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ON AMBARZUMIAN TYPE THEOREMS FOR TREE DOMAINS

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Abstract. It is known that the spectrum of the spectral Sturm–Liouville problem on an equilateral tree with (generalized) Neumann's conditions at all vertices uniquely determines the potentials on the edges in the unperturbed case, i.e. case of the zero potentials on the edges (Ambarzumian's theorem). This case is exceptional, and in general case (when the Dirichlet conditions are imposed at some of the pendant vertices) even two spectra of spectral problems do not determine uniquely the potentials on the edges. We consider the spectral Sturm–Liouville problem on an equilateral tree rooted at its pendant vertex with (generalized) Neumann conditions at all vertices except of the root and the Dirichlet condition at the root. In this case Ambarzumian's theorem can't be applied. We show that if the spectrum of this problem is unperturbed, the spectrum of the Neumann-Dirichlet problem on the root edge is also unperturbed and the spectra of the problems on the complimentary subtrees with (generalized) Neumann conditions at all vertices except the subtrees' roots and the Dirichlet condition at the subtrees' roots are unperturbed then the potential on each edge of the tree is 0 almost everywhere.

Keywords: Sturm–Liouville equation, eigenvalue, equilateral tree, star graph, Dirichlet boundary condition, Neumann boundary condition.

Mathematics Subject Classification: 34B45, 34B24, 34L20.

1. INTRODUCTION

The following uniqueness theorem was proved by V. Ambarzumian in 1929 [1].

Theorem 1.1 (Ambarzumian)**.** *If the spectrum of the boundary value problem*

$$-y'' + q(x)y = \lambda y,$$

$$y'(0) = y'(\pi) = 0$$

with $q(x) \in C[0,\pi]$ *real-valued is* $0 \cup \{k^2\}_{k=1}^{\infty}$ *then* $q(x) \equiv 0$.

This theorem was the starting point of the Sturm–Liouville inverse spectral theory. The next step was made by G. Borg [3] who proved that the case considered by V. Ambarzumian was exceptional and in general two spectra are needed to determine the potential. Borg's theorem is as follows. Denote by $\{\vartheta_k\}_{k=1}^{\infty}$ the spectrum of the problem

$$-y'' + q(x)y = \lambda y,$$

$$y'(0) - hy(0) = 0, \quad y'(\pi) + Hy(\pi) = 0,$$

where $h \in \mathbb{R}$, $H \in \mathbb{R}$ and by $\{\xi_k\}_{k=1}^{\infty}$ the spectrum of the problem

$$-y'' + q(x) = \lambda y,$$

$$y'(0) - hy(0) = 0, \quad y(\pi) = 0.$$

Theorem 1.2 ([3])**.** *Let $q(x) \in L_1(0,a)$ be real-valued. Then the two spectra $\{\vartheta_k\}_{k=1}^{\infty}$ and $\{\xi_k\}_{k=1}^{\infty}$ determine uniquely $q(x)$ and the numbers h and H.*

There is vast literature on generalizations of Borg's theorem (see, e. g. [6, 11, 12]). In some sense similar situation occurs in the case of the Sturm–Liouville problem on a metric tree graph. There are generalizations on Borg's theorem for trees [5, 20] and generalizations of Ambarzumian's theorem for trees [7]. However, there is a difference between Borg's theorems for an interval and for a tree. If we choose a pendant vertex as the root of the tree and consider the spectral problems with the Neumann condition at the root and the spectral problem with the Dirichlet condition at the root then according to [5, 20] the spectra of the Neumann and the Dirichlet problems uniquely determine the potential only on the root edge. Thus, even knowledge of two spectra of problems for the whole tree is not sufficient to determine uniquely the potentials on the edges except of the "Ambarzumian's" case [7]. Ambarzumian's theorem for a tree is true only in the case of Neumann conditions at all the pendant vertices. Due to results of [7] it is clear that the spectra of problems with the Neumann conditions at the pendant vertices of a graph contain information on the form of the graph (see [2, 13] and [9] for the so-called geometric Ambarzumian's theorem).

