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ON TEACHING OF GEOMETRIC TRANSFORMATIONS IN SCHOOL

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Abstract

The current core curriculum in mathematics for lower secondary school (3-rd educational level in Poland) omits formal definitions of concepts related to geometric transformations in the plane and is based on their intuitive sense. Practice shows that the current approach makes teaching very difficult and the students solve the typical tasks, not understanding the meaning of geometrical concepts. The article contains basic concepts connected with geometric transformations and examples of geometric tasks that are solved in the third and also in the fourth educational level in an intuitive way, sometimes deviating or even incompatible with the mathematical definition. We show how they could be solved in easier way with introducing definitions of geometric transformations in a simple and understandable for students way sometimes using vector calculus. We take into account isometries: reflection and point symmetry, rotation and translation and similarities with particular consideration on homothetic transformation.

1. Introduction

As the result of permanent reduction of core curriculum in mathematics in past few years (see [8],[9]) and practical attitude to mathematical tasks (an old gymnasium exam: from 2002 to 2011), the big problems with teaching geometric transformations appeared. The three main reasons are:

(1) reduction of contents connected with forming concept of a function;
(2) eliminating vectors;
(3) drastic reduction of contents connected with geometric constructions.

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The above reasons make teaching geometric transformations very difficult. In this case education is based on an intuitive understanding of concepts like symmetry, similarity etc. that may cause numerous misunderstandings. Of course, we do not want to discourage students introducing difficult mathematical formalism but primary theory is also necessary for them. The teachers realize that good theoretical bases help to solve mathematical tasks ([3], [4], [5], [6], [7]). Some of the educators suppose that students of the 3-rd educational level are able to understand geometrical definitions and relations between geometric concepts. According to Dutch psychologist P. M. Van Hiele model of learning geometry by students one can say that pupils of lower secondary school are on level 2 (Abstraction). At this level, properties are ordered. Students understand that properties are related and one set of properties may imply another property ([1]).

The main goal of this article is to remind the most important concepts connected with geometric transformations and discuss the possibility of implementing more theory to lessons instead of using intuitive sense. It is very important now because mathematics became separate subject on the new gymnasium exam (since 2012) and we hope its role will be increasing. Today it is a chance to return to less practical but more mathematical attitude to solving tasks.

2. Geometric transormations

Let us consider the idea of geometric transformation. If we understand it intuitively, it is the change of the location of the points that happens according to some strict rules. So it is a function that relates points of the plane (or space) to the other points of it. In this article we assume traditional definition used in polish school mathematics since the sixties years of the XX century introduced by professor Zofia Krygowska. More precisely:

Definition 1. *Denote as π the arbitrary plane or space. Every bijection $f : \pi \to \pi$ is called the geometric transformation.*

If we accept the above definition it may cause some problems. The function in this definition is not numerical. Domain and range of the function are the sets of points. The students rarely realize that in every geometric transformation the locations of all points change. For example, in school practice they often find only the images of favoured points (vertices of a polygon) forgetting that the other points also change location. Similar approach causes misunderstanding of other important facts. It seems to be obvious that geometric transformations should be taught after the functions and be the excellent example of non-numerical ones. It is also problematic to explain how to understand the word "bijection" in this case. We suggest such an explanation:

(1) for the arbitrary two different points X, Y from π their images $f(X)$, $f(Y)$ are also different,
(2) every point $Y \in \pi$ has to be the image of some point $X \in \pi$ (i.e. $Y = f(X)$ for some point X).

It can be helpful to create some drawing of simple transformation (reflection symmetry) and show that conditions (1) and (2) are satisfied.

