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Note on describing function analysis of fractional order nonlinear control systems*

by

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Abstract: This paper presents extensions of some results, obtained for the analysis of classical nonlinear control systems, to the nonlinear fractional order systems. It is shown that the results related to limit cycle prediction using describing function method can be applied to the fractional order plants. The frequency and the amplitude of the limit cycle are used for auto-tuning of the PID controller for nonlinear control systems with fractional order transfer functions. Fractional order control system with parametric uncertainty is also considered for the nonlinear case. On the other hand, a new method is provided for stability margin computation for fractional order nonlinear control system with parametric uncertainty structure using the Nyquist envelopes of the fractional order uncertain plant and the describing function that represents the nonlinearity of the system. Maximum perturbation bounds of the parameters of the fractional order plant are computed. Numerical examples are included to illustrate the methods presented.

Keywords: describing function, fractional order control, limit cycle, nonlinear systems, relay auto-tuning, stability margin

1. Introduction

Increasing popularity of the fractional calculus encouraged many researchers to investigate new features of the fractional order systems (Samko et al., 1993; Oustaloup et al., 2000; Oldham and Spanier, 1974; Miller and Ross, 1993). In this regard, considerable attention has been paid to the fractional order control systems (FOCS) (Podlubny, 1999). Some important results dealing with the applications of the fractional calculus to the control systems have been extensively studied (Oustaloup et al., 1999). Fractional differentiation becomes an important tool in scientific and industrial applications due to the development in the use of fractional differentiation in various fields during last two decades

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(Hwang and Cheng, 2006; Sabatier et al., 2004; Malti et al., 2008; Nataraj, 2008; Bettou et al., 2008). Thus, the extension of the results obtained for classical control systems to the fractional order one, will be important (Yeroglu et al., 2010). In this paper, some results related to the limit cycle analysis of the classical nonlinear systems are extended to nonlinear FOCS and some results related to the stability margin computations, obtained for classical nonlinear control, are extended to fractional order interval control system (FOICS) in the presence of nonlinearity.

The stability analysis of the nonlinear control systems is often carried out under the conditions of existence of sustained oscillations known as limit cycles (Glad and Ljung, 2000). The most powerful method to analyze the limit cycle is known as the Describing Function method (Oliveira et al., 2006; Nataraj and Kalla, 2009). Theoretically, the frequency at the intersection point of the Nyquist plot of the system and the complex plot of the negative inverse of describing function is used in limit cycle analysis (Glad and Ljung, 2000). Some new results for determining the limit cycles using the approximate describing function method and an exact method have been lately announced in Atherton et al. (2014). In the present study, oscillation frequency of the limit cycle of nonlinear system is calculated at the intersection point of the Nyquist plot of the FOCS and the negative inverse plot of the describing function. Simulation results of the fractional order nonlinear systems with static nonlinearities are verified with the theoretical values of the oscillation frequency.

Another important subject in classical control is the auto-tuning of the controller. The difficulty of the traditional frequency response method of auto-tuning is that the appropriate frequency of the input signal must be chosen. But the nonlinear feedback of the relay method can generate a limit cycle oscillation. Using an ideal relay, this method gives an input signal to the process with a period close to the ultimate frequency of the open loop system. The period and amplitude of the oscillation give the ultimate period and the ultimate gain. Then, the parameters of the PID controller can be determined from these values (Åström and Hägglund, 2006) and the auto-tuning of controllers in classical nonlinear control can be extended to FOCS. Some studies related to the auto-tuning of the FOCS have been presented in Monje et al. (2007, 2008). The results presented in this paper for obtaining the limit cycle are used for auto-tuning of PID controllers. The frequency and amplitude of the oscillation is determined and the parameters of the PID controller are obtained using the classical Zigler-Nichols and Åström–Hägglund methods. Robust performance of the PID controllers in controlling the fractional order system and the fractional order plant with parametric uncertainty structure are demonstrated via illustrative examples.

On the other hand, the frequency domain analysis of systems is also an important topic in control theory. It is clear that the extensions of the frequency domain analysis of the classical control to the FOICS will provide useful tools for analysis and design. The Nyquist and Bode envelopes have been obtained for frequency response analysis of FOICS in recent studies (Yeroglu et al., 2010). In this paper, the Nyquist envelope is used for the stability margin computation of nonlinear FOICS. As known from the classical control theory, the computation of the stability margin is important in the analysis and design of the control systems (Ogata, 2002). Due to the importance of this issue, several studies have been published for the stability margin computation (Datta et al., 2009). For example; a numerical method for computing the stability margin of time-delay linear time-invariant systems with delay dependence by using a constrained simulated annealing algorithm is provided in Kim and Bae (2006). An algorithm for finding the stability margins and cross over frequencies for an uncertain fractional-order system using the interval constraint propagation technique is proposed in Nataraj and Kalla (2010). Nyquist robust stability margin and its application to systems with real affine parametric uncertainties are discussed in (Baab et al., 2001). Most of the studies, related to the stability margin computation, reported in the literature, have been performed for classical control systems. Extension of the results of the stability margin computation in classical nonlinear control to the FOICS in the presence of nonlinearity shall, as well, contribute to the studies in this field.

Consequently, one of the contributions of the present paper is to use the describing function method, combined with the auto-tuning technique, for nonlinear FOCS and FOICS. The procedure, given for the stability margin computation is another contribution of the paper. Here, the Nyquist envelope of the fractional order interval plant is combined with the classical describing function method to compute the maximum allowable parameter perturbations while preserving the stability.

The rest of the paper is organized as follows: in Section 2, mathematical background is briefly summarized. Describing function analysis for nonlinear fractional order control systems is given in Section 3. Relay auto-tuning of the PID controller for fractional order nonlinear systems is presented in Section 4. Stability margin computation of the fractional order nonlinear interval system is discussed in Section 5. Section 6 contains the concluding remarks.

