numerically only, perhaps using MATLAB functions. An example of such a stability test is presented below.

From remarks 1 and 2 it can be noted that the stability of the whole system can be tested via tests of stability of its N separated, scalar modes. To do this Theorems 1 and 2 are used. Before we start it, the positivity of the considered system needs to be shortly discussed.

At the beginning it is important to notice that the stability analysis requires consideration of state $Q^{+}(k)$ behaviour only, input and output of the system, described by the operators B and C are not required to be analyzed. It can be seen immediately that the state operator for the time continuous system (31), (32) is the Metzler matrix (definition of the Metzler matrix is given for example in [3]). This implies that the time-continuous state of our system is positive and asymptotically stable.

Theorems 1 and 2 are formulated with refer to the standard systems. They can be easily constructed for each mode of our decomposed system separately. The n-th decomposed, standard system takes the form for fixed memory length L as follows:

$$q_n^{+}(k+1) = \left(\lambda_{\beta_n}^{+} + \sum_{l=2}^{L} d_l\right) q_n^{+}(k), \qquad n = 1,..,N. \tag{49}$$

The system (49) is the first order scalar system with one eignevalue $\lambda_{sL_n}^{+}$:

$$\lambda_{sL_n}^{+} = \lambda_{\beta_n}^{+} + \sum_{l=2}^{L} d_l \tag{50}$$

and for each memory length:

$$q_n^{+}(k+1) = \left(h^{\alpha}\lambda_{\beta_n} + 1\right) q_n^{+}(k), \qquad n = 1,..,N. \tag{51}$$

The eigenvalue of the system (51) is equal:

$$\lambda_{s_n}^{+} = h^{\alpha}\lambda_{\beta_n} + 1. \tag{52}$$

The n-th mode of decomposed system (49) or (51) is asymptotically stable if and only if: $|\lambda_{sL_n}^{+}| < 1$ or $|\lambda_{s_n}^{+}| < 1$ respectively. The whole system is stable if and only if all its modes are asymptotically stable. This allows to formulate analytical conditions associating the practical stability with the size of model N, sample time h and memory length L. These conditions are formulated in the following propositions:

Proposition 1 *(Maximal size of model N assuring the practical stability of the discrete model for fixed memory length L)*

1. *Consider the discrete model of heat transfer process described by* (41)*,*
2. *the solution of n-mode of the decomposed system is expressed by* (45)*,*
3. *the standard FO system associated with n-th mode has the form* (49)*.*

The size N of finite-dimensional approximation assuring the practical stability of the discrete model (41) *meets the following inequality:*

$$N \leqslant Int\left(\left(\frac{1+\alpha-h^{\alpha}R_a+\sum\limits_{l=2}^{L+1} d_l}{h^{\alpha}a\pi^{\beta}}\right)^{\frac{1}{\beta}}\right). \tag{53}$$

Proposition 2 *(Maximal size of model N assuring the practical stability of the discrete model for each memory length)*
Assumptions:

1. *Consider the discrete model of heat transfer process described by* (41)*,*
2. *the solution of n-mode of the decomposed system is expressed by* (47)*,*
3. *the standard FO system associated with n-th mode has the form* (51)*.*

The maximal size N of finite-dimensional approximation assuring the practical stability of the discrete model (41) *meets the following inequality:*

$$N \leqslant Int\left(\left(\frac{2-h^{\alpha}R_a}{h^{\alpha}a\pi^{\beta}}\right)^{\frac{1}{\beta}}\right). \tag{54}$$

In (53) and (54) $Int(..)$ denotes the nearest integer. Both conditions follow directly from (49) and (51) and known asymptotic stability condition for discrete systems.

From (15) it turns out that condition (54) is the limit case of (53) for $L \to \infty$. Results of numerical calculations show that the use of both propositions gives practically the same result.

Another important problem arising while using and implementing the considered discrete FO model of heat transfer process is to estimate the value of sample time h necessary to keep the practical stability of model for fixed, finite order N. Remember that increasing N improves the accuracy of the model. This allows to use a high order N independently on other limitations. Then a solution is to use a shorter sample time h. Suitable conditions can be also formulated using the approach considered here. They have a form of the following propositions, analogically as above:

Proposition 3 *(Maximal size of sample time h assuring the practical stability of the discrete model for fixed memory length L)*
Assumptions:

1. *Consider the discrete model of heat transfer process described by* (41)*,*
2. *the solution of n-mode of the decomposed system is expressed by* (47)*,*
3. *the standard FO system associated with n-th mode has the form* (51)*.*