3. Vectors and their applications

It is alarming, from the point of teaching geometrical transformations, that vector calculus was eliminated from core curriculum. Vectors make definitions of some geometric transformation simpler and shorter. Vector calculus is also the best and the quickest instrument used in solving mathematical tasks from the analytic geometry. The problem is that today students are supposed to show the result – not to compute it. This way is not correct. First of all, we are not always able to read the result (from the drawing). Secondly, tasks often have more than one solution – vector calculus lets us find all the solutions. Of course, free vector is too complicated question to use at this stage of education but we may identify the vector with its coordinates and then use this helpful tool. Here are the examples of using vector calculus in tasks addressed to students in the third educational level.

Example 1. *Check whether the triangle ABC is right-angled, if $A = (-1, 2)$, $B = (0, 0)$, $C = (6, 3)$.*

Standard analytic solutions are:

- computing distances $d(A, B), d(A, C), d(B, C)$ and checking if the square of one of the distances is equal to the sum of squares of the other distances (Pythagorean theorem) or
- finding the equations of straight lines AB, BC and AC and checking if one of the straight lines is perpendicular to the other one.

But the two above solutions are numerically complicated because they require many various tough calculations. Using vector calculus, we can solve the problem in easy and quick following way:

We compute the coordinates of the vectors $\vec{AB}$, $\vec{BC}$ and $\vec{AC}$:

$$\vec{AB} = [1, -2], \quad \vec{BC} = [6, 3], \quad \vec{AC} = [7, 1].$$

We now notice that $\vec{AB} \circ \vec{BC} = 0$ (scalar product) so these two vectors are perpendicular and finally the triangle ABC is right-angled.

Example 2. *Check if the tetragon $ABCD$ is a rhomboid, if $A = (1,4)$, $B = (6,5)$, $C = (56,105)$, $D = (51,104)$.*

Standard solutions are:

- computing the distances $d(A,B)$, $d(B,C)$, $d(C,D)$ and $d(D,A)$ and checking if we obtain two pairs of equal distances (more precisely: $d(A,B) = d(C,D)$ and $d(A,D) = d(B,C)$) or
- finding the equations of straight lines AB, BC, CD and AD and checking if we have two pairs of parallel lines.

Of course these two ways are not the fastest methods. Let us notice that:

$$\vec{AB} = \vec{DC} = [5,1].$$

This fact proves that tetragon $ABCD$ is a rhomboid because the equality of coordinates of vectors causes not only the equality of the lengths of line segments but also their parallelism.

Example 3. *Calculate the area of a triangle ABC, if $A = (1,6)$, $B = (2,8)$, $C = (9,-1)$.*

If we try to solve the above problem in the third or fourth educational level we have to face with complicated calculations as standard solutions are:

- calculating the lengths of all sides of the triangle and then use Heron's formula or
- calculating the length of one side of the triangle and then the proper height of a triangle (as a distance between the straight line and the point).

The fastest method (possible even in the third educational level) is to calculate coordinates of two vectors that have common beginning and then use well known formula that lets compute the area of a triangle:

$$\vec{AB} = [1,2], \vec{AC} = [8,-7].$$

$$P_{ABC} = \tfrac{1}{2} \cdot \left| \begin{vmatrix} 1 & 2 \\ 8 & -7 \end{vmatrix} \right| = 11.5.$$

The other examples showing applications of vector calculus will be discussed in the sections connected with point symmetry, homothetic transformation and examples of tasks for students.

4. Isometries

Content ISOMETRY is rarely used in school mathematics. If the teachers talk about isometry they explain to the students that it is a geometric transformation that does not change the size and the shape of geometric figures. It is intuitively clear. Of course, isometry can change only the location of the figures: it transforms them into another place or only replaces the points (the whole figure stays in the same place). In addition, only two isometric transformations are discussed: reflection symmetry and point symmetry. There is no information about translation and rotation that is puzzling since particularly translation is one of the most important transformations that is used in the whole fourth educational level when the teachers explain the transformations of the functions' graphs. We think that the definition of an isometry is not too difficult for students. If we define the distance between the points X, Y as follows: $d(X,Y) = \left|\vec{XY}\right|$, we may define the formal but simple definition of an isometry.