In present paper we consider the case where the Neumann conditions are imposed at all but one pendant vertices of a tree, the Dirichlet condition at one of the pendant vertices (at the root) and the generalized Neumann (continuity and Kirchhoff's) conditions at all the interior vertices. Together with this spectral problem, we consider spectral problems for the subtrees obtained from the initial tree by deleting the root and its incident edge. Also we consider the Neumann-Dirichlet problem at the root edge. The aim is to recover the potentials of the Sturm–Liouville equations on the edges of the initial tree using the spectra of these problems. Up to our knowledge such inverse problem is not investigated at all even for a star graph. Only the case of P_3 graph was considered in [18] (see [4] for three spectral problems). We prove that if the spectra of the above problems are unperturbed (such as in the case of zero potentials on all the edges) then these spectra uniquely determine the potentials on the edges (these potentials are 0).

After this brief review of the uniqueness results for inverse spectral Sturm–Liouville problems important for us, we present in Section 2 the description of our problem and give auxiliary results. In Section 3 we prove the uniqueness theorem which is the main result of this paper. In Section 4 we describe applications of the main theorem to the case of a P_3 graph and to the case of a star graph.

2. STATEMENT OF THE PROBLEM AND AUXILIARY RESULTS

Let T be an equilateral tree with g edges denoted by e_j, each of the length a. We denote the vertices by v_i and chose an arbitrary pendant vertex v_1 as the root and direct all the edges away from this root. Let us describe the *Neumann* spectral problem as follows. We consider the Sturm–Liouville equations on the edges

$$-y_j'' + q_j(x)y_j = \lambda y_j, \quad j = 1, 2, \ldots, g, \tag{2.1}$$

where $q_j \in L_2(0, 1)$ are real.

For an edge e_j incident with a pendant vertex which is not the root, we impose the Neumann condition at the pendant vertex:

$$y_j'(a) = 0. \tag{2.2}$$

At each interior vertices we impose the continuity conditions

$$y_j(a) = y_k(0) \tag{2.3}$$

for the incoming to v_i edge e_j and for all e_k outgoing from v_i and the Kirchhoff's conditions

$$y_j'(a) = \sum_k y_k'(0), \tag{2.4}$$

where the sum is taken over all edges e_k outgoing from v_i.

At the root we impose the Neumann conditions

$$y_1'(0) = 0. \tag{2.5}$$

The following theorem was proved in [7].

Theorem 2.1. *Suppose T is a finite tree with all edges of length a. For $r = 1, 2, \ldots$, let $\{m_r\}$ be a sequence of integers with $\lim_{r\to\infty} m_r = \infty$. If the set of eigenvalues for problem (2.1)–(2.5) is nonnegative, and contains a subsequence $\{\lambda_r\}$ with*

$$\lim_{r\to\infty}\left(\lambda_r - \left(\frac{\pi}{a}m_r\right)^2\right) = 0,$$

then $q_j(x) \stackrel{a.e.}{=} 0$ for each $j = 1, 2, \ldots, g$.

It is known also that this "Ambarzumian's" case is exceptional. In other cases knowledge of one spectrum is not sufficient to determine the potentials on the edges. For example, in case of the Dirichlet conditions at the pendant vertices of a star graph one needs additional information, e.g. the spectra of the Dirichlet problems on the edges (see, e.g. [19]).

Let us consider another spectral problem on the same tree (which we call the Dirichlet problem). We choose a pendant vertex v_1 as the root of the tree T and impose the Dirichlet condition at the root

$$y_1(0) = 0. \tag{2.6}$$

The conditions at all the other vertices are such as in (2.2)–(2.4).

If the potentials are the same on all edges and are symmetric with respect to the midpoint of an edge: $q_j(a-x) \overset{a.e.}{=} q_j(x)$ we have the following result (see Corollary 6.3.10 in [17]).