2. Brief mathematical background

Fractional calculus can be considered to be the generalization of integration and differentiation of the integer order expressions to the non-integer order ones. The most frequently used integro-differential definitions are Grünwald-Letnikov, Riemann-Liouville and Caputo expressions (Xue et al., 2007; Caponetto et al., 2010). Generally, dynamic behaviors of systems can be analyzed using the transfer function of the control system. Thus, the Laplace transformations of the integro-differential expressions for fractional order control systems are important. Fortunately, the difference between the Laplace transformation of the fractional order expression and the integer order one is not very important. The most general formula for the Laplace transformations of the integro-differential expressions can be given as (see Caponetto et al., 2010),

$$L\left\{\frac{d^m f(t)}{dt^m}\right\} = s^m L\left\{f(t)\right\} - \sum_{k=0}^{n-1} s^k \left[\frac{d^{m-1-k} f(t)}{dt^{m-1-k}}\right]_{t=0} \tag{1}$$

where n is an integer number and m satisfies $n-1 < m < n$. The transfer function of a fractional order plant can be obtained using Eq. (1) as follows:

$$G(s) = \frac{Y(s)}{X(s)} = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + ... + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + ... + a_0 s^{\alpha_0}} \tag{2}$$

where $\alpha_n > \alpha_{n-1} > \cdots > \alpha_0 \geq 0$, $\beta_m > \beta_{m-1} > \cdots > \beta_0 \geq 0$, a_k $(k = 0, 1, 2, ..., n)$, and b_l $(l = 0, 1, 2, ..., m)$, are constants (Xue et al., 2007). Upon substituting $s = j\omega$ in the transfer function of the control system in Eq. (2), the frequency domain analysis of the fractional order control system can be performed.

3. Describing function analysis for nonlinear fractional order control systems

Describing function analysis is a widely known technique for studying the frequency response of nonlinear systems. It is an extension of the linear frequency response analysis. In linear systems, transfer functions depend only on the frequency of the input signal. In nonlinear systems, when a specific class of input signals, such as sinusoidal ones, is applied to a nonlinear element, one can represent the nonlinear element by a function that depends not only on frequency, but also on input amplitude. This function is referred to as a describing function. Describing function analysis has a wide area of applications, which is covered in many books and papers (see, for instance, Vukic et al., 2003; Khalil, 1996). Consider the nonlinear control system, given in Fig.1.

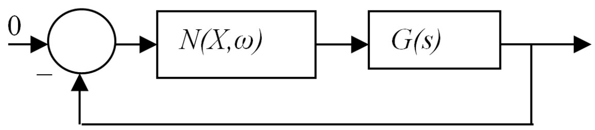

Figure 1. Nonlinear control system

The nonlinearity of the system is represented with $N(X, \omega)$, and the plant is represented with $G(s)$. X is the amplitude of the input signal of the nonlinear element and ω is the frequency of the oscillation. The characteristic equation of the system can be given as

$$\Delta(j\omega) = 1 + N(X, \omega)G(j\omega) = 0. \tag{3}$$

The plant $G(s)$ in Eq. (3) is the linear part of the characteristic equation of the system. One can conclude that the existence of the limit cycle can be predicted with the following relationship, if the input is taken to be zero (Glad and Ljung, 2000):

$$G(j\omega) = \frac{-1}{N(X,\omega)}. \tag{4}$$

The negative inverse plot of the describing function $N(X,\omega)$ in the complex plane and the Nyquist plot of the plant $G(j\omega)$ can be obtained as defined in Xue et al. (2007). It is clear from Eq. (4) that the occurrence of the possible limit cycle can be predicted, if there is any intersection between these two curves. Namely, frequency value at the intersection point of these two curves will be the oscillation frequency of the limit cycle. Frequency domain analysis of the transfer function enables us to design a suitable controller or compensator for the desired system. The frequency domain analysis of the FOCS can be conducted in the similar way as for the integer order one. The frequency domain expression can be easily obtained by substituting $s = j\omega$ in the Laplace transform of the transfer function. Since the describing function method is a frequency domain approach, it can be applied to the FOCS to analyze the nonlinearity. Thus, the extension of the results of frequency response analysis in classical nonlinear control to the nonlinear FOCS can be implemented. Let the nonlinearity in the negative feedback control system of Fig. 1 be saturation nonlinearity $N(X,\omega)$ and the plant $G(s)$ be a fractional order transfer function. The saturation nonlinearity can be expressed as (Glad and Ljung, 2000),

$$N = \frac{2k}{M\pi}\left[\arcsin(\frac{S}{X}) + \frac{S}{X}\sqrt{1-\left(\frac{S}{X}\right)^2}\right] \tag{5}$$

where k is the slope of saturation, X is the amplitude of oscillation, M and S are the magnitudes of the saturation (see Fig. 2). A plot of the describing function of Eq. (5), normalized with slope, plotted versus X/S, is shown in Fig. 3. The Nyquist plot of the $G(s)$ can be obtained using toolbox in Yeroglu and Tan (2009) and the negative inverse of the describing functions for saturation nonlinearity can be computed using Eq. (5). Now, consider the transfer function with a fractional order representation as follows,

$$G(s) = \frac{K}{s^{3.2} + 4s^{2.2} + 4s^{1.2}}. \tag{6}$$

Consider that the describing function represents the saturation nonlinearity. The slope of the saturation and values of the S are taken as 1 for all cases, X/S values are calculated with respect to the N/k values from Fig. 3 and the magnitudes of the oscillation X are obtained for $S = 1$.

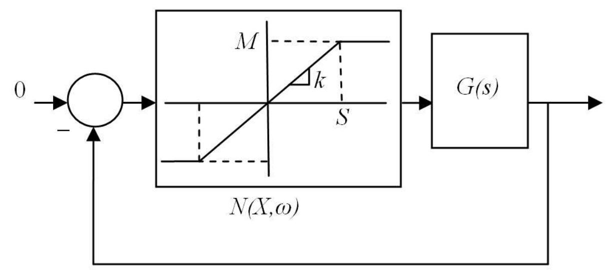

Figure 2. Control system with saturation nonlinearity

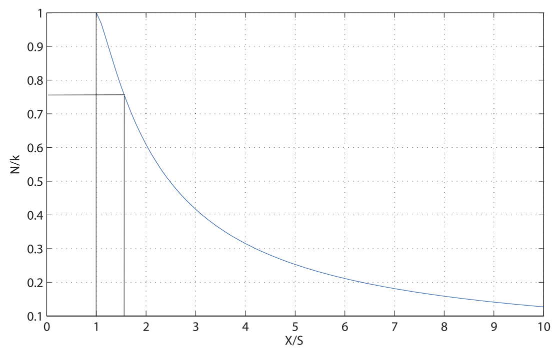

Figure 3. Plot of N/k versus X/S

Fig. 4 presents the Nyquist plot of the plant $G(s)$ for different values of the gain K and the negative inverse of the describing function in Eq. (5). One can compute the oscillation frequency using Fig. 4 and Eq. (4) as $\omega = 1.453\ rad/sec.$ for all K values. The value of the describing function, values of X/S and magnitude of the oscillations for $K = 8, 9.571, 12$ are given in Table 1.