4.1. **Formal definition.**

Definition 2. *Geometric transformation $f : \pi \to \pi$ is called an isometry if for the arbitrary points $X, Y \in \pi$ we have the following equality:*

$$d(f(X), f(Y)) = d(X,Y),$$

i.e. the distance between two arbitrary points is equal to the distance between their images.

From the above definition we can notice at once that the sizes of the figures and their shapes cannot change because the distances between proper points stay the same. We now can explain why in isometric transformations we may only find the images of all vertices of a polygon in order to find an image of the whole polygon. Moreover, we also have to discover only an image of the centre of the circle in order to find an image of the whole circle (because the radius stays the same) although above figures contain infinite number of points. Of course, it is not possible for non-regular figures.

In school mathematics we should talk about the following examples of isometric transformations:

(1) REFLECTION SYMMETRY;
(2) POINT SYMMETRY;
(3) TRANSLATION;
(4) ROTATION.

Above transformations will be discussed in the next subsections.

4.2. **Reflection symmetry.**

If we introduce the distance between the point X and straight line l as follows:

$$d(X,l) = \inf\{d(X,Y) : Y \in l\}$$

and the following definition:

Definition 3. *$\vec{u} \perp l$ if and only if for every two different points $X, Y \in l$ we have the condition: $\vec{u} \perp \vec{XY}$,*

then we can introduce the following definition.

Definition 4. *Assume that l is any straight line in the plane π. Geometric transformation $f : \pi \to \pi$ is called reflection symmetry towards straight line l if for any point $X \in \pi$ the following conditions are satisfied:*

$$d(X, l) = d(f(X), l), \tag{1}$$

$$\overrightarrow{Xf(X)} \perp l. \tag{2}$$

Let us notice that Definition 4. holds also for $X \in l$. In school practice we modify Definition 4. in the following way:

Definition 5. *Assume that l is any straight line in the plane π and point X does not belong to l. Point X' is symmetric to point X towards l if it satisfies the following conditions:*

- *Distances $d(X, l)$ and $d(X', l)$ are the same,*
- *Line segment XX' is perpendicular to the straight line l.*

In addition we assume that if $X \in l$, then $X' = X$.

Let us notice that Definition 5. has also two conditions, we do not need to add that points X, X' are located on both sides of the straight line l (as it is added in many handbooks).

4.3. **Point symmetry.**

Definition 6. *Assume that S is any point in the plane π. Geometric transformation $f : \pi \to \pi$ is called the point symmetry towards point S if for the arbitrary point $X \in \pi$ the following condition is satisfied:*

$$\vec{XS} = \overrightarrow{Sf(X)}. \tag{3}$$

The above definition holds also for $X = S$.

In school practice we modify Definition 6. in the following way:

Definition 7. *Assume that S is an arbitrary point in the plane π and point X is different from point S. Point X' is symmetric to point X towards point S if it satisfies the following condition: Point S is the centre of the line segment XX'. In addition, if $X = S$ then we assume $X' = X$.*

Let us notice that above definition contains only one condition. We do not have to talk about the same distances and we do not need to add that points X, S, X' are located on the same straight line and points X and X' are different (as it is in many handbooks – then we have three conditions). From Definition 7. we also conclude that vector calculus is the best way of solving tasks connected with point symmetry.

Example 4. *Find an image of point $A = (3, 4)$ in the symmetry towards point $S = (5, 6)$.*

The image of point A can be denoted as $A' = (x, y)$. If we use the definition of point symmetry and vector calculus we obtain:

$\vec{AS} = [5 - 3, 6 - 4] = [2, 2]$,

$\vec{SA'} = [x - 5, y - 6]$.

Because the above vectors are equal we obtain: $x - 5 = 2$ and $y - 6 = 2$. So $x = 7$ and $y = 8$. Then finally $A' = (7, 8)$.