Theorem 2.2. *Let the potentials on the edges be the same and symmetric with respect to the midpoint of the edge. Then eigenvalues of the Neumann problem* (2.1)–(2.5)*, counted with multiplicities are the zeros of the entire function*

$$\phi(\lambda) = s^{-1}(\lambda, a)P(c(\lambda, a)),$$

where $s(\lambda, x)$ *is the solution of equation* (2.1) *which satisfies the initial conditions* $s(\lambda, 0) = s'(\lambda, 0) - 1 = 0$*,* $c(\lambda, x)$ *is the solution of equation* (2.1) *which satisfies the initial conditions* $c(\lambda, 0) - 1 = c'(\lambda, 0) = 0$*,*

$$P(z) := det(zR - A),$$

A *is the adjacency matrix and* $R := diag\{d(v_1), d(v_2), \ldots, d(v_p)$ *is the degree matrix.*

In case of $q_j = 0$, we have $s(\lambda, a) = \frac{\sin\sqrt{\lambda}a}{\sqrt{\lambda}}$ and $c(\lambda, a) = \cos\sqrt{\lambda}a$, and Theorem 2.2 implies

Corollary 2.3. *If in addition to the conditions of Theorem 2.2,* $q_j = 0$ *for all* j *then the characteristic function of problem* (2.1)–(2.5) *is*

$$\phi_0(\lambda) = \sqrt{\lambda}\sin^{-1}(\sqrt{\lambda}a)P(\cos(\sqrt{\lambda}a)).$$

Definition 2.4. Let v_i be an interior vertex. Then we call conditions (2.3), (2.4) the generalized Neumann condition.

If e_j is the edge incoming into v_i and e_k are the edges outgoing from v_i. Then we call

$$y_j(a) = y_k(0) = 0 \text{ for all } k.$$

the generalized Dirichlet condition at v_i.

Now we consider the subtree $\hat{T}$ obtained from the tree T by deleting the vertex v_1 together with its incident edge. Denote by $\hat{R} := diag\{\hat{d}(v_2), \hat{d}(v_3), \ldots, \hat{d}(v_p)\}$ where $\hat{d}(v_i)$ is the degree of v_i as a vertex in T (see Figure 1).

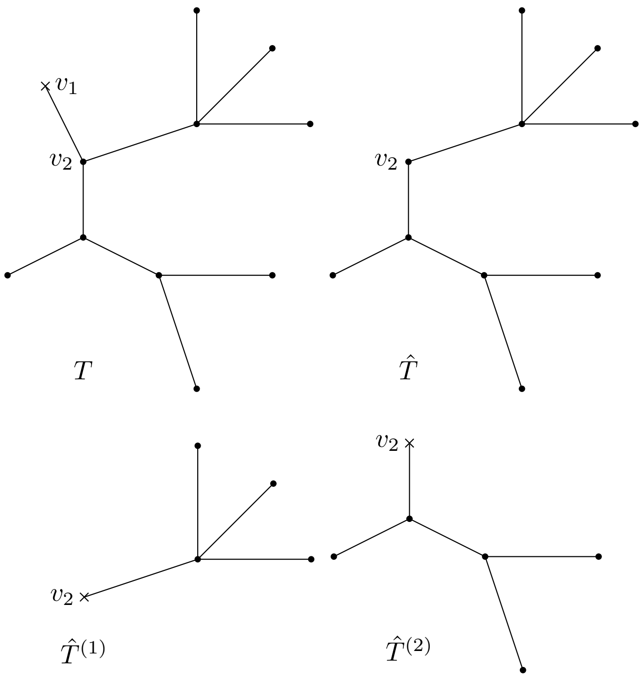

Fig. 1. A tree T and its complementary subtrees. x - vertices with Dirichlet conditions, · - vertices with Neumann conditions

Theorem 6.4.2 of [17] applied to an equilateral tree T with symmetric with respect to the midpoint potentials is the following.

Theorem 2.5. *The eigenvalues of problem* (2.1)–(2.4), (2.6) *counted with multiplicities are the zeros of the entire function*

$$\psi(z) = P_1(c(\lambda, a)),$$

where

$$P_1(z) := det(z\hat{R} - \hat{A}),$$

and $\hat{A}$ is the adjacency matrix of $\hat{T}$.