The maximal size h of sample time assuring the practical stability of the discrete model (41) *meets the following inequality:*

$$h < \left(\frac{1+\alpha+\sum_{l=2}^{L} d_l}{a\pi^{\beta} n^{\beta} + R_a} \right)^{\frac{1}{\alpha}}. \tag{55}$$

Proposition 4 *(Maximal size of sample time h assuring the practical stability of the discrete model for each memory length L)*
Assumptions:

1. *Consider the discrete model of heat transfer process described by* (41)*,*
2. *the solution of n-mode of the decomposed system is expressed by* (47)*,*
3. *the standard FO system associated to n-th mode has the form* (51)*.*

The maximal size h of sample time assuring the practical stability of the discrete model (41) *meets the following inequality:*

$$h < \left(\frac{2}{a\pi^{\beta} n^{\beta} + R_a} \right)^{\frac{1}{\alpha}}. \tag{56}$$

Analogically as above, the condition (56) is the limit case of condition (55) .

5. An Example

As an example let us consider the heat plant presented in the previous sections. Numerical values of its parameters were calculated numerically (see [20]) with the use of experimental results and MSE (Medium Square Error) cost function. They are given in table 1.

Table 1: Parameters of heat plant

Parameter	α	β	a	R_a
value	0.9430	1.9847	0.0005	0.0591

For the system with parameters given in table 1 the maximal size of model N for different h estimated with respect to conditions (53)–(54) are given in table 2. The maximal values of sample time h for fixed values of model size N are given in table 3.

Table 2: Maximal size of model N for different sample times h

Sample time h [s]	0.5	1	2	5
$N_{\max}$ for L=50	29	20	15	9
$N_{\max}$ for each L	29	20	15	9

Table 3: Maximal size of sample time h for fixed model size N

Model size N	10	20	30	40
$h_{\max}$ [s] for L=50	3.7971	1.0471	0.4764	0.2708
$h_{\max}$ [s] for each L	3.7934	1.0461	0.4760	0.2706

To verify the above results in figures 2 and 3 are shown exemplary spectra of standard systems, calculated as sets of eigenvalues (50) and, to verify resulting spectra of the whole discrete system, calculated as roots of characteristic polynomials (46).

Both diagrams 2 and 3 confirm the corectness of proposed stability conditions.