On the base of definitions of reflection and point symmetry we can define the concepts of symmetry axis and symmetry centre of a geometric figure.

Definition 8. *Symmetry axis of a figure is a straight line towards which the figure is symmetric to itself.*

Geometric figures may have no symmetry axis, infinite number of symmetry axes or infinite number of symmetry axes.

Definition 9. *Symmetry centre of a figure is the point towards which the figure is symmetric to itself.*

Geometric figures may have no symmetry centre, one symmetry centre or infinite number of symmetry centres.

Let us notice that point symmetry towards point S is an identical transformation like a U-turn around point S. The above description is the best way to find symmetry centre of a figure. If there exists such a point S around which U-turn gives the same figure (understood as the same set of points) then point S is a symmetry centre of a figure.

The aforementioned concepts have nothing in common. There exist figures that have symmetry centre and have no symmetry axes and figures that have symmetry axes and have no symmetry centre. In addition, we notice that point symmetry can be understood as the superposition of two reflection symmetries in which axes are perpendicular.

4.4. **Translation.**

Definition 10. *Assume that $\vec{u}$ is an arbitrary vector in the plane (or space) π. A geometric translation $f : \pi \to \pi$ is called the translation by a vector $\vec{u}$ if for every point $X \in \pi$ it satisfies the following condition:*

$$\overrightarrow{Xf(X)} = \vec{u}. \tag{4}$$

In school practice we modify the above definition the following:

Definition 11. *Assume that $\vec{u}$ is an arbitrary vector in the plane (or space) π. The image of a point $X \in \pi$ in translation by a vector $\vec{u}$ is the point X' that satisfies the condition:*

$$\vec{XX'} = \vec{u}.$$

4.5. **Rotation.**

Definition 12. *Assume that S is an arbitrary point in the plane π. The rotation around point S of an angle α is a geometric transformation f of the plane π that for every point $X \in \pi$ satisfies the following conditions:*

$$d(X,S) = d(f(X),S), \tag{5}$$

$$|\angle XSf(X)| = \alpha. \tag{6}$$

And in school practice:

Definition 13. *Assume that S is an arbitrary point in the plane π. The image of a point $X \in \pi$ in rotation around point S of angle α is the point X' that satisfies the following conditions:*

$$d(X,S) = d(X',S),$$

$$\left|\angle XSX'\right| = \alpha.$$

Remark. If we accept these definitions 12. and 13. we additionally have to determine in which way we measure the angle (it is a directed angle). In school practice we usually measure the angle anticlockwise.

5. Similarities

Similarities are colloquially understood as transformations that do not change the shapes of the figures but they may change their sizes. Therefore the difference between similarities and isometries is that similarities may not only change the location of a figure but likewise decrease or increase it in a given scale. Of course, similarities do not change the measures of the angles and if the scale is equal to 1, then the similarity is an isometry. The formal definition is not too difficult for students so we can introduce it in the following way.

5.1. **Formal definition.**

Definition 14. *Assume that k is an arbitrary positive number ($k > 0$). Similarity of the scale k is a geometric transformation $f : \pi \to \pi$ that for all points $X, Y \in \pi$ satisfies the following equation:*

$$d(f(X), f(Y)) = k \cdot d(X,Y).$$

Remark. If the figure F is similar to figure G in the scale k then G is similar to F in the scale $\frac{1}{k}$.

5.2. **Homothetic transformation.** Homothetic transformations have not been discussed in lower secondary schools for a long time. It is mainly caused by eliminating vector calculus. It is alarming because homothetic transformations are special cases of similarities that additionally keep the line segments parallelism and they have many various practical applications in various domains (physics, architecture). We also use homothetic transformations in everyday usage to constructional decreasing or increasing figures in given scale. This is the formal definition.