Table 1. Parameters of the describing function for different gains of the plant at intersection point

Gain	-1/N values at intersection	N/k	$X/S(S = 1)$
$K = 8.0000$	No intersection	1.1962	-
$K = 9.5710$	-1.0000	1.0000	1.0000
$K = 12.0000$	-1.254	0.7974	1.4700

The value of the describing function can be computed as $N = 1$ for $X/S = 1$ and the slope $k = 1$ in Fig. 3. As a result, $-1/N = -1/1 = -1$, which means that the negative inverse of the describing function starts at the point -1 and goes through $-\infty$. One can make the simulation of the system using integer approximation of the fractional order transfer function (Ozyetkin and Tan, 2009). Fig. 5 shows that the simulation results of the nonlinear FOCS are in agreement with the calculated values for all cases in Table 1. As seen from Fig. 4, the Nyquist plot of $G(s)$ crosses the negative real axis at -1 point for $K = 9.5710$. We observed that $K = 9.5710 = K_{cr}$ is the critical gain value. The Nyquist curve of the FOCS crosses the negative inverse of the describing function for $K > K_{cr}$ and the limit cycle occurs for the system. One can conclude from Fig. 6 that the system will be stable for the gain $K < K_{cr}$ and becomes more stable for decreasing gain. Figs. 4 and 5 show that there will be a limit cycle for the value of $K > K_{cr}$. Parameters of the limit cycle can be used for the analysis and design of FOCS. In order to obtain the time response characteristics of the system in Figs. 5 and 6, 4^{th} order integer approximation of the fractional order plant, which is obtained using CFE (Continuous Fractional Expansion) method, are used in the Simulink model of the negative feedback system (Ozyetkin and Tan, 2009). On the other hand, obtaining of the exact time response of the fractional order systems from the frequency response data has been first studied in Atherton et al. (2015).

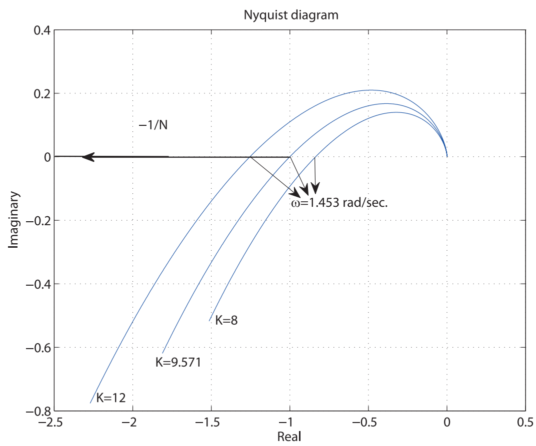

Figure 4. The Nyquist plots of $G(j\omega)$ and the $-1/N$ plot for saturation nonlinearity for $K = 8, 9.571, 12$

4. Relay auto-tuning of PID controller for fractional order nonlinear systems

Several methods for auto-tuning of PID controllers to control integer order control systems can be found in the literature (Chen and Moore, 2005; Chen et al., 2004). However, only few methods for auto-tuning of controllers have been proposed for fractional order control systems (Monje et al., 2007, 2008). In this sec-

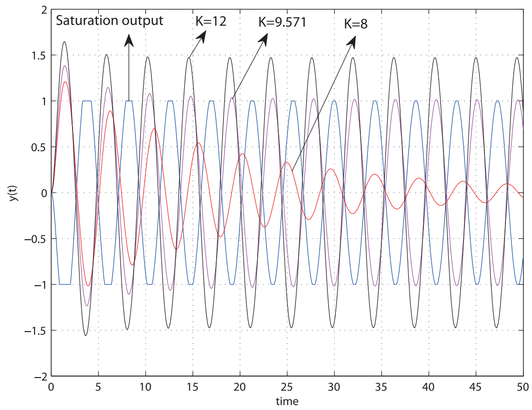

Figure 5. Limit cycle of fractional order system with saturation nonlinearity for $K = 8, 9.571, 12$

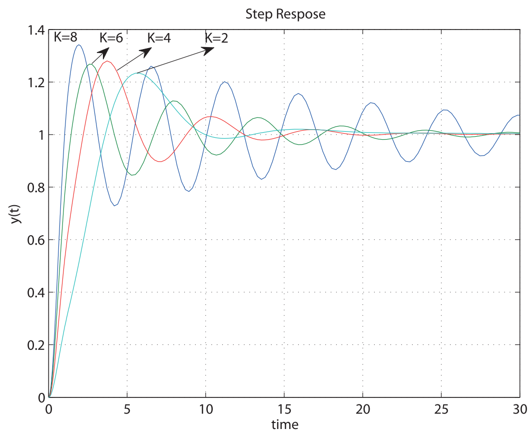

Figure 6. Step responses of the system for $K = 8, 6, 4, 2$

tion, the relay auto-tuning of fractional order nonlinear system is demonstrated using the idea of the describing function method for the nonlinear FOCS, which was presented in Section 3. In the classical Zigler–Nichols method, the critical gain and critical frequency were determined manually. In order to determine the critical gain and frequency, a better method was proposed by Åström and Hägglund. In this method, a relay was connected in a feedback loop with the plant as given in Fig. 7. The relay is connected to the loop in tuning mode. In this case, the error signal e is a periodic signal. The parameters, such as the critical gain K_c and the oscillation frequency ω_c, can be determined using the describing function method. The characteristic equation of the system can be obtained in the form of Eq. (3) and condition of the oscillation can be given as in Eq. (4). Thus, the frequency and amplitude of the oscillation of the limit cycle can be determined by using the following equations,

$$\mathrm{Re}\{1 + N(X,\omega)G(j\omega)\} = 0 \tag{7}$$

$$\mathrm{Im}\{1 + N(X,\omega)G(j\omega)\} = 0. \tag{8}$$

The critical gain K_c and oscillation period T_c are determined, respectively, as

$$K_c = \frac{4M}{\pi X} \quad \text{and} \quad T_c = \frac{2\pi}{\omega_c} \tag{9}$$

where X is the amplitude of the input signal of the nonlinear element and, generally, $M = 1$. Then, the parameters of the PID controller can be obtained using the classical Zigler–Nichols tuning rules as

$$k_p = 0.6K_c\,, \quad T_i = 0.5T_c \quad \text{and} \quad T_d = 0.125T_c. \tag{10}$$