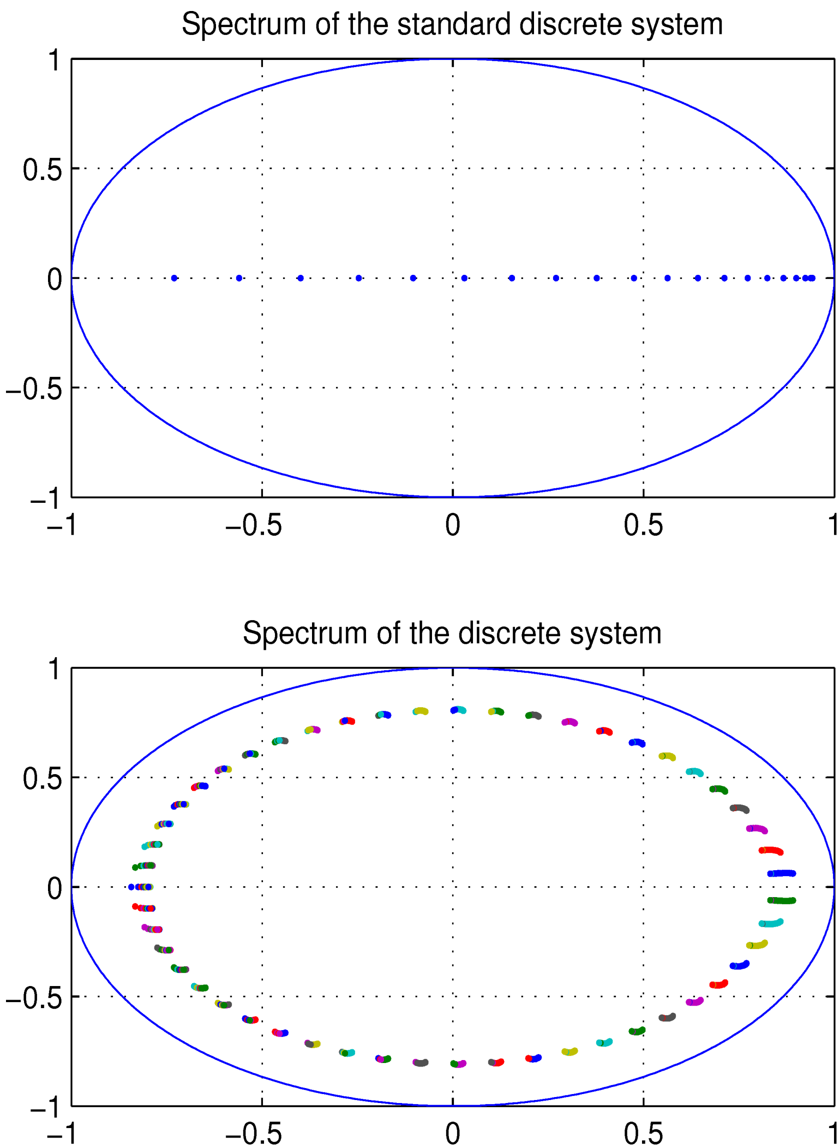

Figure 2: The spectra of stable system: $h = 1$ [s], $N = 20$

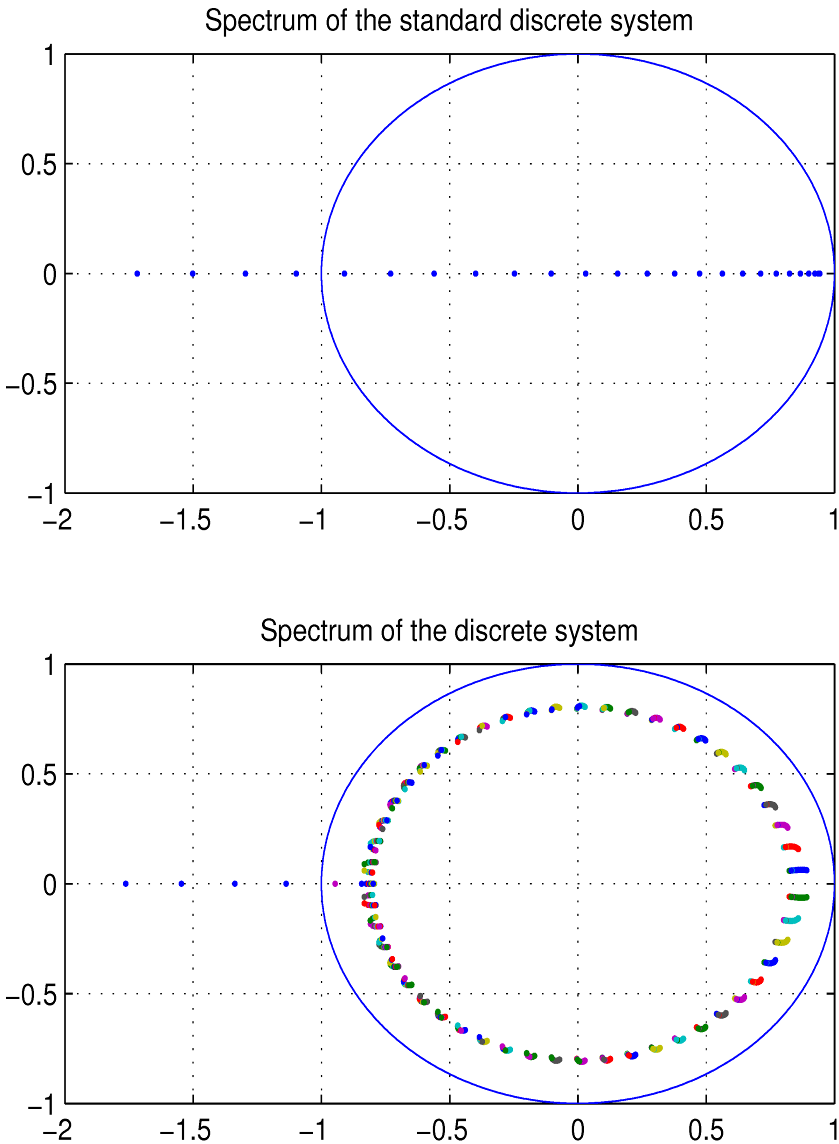

Figure 3: The spectra of unstable system: $h = 1$ [s], $N = 25$

6. Final conclusions

Final conclusions of the paper can be formulated as follows.

- Results presented in this paper can be used in digital implementation of the discussed discrete non integer order model in PLC. Such a model is always a compromise between accuracy and capabilities of digital platform.
- The approach proposed in this paper can be also generalized to other FO systems described by state equation with diagonal state matrix. Such a model can be obtained by transformation to Jordan canonical form.

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