Definition 15. *Assume that k is an arbitrary non-zero number ($k \neq 0$). Homothetic transformation with centre S and ratio k is a geometric transformation $f : \pi \to \pi$ that for every point $X \in \pi$ satisfies the following condition:*

$$\overrightarrow{Sf(X)} = k \cdot \vec{SX}. \tag{7}$$

Definitions of similarity and homothetic transformation are completely different. First of them is based on distances and the second is based on vector calculus. So the next theorem seems to be very interesting for students.

Theorem 1. *Every homothetic transformation with scale k is a similarity of scale $|k|$.*

We present well known proof of this theorem (see [2]). Denote a homothetic transformation of centre S and scale k by f. Let us assume that X, Y are two arbitrary points in the plane (or space) π. Then we have the following equalities, that prove the theorem:

$$d(f(X), f(Y)) = \left|\overrightarrow{f(X)f(Y)}\right| = \left|\overrightarrow{f(X)S} + \overrightarrow{Sf(Y)}\right| = \left|-\overrightarrow{Sf(X)} + \overrightarrow{Sf(Y)}\right| =$$

$$= \left|-k \cdot \vec{SX} + k \cdot \vec{SY}\right| = \left|k \cdot \vec{XS} + k \cdot \vec{SY}\right| = \left|k \cdot (\vec{XS} + \vec{SY})\right| = \left|k \cdot \vec{XY}\right| =$$

$$= |k| \cdot \left|\vec{XY}\right| = |k| \cdot d(X, Y).$$

6. Examples of tasks for students

In this section we want to present some tasks connected with geometric transformations and their possible solutions. We want to draw your attention to solutions in which we use elements of theory of isometries and similarities as well as vector calculus. These solutions are much shorter, more simple and more intelligible than traditional intuitive ones.

Task 1. *Find the centre of line segment AB, if $A = (3, 4)$, $B = (11, 22)$.*

Solution: In the fourth educational level students know the formula for the centre of line segment but it is a conclusion from vector calculus. We can solve the above task as it follows:

Denote as $S = (x, y)$ the centre of a line segment AB. Then we have:

$$\vec{AS} = [x - 3, y - 4], \vec{SB} = [11 - x, 22 - y].$$

Of course $\vec{AS} = \vec{SB}$, so we obtain equations $x-3 = 11-x$ and $y-4 = 22-y$. Finally we obtain $x = 7, y = 13$ and $S = (7, 13)$.

Task 2. *Give the example of the figure that satisfies all the following conditions:*

a) it has infinite numbers of symmetry axes;
b) it has the symmetry centre that does not belong to this figure;
c) has positive and finite area.

Solution: It is a very interesting task because the only one type of figures satisfies all conditions. These are circle rings.

Task 3. *Assume that* $A = (1, 1)$*,* $B = (3, 7)$*. Find coordinates of the points that divide the line segment* AB *to three equal parts.*

Solution: Here the vector calculus is also the best tool. Let us denote by $S = (x, y)$ and $T = (u, v)$ the searched points. We have

$$\vec{AB} = [2, 6], \vec{AS} = [x - 1, y - 1],$$
$$\vec{AS} = \tfrac{1}{3}\vec{AB} = \left[\tfrac{2}{3}, 2\right].$$

So we obtain $x = 1\frac{2}{3}, y = 3$ and $S = \left(1\frac{2}{3}, 3\right)$. We can find the coordinates of point T in the same way but we also can use the fact that point T is an image of point S in the translation by the vector $\vec{AS}$ so we finally have:

$$T = (u, v) = \left(1\tfrac{2}{3}, 3\right) + \left[\tfrac{2}{3}, 2\right] = \left(2\tfrac{1}{3}, 5\right).$$

If we want to divide a line segment into more parts we proceed analogously.