The PID controller can be expressed as

$$C(s) = k_p(1 + \frac{1}{T_i s} + T_d s). \tag{11}$$

As known from the classical Åström and Hägglund tuning rule, the following equations can be written for the parameters of the PID controller for specified phase margin ϕ_m,

$$k_p = K_c \cos\ \phi_m\,, \quad \omega_c T_d - \frac{1}{\omega_c T_i} = \tan\phi_m \quad \textit{and} \quad T_i = \alpha T_d \tag{12}$$

where $\alpha = 4$ is chosen in most cases. Then, the PID controller can be obtained in the form of Eq. (11). Numerous PID tuning methods have been proposed in the literature. However, this paper intends to use critical values of the limit cycle of the fractional order system for controller design. Thus, basic and very well-known tuning methods, namely the classical Zigler-Nichols and Åström and Hägglund tuning rules are used to present the methodology clearly. The following two examples give the explicit applications of this section. Example 1 demonstrates the auto-tuning method for the fractional order nonlinear system and Example 2 extends the method for the same system with parametric uncertainty structure.

Example 1 Consider the plant $G(s)$ in Eq. (6) for $K = 16$ as follows

$$G_1(s) = \frac{16}{s^{3.2} + 4s^{2.2} + 4s^{1.2}}. \tag{13}$$

Theoretically, the value of the describing function can be computed from the intersection of the Nyquist curve of fractional order system and the negative inverse of the describing function using Eq. (5). The describing function for relay nonlinearity in Fig. 7 can be defined as

$$N = \frac{4M}{\pi X} \tag{14}$$

where $M = 1$ for most cases. The amplitude and the frequency of the possible limit cycle of FOCS can be determined easily using Eq. (14) with the information obtained from the intersection point of the Nyquist plot of the plant $G_1(s)$ and the negative inverse of the describing function for relay nonlinearity of Fig. 8. The simulation result for the same system, which is given in Fig. 9, verifies the theoretical values. The limit cycle parameters of the fractional order nonlinear system for relay nonlinearity can be calculated using Eq. (14) with the information from Figs. 8 and 9. In this example, the magnitude and the frequency of the limit cycle oscillation are computed, respectively, as, $\omega = 1.453$ *rad/sec.* and $X = 2.2$. Using Eqs. (9), (10) and (11) with the parameters of the limit cycle, the PID controller can be obtained as follows,

$$C_1(s) = 0.3446(1 + \frac{1}{2.1622s} + 0.5405s). \tag{15}$$

Different values of parameters of the controller $C_1(s)$ for different values of gain K can be obtained. The PID controller for $\phi_m = 45°$ phase margin can be obtained using Eqs. (9), (11) and (12) as follows,

$$C_2(s) = 0.3017(1 + \frac{1}{4.3504s} + 1.0876s). \tag{16}$$

The step responses of $C_1(s)G_1(s)$ and $C_2(s)G_1(s)$ are given in Fig. 10.

Example 2 Consider the plant in Eq. (6) with parameter uncertainty structure as follows,

$$G_2(s) = \frac{K}{a_2 s^{3.2} + a_1 s^{2.2} + a_0 s^{1.2}} \tag{17}$$

where $a_2 \epsilon\, [0.8, 1.2]$, $a_1 \epsilon\, [3.5, 4.5]$ and $a_0 \epsilon\, [3.5, 4.5]$. Different transfer functions of $G_2(s)$ can be obtained using lower and upper values of the uncertain parameters

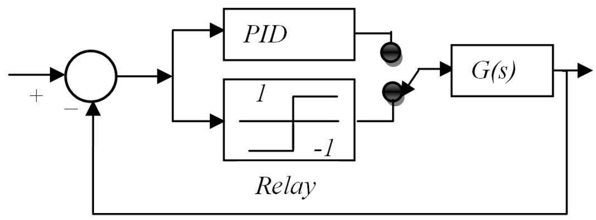

Figure 7. Relay auto-tuning system

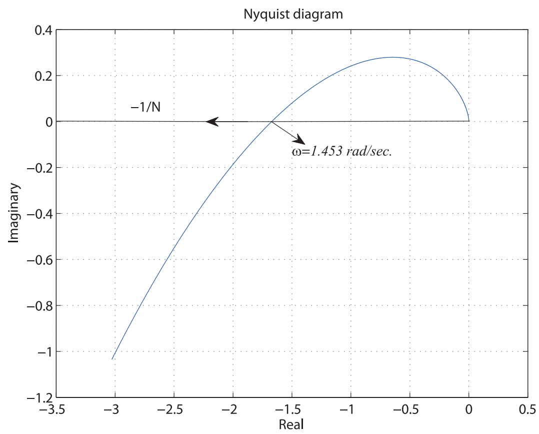

Figure 8. The Nyquist plots of $G_1(j\omega)$ and the $-1/N$ plot for relay nonlinearity (for $K = 16$)

of the plant. The parameters of the limit cycle can be obtained for the lower and upper values of the uncertain parameters as in Table 2.

It is possible to obtain different controllers for different values of the critical gain, period, amplitude and frequency of the limit cycle, as given in Table 2. These controllers satisfy the robust performance of the system for the values of $a_2 \epsilon\, [0.8, 1.2]$, $a_1 \epsilon\, [3.5, 4.5]$ and $a_0 \epsilon\, [3.5, 4.5]$.

Let $C_3(s)$ be one of the controllers given in Table 2. Figs. 11 and 12 illustrate the step responses and control signals of the system, which satisfy the robust performance of $C_3(s)G_2(s)$ for different values of the uncertain plant. These figures also illustrate good disturbance rejection performance of the controllers, the disturbance being applied to the system at the 30^{th} second. The Bode plots of the system $C_3(s)G_2(s)$ in Fig. 13 show that the controllers satisfy the required phase margin and robust performance.