Corollary 2.6. *If in addition to the conditions of Theorem 2.5, $q_j \overset{a.e}{=} 0$ for all j then the characteristic function of problem* (2.1)–(2.4), (2.6) *is*

$$\psi_0(z) = P_1(\cos(\sqrt{\lambda}a)).$$

3. MAIN RESULTS

Let v_2 be the vertex adjacent to the root v_1 in T. We regard v_2 as the root of $\hat{T}$ and together with problems (2.1)–(2.5) and (2.1)–(2.4), (2.6) for T consider the following problems:

(1) the Neumann problem for $\hat{T}$ which consists of the equations

$$-y_j'' + q_j(x)y_j = \lambda y_j, \quad j = 2, 3 \ldots, g, \tag{3.1}$$

for each pendant vertex:

$$y_j'(a) = 0, \tag{3.2}$$

for each interior vertex v_i except of the root with the incoming edge e_j and outgoing edges e_k:

$$y_j(a) = y_k(0) \tag{3.3}$$

and

$$y_j'(a) = \sum_k y_k'(0), \tag{3.4}$$

where the sum taken over all edges e_k outgoing from v_i, for the root:

$$y_j(0) = y_k(0), \tag{3.5}$$

for all the outgoing from v_2 edges e_j and e_k and

$$\sum_k y_k'(0) = 0, \tag{3.6}$$

(2) the Dirichlet problem which consists of equations (3.1)–(3.4) and of

$$y_j(0) = 0 \tag{3.7}$$

for all e_j incident with v_2 in $\hat{T}$.

We use the following notation: ϕ is the characteristic function of the Neumann problem for T (problem (2.1)–(2.5)), ψ is the characteristic function of the Dirichlet problem for T (problem (2.1)–(2.4)), (2.6), $\hat{\phi}$ is the characteristic function of the Neumann problem (3.1)–(3.6) for $\hat{T}$ and $\hat{\psi}$ is the characteristic function of the Dirichlet problem (3.1)–(3.4), (3.7). We denote be $\hat{T}^{(r)}$ the subtrees of $\hat{T}$ rooted at v_2.

Using Theorem 2.1 in [14] we obtain

$$\psi(\lambda) = s_1'(\lambda, a)\tilde{\psi}(\lambda) + s_1(\lambda, a)\tilde{\phi}(\lambda) \tag{3.8}$$

and

$$\phi(\lambda) = c_1'(\lambda, a)\tilde{\psi}(\lambda) + c_1(\lambda, a)\tilde{\phi}(\lambda). \tag{3.9}$$

It is clear (see Corollary 2.2 in [14]) that

$$\tilde{\psi}(\lambda) = \prod_{r=1}^{d(v_2)-1} \tilde{\psi}^{(r)}(\lambda),$$

where $\tilde{\psi}^{(r)}$ $(r = 1, 2, \ldots, d(v_2) - 1)$ is the characteristic function of Dirichlet problem on the subtree T_r rooted at v_2.

Multiplying (3.8) by $c_1(\lambda, a)$ and (3.9) by $s_1(\lambda, a)$ and then subtracting the second obtained equation from the first one and using the Lagrange identity

$$c_1(\lambda, a)\, s_1'(\lambda, a) = 1 + s_1(\lambda, a)\, c_1'(\lambda, a)$$

we arrive at

$$c_1(\lambda, a)\psi(\lambda) = s_1(\lambda, a)\phi(\lambda) + \tilde{\psi}(\lambda). \tag{3.10}$$

In case of $q_j \overset{a.e.}{=} 0$ for all j we have $s_1(\lambda, a) = \frac{\sin(\sqrt{\lambda}a)}{\sqrt{\lambda}}$, $c_1(\lambda, a) = \cos\sqrt{\lambda}a$ and by Corollary 2.6

$$\psi_0(\lambda) = P_1(\cos(\sqrt{\lambda}a))$$

and by Corollary 2.3

$$\phi_0(\lambda) = \sqrt{\lambda}\sin^{-1}(\sqrt{\lambda}a)P(\cos(\sqrt{\lambda}a)).$$

Then it follows from (3.10) that

$$P(\cos(\sqrt{\lambda}a)) = \cos(\sqrt{\lambda}a)P_1(\cos(\sqrt{\lambda}a)) - \tilde{\psi}_0(\lambda), \tag{3.11}$$

where

$$\tilde{\psi}_0(\lambda) = \prod_{r=1}^{d(v_2)-1} \tilde{\psi}_0^{(r)}(\lambda)$$

and $\tilde{\phi}_0^{(r)}$ is the characteristic function of the Dirichlet problem on the subtree $\hat{T}^{(r)}$ rooted at v_2 with $q_j \overset{a.e.}{=} 0$ for all j.