Task 4. *Find an image of point* $A = (2, 7)$ *in the homothetic transformation of centre* $S = (2, 11)$ *and scale* $k = -\frac{1}{2}$*.*

Solution: Denote by $A' = (x, y)$ the image of a point A in the above transformation. We have:

$$\vec{SA'} = [x - 2, y - 11],$$
$$\vec{SA} = [0, -4].$$

By the definition 14. we get:

$$[x - 2, y - 11] = -\tfrac{1}{2} \cdot [0, -4] = [0, 2].$$

So we finally obtain $x = 2, y = 13$ and $A' = (2, 13)$.

Task 5. *Find the images of the rectangle $ABCD$:*

a) in homothetic transformation with scale $k = 2$ and centre S located inside the rectangle;

b) in homothetic transformation with scale $k = -\frac{1}{4}$ and centre S located outside the rectangle;

c) in homothetic transformation with scale $k = -3$ and centre S being any vertex of the rectangle.

Remark: It is a very important task. Finding the images of figures in homothetic transformations (especially if the scale is negative) makes possible to notice very interesting facts. It can be easily shown (even on drawings) that if the scale k of homothetic transformation is negative then this transformation is a superposition of homothetic transformation of a positive scale $-k$ and point symmetry towards point S (or U-turn). Students are used to positive scale. In homothetic transformation we also have negative scales thus students may find it difficult to understand this fact.

Task 6. *Which sentences are true?*

a) If the figure has symmetry axis it also has the symmetry centre;

b) If the figure has more than one symmetry centre then it has the infinite number of symmetry centres;

c) If the tetragon has a symmetry centre then it is a rhomboid;

d) Every regular polygon has the symmetry centre;

e) Symmetry centre of a figure always belongs to it;

f) If the triangle has more than one symmetry axis then it is regular;

g) Perpendicular bisector of a line segment is the only one symmetry axis of a line segment;

h) Regular pentagon has the symmetry centre.

Remark: Some of those problems are very important. For example regular polygon has the symmetry centre if and only if the number of its sides is even and every tetragon that has a symmetry centre is a rhomboid.

7. Final remarks

If we analyse the above considerations and examples of tasks we can draw the conclusion that introduction the formal definitions of geometric transformations in 3-rd educational level in proper and intelligible way is possible and makes many geometric tasks easy. It is also significant that formal base causes good comprehension of the above geometric concepts hence a chance for easier work in the third and fourth educational level. In addition, it should be emphasized that vector calculus is the best and fastest tool to solve problems from analytic geometry.

References

[1] U. Trelińska, G. Treliński, *Kształtowanie pojęć geometrycznych na etapie przeddefinicyjnym*, MAT & MET, Kielce 1996.
[2] J. Knop, *Geometria*, Wydawnictwo AJD w Częstochowie, Częstochowa 2007.
[3] J. Jędrzejewski, A. Vizváry, M. Ziółkowski, *Matematyka – Świat liczb* – poradnik dla nauczyciela gimnazjum klasa I, Res Polona, Łódź 2012.
[4] J. Jędrzejewski, A. Vizváry, M. Ziółkowski, *Matematyka – Świat liczb* – podręcznik dla uczniów gimnazjum klasa I, Res Polona, Łódź 2012.
[5] J. Jędrzejewski, A. Vizváry, M. Ziółkowski, *Matematyka – Świat liczb* – zbiór zadań dla uczniów gimnazjum klasa I, Res Polona, Łódź 2012.
[6] J. Jędrzejewski, A. Vizváry, M. Ziółkowski, *Repetytorium z matematyki*, Res Polona, Łódź 2012.
[7] J. Jędrzejewski, A. Vizváry, M. Ziółkowski, *Matematyka – Świat liczb* – program nauczania matematyki dla gimnazjum klasa I - III, Res Polona, Łódź 2012.
[8] `http://www.bip.men.gov.pl/men_bip/akty_prawne/rozporzadzenie_20081223_zal_4.pdf` (2014, June 20$^{\text{th}}$)
[9] `http://www.oke.krakow.pl/inf/filedata/files/men_tom_6.pdf` (2014, June 20$^{\text{th}}$)

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