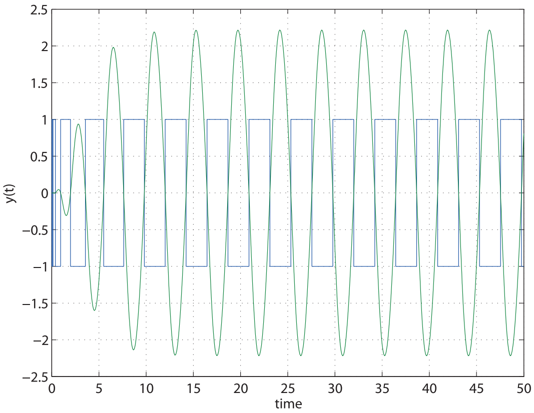

Figure 9. Limit cycle of fractional order system $G_1(s)$ with relay nonlinearity (for K=16)

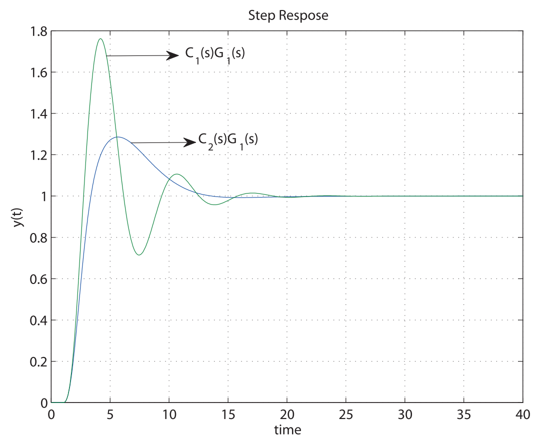

Figure 10. Step responses of $C_1(s)G_1(s)$ and $C_2(s)G_1(s)$

5. Stability margin computation of fractional order nonlinear interval system

The purpose of this section is to present the extension of some results, related to the stability margin computation in classical nonlinear control, to the FOICS in the presence of nonlinearity. The computation of the frequency responses of uncertain transfer functions plays an important role in the application of frequency domain methods for the analysis and design of robust control systems. For example, Nyquist envelope of a fractional order interval transfer function

Table 2. Values of the parameters of the limit cycle and the parameters of the PID controller in the form of $C_2(s)$ for different values of a_2, a_1 and a_0 (for $K = 16$ and $\phi_m = 45°$)

	X	K_c	T_c	ω_c	k_p	T_i	T_d
$a_2 = 0.8,\ a_1 = 3.5,\ a_0 = 3.5$	2.37	0.5372	4.1944	1.498	0.2822	4.7040	1.1760
$a_2 = 0.8,\ a_1 = 3.5,\ a_0 = 4.5$	1.64	0.7764	3.5599	1.765	0.4079	3.9924	0.9981
$a_2 = 0.8,\ a_1 = 4.5,\ a_0 = 3.5$	2.27	0.5609	4.5896	1.369	0.2947	5.1472	1.2868
$a_2 = 0.8,\ a_1 = 4.5,\ a_0 = 4.5$	1.55	0.8214	3.8595	1.628	0.4315	4.3284	1.0821
$a_2 = 1.2,\ a_1 = 3.5,\ a_0 = 3.5$	3.22	0.3954	4.8369	1.299	0.2077	5.4248	1.3562
$a_2 = 1.2,\ a_1 = 3.5,\ a_0 = 4.5$	2.25	0.5659	4.1337	1.520	0.2973	4.6360	1.1590
$a_2 = 1.2,\ a_1 = 4.5,\ a_0 = 3.5$	2.99	0.4258	5.2186	1.204	0.2237	5.8528	1.4632
$a_2 = 1.2,\ a_1 = 4.5,\ a_0 = 4.5$	2.07	0.6151	4.4217	1.421	0.3234	4.9592	1.2398

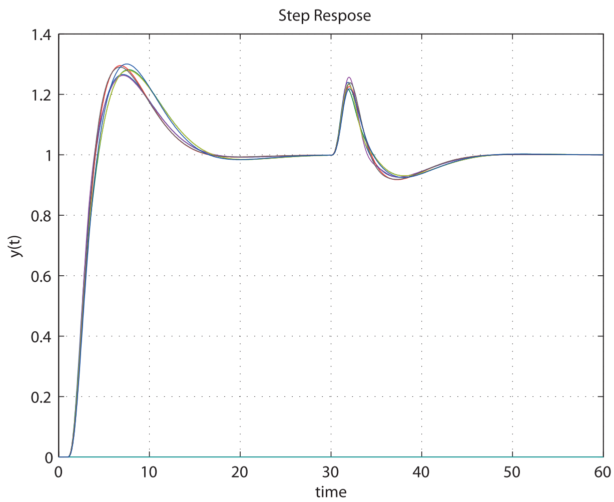

Figure 11. Step responses of the system $C_3(s)G_2(s)$ for different values given in Table 2

(FOITF) can be used for the stability margin computation of the nonlinear FOICS.

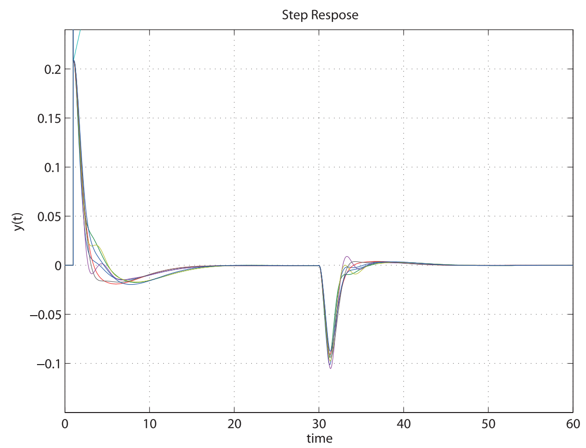

Figure 12. Control signals of the system $C_3(s)G_2(s)$ for different values given in Table 2