Now we present the main result of this paper.

Theorem 3.1. *Let the spectrum of problem* (2.1)*)–* (2.4), (2.6) *coincide with the set of zeros of* $P_1(\cos\sqrt{\lambda}a)$*, the spectrum of the Dirichlet problem on the subtree* $\hat{T}^{(r)}$ *coincide with the set of zeros of* $\tilde{\psi}_0^{(r)}$ *for all* $r \in \{1, 2, \ldots, d(v_2) - 1\}$*, the spectrum of the problem*

$$-y_1'' - q_1(x)y_1 = \lambda y,$$

$$y_1'(0) = y_1(a) = 0$$

on the root edge coincide with the set of zeros of $\cos(\sqrt{\lambda}a)$*.*

Then $q_j(x) \overset{a.e.}{=} 0$ *for all* j*.*

Proof. Under the conditions of this theorem equation (3.10) attains the form

$$s(\lambda, a)\psi(\lambda) = \cos\sqrt{\lambda}aP_1(\cos(\sqrt{\lambda}a)) - \tilde{\phi}_0(\lambda). \tag{3.12}$$

Using identity (3.11) we obtain from (3.12):

$$s(\lambda, a)\psi(\lambda) = P(\cos(\sqrt{\lambda}a)). \tag{3.13}$$

It is known (see e.g. [15, Lemma 3.4.2] or [16, Corollary 12.5.1]) that the zeros of $s_1(\lambda, a)$ are simple and behave asymptotically as follows:

$$\sqrt{\lambda_k^{(1)}} = \frac{\pi k}{a} + \frac{A_1}{k} + o\left(\frac{1}{k}\right),$$

where A_1 is a constant.

Using results of [8] it was proved in [9] (Theorem 5.3) that even when q_js are different and not symmetric with respect to the midpoints of the edges there is a subsequence of the sequence of zeros of ψ which behave asymptotically as

$$\sqrt{\lambda_k^{(2)}} = \frac{\pi(k-1)}{a} + \frac{A_2}{k} + o\left(\frac{1}{k}\right), \tag{3.14}$$

where A_2 is a constant.

By Lemma 1.7 (iv) in [10], $z = 1$ is a simple zero of $P(z)$. Since the graph is bipartite, the zeros of $P(z)$ are located symmetrically with respect to the origin (see e.g. Lemma C.5.4 in [17]), what means that $z = -1$ is also a zero of $P(z)$. Therefore,

$$\begin{aligned} P(\cos(\sqrt{\lambda}a)) &= (1 - (\cos(\sqrt{\lambda}a))^2 \tilde{P}(\cos(\sqrt{\lambda}a)) \\ &= \sqrt{\lambda}\sin(\sqrt{\lambda}a)\frac{\sin(\sqrt{\lambda}a)}{\sqrt{\lambda}}\tilde{P}(\cos(\sqrt{\lambda}a)), \end{aligned} \tag{3.15}$$

where $\tilde{P}(z)$ is a polynomial. It is known that all the zeros of $P(z)$ lie in the interval $[-1, 1]$ (see [10]) and, therefore, all the zeros of $P(\cos(\sqrt{\lambda}a))$ are nonnegative. Comparing (3.13) with (3.15) and taking into account that all the zeroes of $\sin(\sqrt{\lambda}a)$ are simple we conclude that there exists a subsequence in the sequence of zeros of $\psi(\lambda)$ of the form

$$\sqrt{\lambda_k^{(2)}} = \frac{\pi(k-1)}{a}.$$

Thus, this subsequence satisfies the conditions of Theorem 2.1 and we arrive at $q_j(x) \stackrel{a.e.}{=} 0$ for all j. □