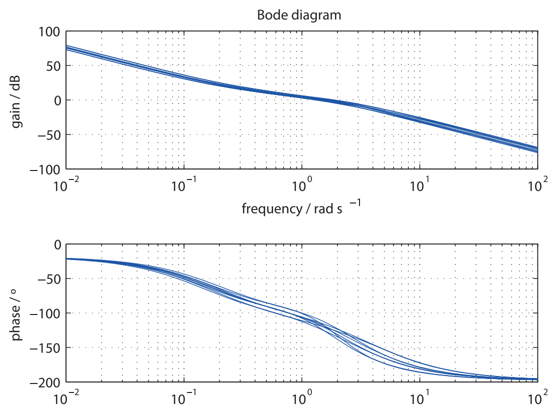

Figure 13. Bode plots of the system $C_3(s)G_2(s)$ for different values given in Table 2

Nyquist Envelope of FOITF

The numerator and denominator polynomials of a FOITF are the fractional order interval polynomials (FOIP) of the form,

$$P(s,q) = q_0 s^{\alpha_0} + q_1 s^{\alpha_1} + q_2 s^{\alpha_2} + q_3 s^{\alpha_3} + \cdots + q_n s^{\alpha_n} \tag{18}$$

where $\alpha_0 < \alpha_1 < \cdots < \alpha_n$ are generally real numbers, $q = [q_0, q_1, q_2, \cdots, q_n]$ is the uncertain parameter vector, and the uncertainty box is
$Q = \left\{ q : q_i \epsilon \left[\underline{q}_i, \overline{q}_i \right], i = 0, 1, 2, ..., n \right\}$.
Here, $\underline{q}_i$ and $\overline{q}_i$ are specified lower and upper bounds of i^{th} perturbation q_i, respectively. Thus, a FOITF can be represented as,

$$G(s, a, b) = \frac{N(s, b)}{D(s, a)} = \frac{b_0 s^{\alpha_0} + b_1 s^{\alpha_1} + b_2 s^{\alpha_2} + ... + b_m s^{\alpha_m}}{a_0 s^{\beta_0} + a_1 s^{\beta_1} + a_2 s^{\beta_2} + ... + a_n s^{\beta_n}} \tag{19}$$

where $\alpha_0 < \alpha_1 < \ldots < \alpha_m$ and $\beta_0 < \beta_1 < \cdots < \beta_n$ are generally real numbers, $a = [a_0, a_1, a_2 \cdots, a_n]$ and $b = [b_0, b_1, b_2 \cdots, b_m]$ are uncertain parameter vectors, $A = \{a : a_i \epsilon [\underline{a}_i, \overline{a}_i], i = 0, 1, 2, ..., n\}$ and $B = \left\{ b : b_i \epsilon \left[\underline{b}_i, \overline{b}_i \right], i = 0, 1, 2, ..., m \right\}$ are uncertainty boxes. It is first shown that the value set of the family of polynomials of Eq. (18) can be constructed using the upper and lower values of uncertain parameters. Then, using the geometric structure of the value set, the Nyquist envelopes of FOITF, represented by Eq. (19), can be computed. For FOIP of Eq. (18), substituting $s = j\omega$ gives,

$$\begin{aligned} P(j\omega, q) &= q_0(k_{0r} + jk_{0i})\omega^{\alpha_0} + \cdots + q_n(k_{nr} + jk_{ni})\omega^{\alpha_n} \\ &= (q_0 k_{0r}\omega^{\alpha_0} + \cdots + q_n k_{nr}\omega^{\alpha_n}) + j(q_0 k_{0i}\omega^{\alpha_0} + \cdots + q_n k_{ni}\omega^{\alpha_n}) \end{aligned} \tag{20}$$

where k_{lr} and k_{li}, $l = 1, 2, ..., n$ are constants. It is clear from Eq. (20) that the uncertain parameters appearing in both the real and imaginary parts are linearly dependent on each other. The value set of such a polynomial in the complex plane is a polygon. Thus, the corresponding polytope of a family of Eq. (18) in the coefficient space has $2^{(n+1)}$ vertices and $(n+1)2^n$ exposed edges, since the polynomial family has $(n+1)$ uncertain parameters. All of the $2^{(n+1)}$ vertex polynomials of $P(s, q)$ can be written using the upper and lower values of the uncertain parameters, in the following pattern

$$\begin{aligned} &v_1(s) = \underline{q_0} s^{\alpha_0} + \underline{q_1} s^{\alpha_1} + \underline{q_2} s^{\alpha_2} + \cdots + \underline{q_n} s^{\alpha_n} \\ &v_2(s) = \overline{q_0} s^{\alpha_0} + \underline{q_1} s^{\alpha_1} + \underline{q_2} s^{\alpha_2} + \cdots + \underline{q_n} s^{\alpha_n} \\ &\vdots \\ &v_{2^{(n+1)}}(s) = \overline{q_0} s^{\alpha_0} + \overline{q_1} s^{\alpha_1} + \overline{q_2} s^{\alpha_2} + \cdots + \overline{q_n} s^{\alpha_n}. \end{aligned} \tag{21}$$

From these vertex polynomials the exposed edges can be obtained. For example, the vertex polynomials $v_1(s)$ and $v_2(s)$ have the same structure, except that the parameter q_0 is its lower value $\underline{q}_0$ in $v_1(s)$ and its upper value $\overline{q}_0$ in $v_2(s)$. Thus, one of the exposed edges can be expressed as

$$e(v_1, v_2) = (1 - \lambda)\, v_1(s) + \lambda\, v_2(s) \tag{22}$$

where $\lambda \epsilon [0, 1]$. The numerator and denominator polynomials of FOITF of Eq. (19) are in the form of $P(s, q)$ of Eq. (18). Therefore, the results given above can be used to obtain the Nyquist envelope of FOITF. The detailed algorithm for obtaining the Nyquist envelope of FOITF can be found in Yeroglu et al. (2010).

Stability margin computation

Parameters of all real systems include different types of uncertainties due to the tolerance values of the components, environmental conditions and nonlinear effects. Parameter uncertainty is, namely, inevitable in the real systems. In control theory, robust control methods have been developed for the analysis and design of the systems with parametric uncertainty structure. Determination of the maximum allowable perturbation bounds of the parameters of the control system which preserves stability is one of the important problems in robust analysis of the system (Tan and Atherton, 2002). In this section, a graphical procedure for the stability margin computation of the fractional order nonlinear interval control system is demonstrated.

Consider the feedback closed loop nonlinear control system of Fig. 2 with saturation nonlinearity. In this case, the nonlinearity of the system can be expressed with Eq. (5). In the case of $M = 1$, $S = 1$, $X = 1$ and $k = 2$ in Eq. (5), the value of the describing function can be calculated as $N = 2$. The negative inverse of the describing function is $-1/N = -1/2 = -0.5$, namely, the negative inverse of the describing function starts at the point -0.5 and goes through $-\infty$. If $G(s)$ of Fig. 2 represents the fractional order interval plant, the Nyquist envelope of the plant can be constructed using the procedure given in Yeroglu et al. (2010). The Nyquist envelope of the fractional order interval plant and the plot of the negative inverse of the describing function in complex plane can be obtained as given in Fig. 14. If there is any intersection between these two plots, one can conclude from the results of Section 3 that there is a limit cycle oscillation in the system. The frequency and amplitude of this limit cycle can be computed from the intersection point of the Nyquist envelope of FOITF and the negative inverse of the describing function. If the Nyquist curve of the system does not cross the negative inverse of the describing function, there will be no limit cycle in the system and the nonlinear fractional order system becomes stable for the values of the parameters of the fractional order interval plant. The question is how big a perturbation of the parameters of fractional order plant can be applied while preserving the stability of the fractional order nonlinear interval system. Perturbation bounds of the parameters of the plant can be identified using the graphical approach of Fig. 14.

The procedure given below can be followed for stability margin computation of the fractional order nonlinear interval control system:

- Specify the nonlinearity of the system and select the appropriate describing function that represents the related nonlinearity.
- Obtain the complex plot of the negative inverse of the describing function in a complex plane.
- Construct the Nyquist envelope of the fractional order interval plant.

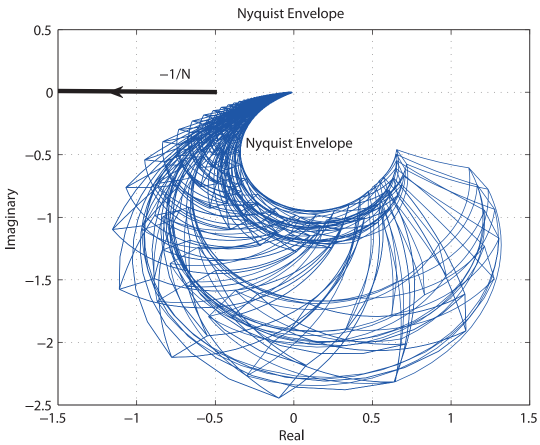

Figure 14. The Nyquist envelopes of the fractional order interval plant and the complex plot of $-1/N$

- If there is an intersection between these two plots, there is oscillation in the system.
- If there is no intersection between these two plots, then change the interval of the parameters of the fractional order plant and reconstruct the Nyquist envelope. Repeat this step until the Nyquist envelope of the fractional order interval plant just touches the plot of the negative inverse of the describing function.
- The stability margin of the system can be computed from the magnitude of perturbation of the parameters of the fractional order plant.

Three different fractional order plants are considered in this section to illustrate the method. Examples 3 and 4 concern the fractional order interval plants, while Example 5 concerns the fractional version of a first order plus dead time system. Saturation nonlinearities are applied to the systems in the examples. Maximum perturbation bounds of the parameters of the fractional order plant, which preserve stability of the nonlinear systems, are investigated.

Example 3 Consider that the transfer function $G(s)$ in Fig. 2 is given as follows,

$$G_3(s) = \frac{b_0 s^{2.2} + b_1 s^{1.1} + b_2}{a_0 s^{4.3} + a_1 s^{3.2} + a_2 s^{2.1} + a_3 s^{0.9} + a_4}. \tag{23}$$

Nominal values of the parameters are given as: $b_0 = 1$, $b_1 = 3$, $b_2 = 60$, $a_0 = 1$, $a_1 = 3$, $a_2 = 35$, $a_3 = 40$, $a_4 = 60$. Let the perturbation of the parameters b_1, b_2, a_3 and a_4 be 1% percent, while the other parameters remain the same as the nominal values. As shown in Fig. 15, the Nyquist envelope can be constructed and the plot of $-1/N$ can be obtained using the parameters of $M = 1$, $S = 1$, $X = 1$ and $k = 2$ in Eq. (5). One can conclude from Fig. 15 that the system still preserves the stability. If one perturbs the parameters b_1, b_2, a_3 and a_4 by 11% percent, while the other parameters remain the same as the nominal values, the Nyquist envelope can be constructed as shown in Fig. 16.

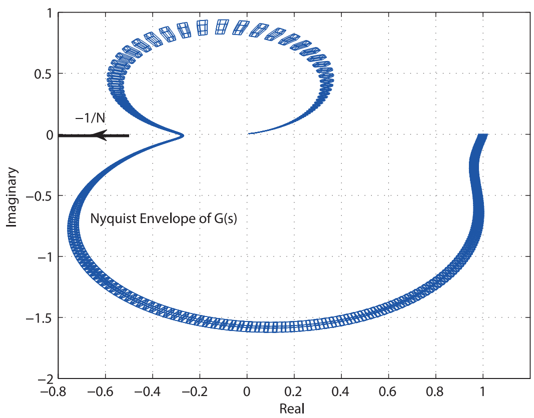

Figure 15. Plot of $-1/N$ and Nyquist envelope of $G_3(s)$ from Eq. (23) for 1% percent perturbations of parameters b_1, b_2, a_3 and a_4

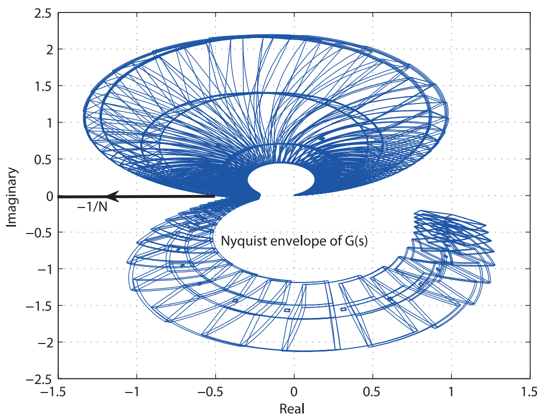

Figure 16. Plot of $-1/N$ and Nyquist envelope of $G_{3u}(s)$ from Eq. (24), for $0.1 < \omega < 10\ rad/sec$

The Nyquist envelope of the fractional order plant in Fig. 16 just touches the plot of $-1/N$ for the frequency $\omega = 4.273\ rad/sec$. One can conclude from Fig. 16 that the stability of the fractional order nonlinear system can be preserved for 11% percent perturbation of the parameters b_1, b_2, a_3 and a_4. Thus, the system is stable for the following uncertain fractional order plant:

$$G_{3u}(s) = \frac{s^{2.2} + [2.67,\ 3.33]s^{1.1} + [53.4,\ 66.6]}{s^{4.3} + 3s^{3.2} + 35s^{2.1} + [35.6,\ 44.4]s^{0.9} + [53.4,\ 66.6]}. \tag{24}$$

For this example, one can conclude that the system still preserves stability until parameter perturbations reach 11% percent. If the parameters are perturbed more than 11% percent, the system will be unstable.

Example 4 Consider that an integrating plant is given as follows

$$G_4(s) = \frac{b_0}{s(a_0 s^{2.4} + a_1 s^{1.3} + a_2 s^{0.2})}. \tag{25}$$

Nominal values of the parameters of $G_4(s)$ are given as $b_0 = 0.5$, $a_0 = 1$, $a_1 = 2$, $a_2 = 1$. For the values of the describing function $M = 1$, $S = 1$, $X = 1$ and $k = 1$ in Eq. (5). The plot of $-1/N$ begins from -1 and goes through $-\infty$. The Nyquist envelope of the plant $G_4(s)$ for 13% percent perturbations of all the parameters and the plot of $-1/N$ are given in Fig. 17. The Nyquist envelope of this system just touches the plot of $-1/N$ at the frequency $\omega = 0.641\ rad/sec$. Consequently, one can conclude that the system preserves stability for the uncertain fractional order plant given in Eq. (26). Thus, the perturbation bound of the parameters is 13% percent for this example. Exceeding the threshold of 13% percent perturbation will drive the system to unstable state.

$$G_{4u}(s) = \frac{[0.435,\ 0.565]}{[0.87,\ 1.13]s^{3.4} + [1.74,\ 2.26]s^{2.3} + [0.87,\ 1.13]s^{1.2}}. \tag{26}$$

Example 5 First order plus dead time (FOPDT) systems provide simple characterization of a process and give valuable information about dynamics of many applications in process control industry (Roy and Iqbal, 2004). The investigation of the stability margin for the FOPDT system will be useful. Let the plant $G(s)$ in Fig. 2 represent the fractional version of the FOPDT system with the following transfer function,

$$G_5(s) = \frac{b_0}{a_0 s^{1.2} + a_1} e^{-0.65s}. \tag{27}$$

Nominal values of the parameters of $G_5(s)$ are given as $b_0 = 1$, $a_0 = 1$ and $a_1 = 1$. The plot of $-1/N$ can also be obtained for the parameters of $M = 1$,

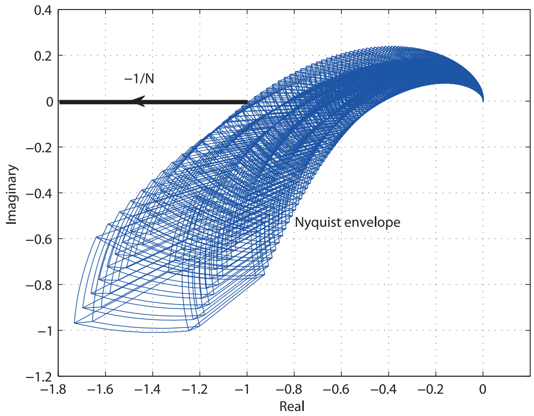

Figure 17. Plot of $-1/N$ and Nyquist envelope of $G_{4u}(s)$ for $0.4 < \omega < 10\ rad/sec$

$S = 1$, $X = 1$ and $k = 2$. Fig. 18 presents the plot of $-1/N$ and the Nyquist envelope of the fractional order time delay system of Eq. (27) for 17% percent perturbations of all parameters of $G_5(s)$. One can conclude from Fig. 18 that the system preserves stability for the fractional order uncertain plant given in Eq. (28). In analogy to the previous two examples, exceeding the 17% percent perturbation bound of the parameters will cause instability.

$$G_{5u}(s) = \frac{[0.83,\ 1.17]}{[0.83,\ 1.17]s^{1.2} + [0.83,\ 1.17]} e^{-0.65s}. \tag{28}$$

6. Conclusions

In this paper, extensions of some results, obtained for classical nonlinear control, to the nonlinear FOCS are studied. A method is presented for prediction of the limit cycle of fractional order nonlinear control systems. Parameters of the limit cycle of the nonlinear FOCS are used for auto-tuning of the PID controller. Auto-tuning method is demonstrated for the plant with the parameter uncertainty structure. On the other hand, the idea of the describing function method is used for stability margin computation of fractional order nonlinear control system with parametric uncertainty structure. Maximum perturbation bounds of the fractional order plant are investigated. The proposed method clearly shows that the parameters of the fractional order system can be perturbed within a certain interval, while preserving the stability of the system. Perturbation interval of each parameter of the plant may be investigated individually or a relative perturbation proportion may be found for all parameters of the plant.

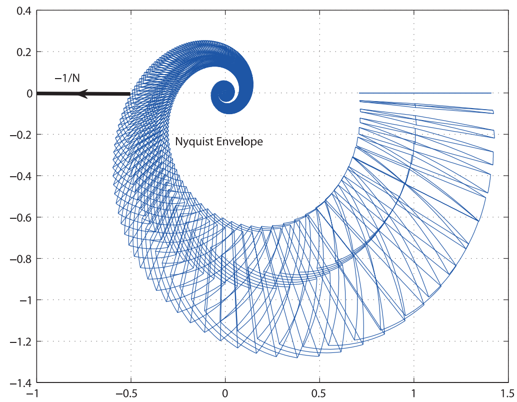

Figure 18. Plot of $-1/N$ and Nyquist envelope of $G_{5u}(s)$ for $0 < \omega < 1000\ rad/sec$

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