4. EXAMPLES

4.1. CASE OF $g = 2$

In [18] the three spectral problems were considered on an interval $[0, a]$ and on its parts $\left[0, \frac{n-1}{n}a\right]$ and $\left[\frac{n-1}{n}a, a\right]$ where $n \in \{2, 3, \ldots\}$ which in our terms look as follows:

$$-y'' + q(x)y = \lambda y, \tag{4.1}$$

$$y'(0) = y(a) = 0, \tag{4.2}$$

$$-y'' + q(x)y = \lambda y,$$

$$y'(0) = y\left(\frac{n-1}{n}a\right) = 0, \tag{4.3}$$

$$-y'' + q(x)y = \lambda y,$$
$$y\left(\frac{n-1}{n}a\right) = y'(a) = 0. \tag{4.4}$$

Denote by $\{\xi_k\}_{k=1}^{\infty}$ the spectrum of problem (4.1), (4.2) and by $\{\mu_k\}_{k=1}^{\infty}$, $(\{\mu_k^{(1)}\}_{k=1}^{\infty})$ the spectrum of problem (4.1), (4.3) (problem (4.1), (4.4), respectively). It is well known that the three spectra consist of real simple eigenvalues only. We enumerate the eigenvalues in the following way: $\xi_{k+1} > \xi_k$, $\mu_{k+1} > \mu_k$, $\mu_{k+1}^{(1)} > \mu_k^{(1)}$, $k \in \mathbb{N}$.

Theorem 4.1. *Let*

$$\xi_k = \frac{\pi^2}{a^2}\left(k - \frac{1}{2}\right)^2, \quad \mu_k = \frac{\pi^2 n^2}{a^2(n-1)^2}\left(k - \frac{1}{2}\right)^2, \quad \mu_k^{(1)} = \frac{\pi^2 n^2}{a^2}\left(k - \frac{1}{2}\right)^2, \ (k \in \mathbb{N}),$$

and let $q(x) \in L_2(0,a)$ *be real-valued. Then* $q(x) \overset{a.e.}{=} 0$.

For the case of $n = 2$ this result follows from Theorem 3.1.

4.2. STAR GRAPH

We consider the spectral problem on an equilateral star graph of g edges rooted at one of its pendant vertices v_1. The generalized Neumann conditions at the central vertex, the Dirichlet condition at the root and the Neumann conditions at the other pendant vertices are imposed:

$$-y_j'' + q_j(x)y_j = \lambda y_j, \tag{4.5}$$

$$y_1(0) = 0, \tag{4.6}$$

$$y_j'(a) = 0, \quad j = 2, 3, \ldots, g, \tag{4.7}$$

$$y_1(a) = y_2(0) = y_3(0) = \ldots = y_g(0), \tag{4.8}$$

$$y_1'(a) - y_2'(0) - y_3'(0) - \ldots - y_g'(0) = 0. \tag{4.9}$$

We also consider the Neumann–Dirichlet problem for the root edge

$$-y_1'' + q_1(x)y_1 = \lambda y_1, \tag{4.10}$$

$$y_1'(0) = y_1(a) = 0, \tag{4.11}$$

and the Dirichlet–Neumann problems for the other edges of the graph

$$-y_j'' + q_j(x)y_j = \lambda y_j, \quad j = 2, 3, \ldots, g, \tag{4.12}$$

$$y_j(0) = y_j'(a) = 0, \quad j = 2, 3, \ldots, g. \tag{4.13}$$

Up to our knowledge, the inverse problem of recovering the potentials q_j using the spectra of problems (4.5)–(4.9), (4.10)-(4.11) and (4.12)–(4.13) is not investigated.

Theorem 4.2. *Let the following conditions be satisfied:*

(1) *the spectrum* $\{\tau_k\}_1^\infty$ *of problem* (4.5)–(4.9) *consist of the subsequences* $\{\tau_k^{(j)}\}_1^\infty$ *of the form*

$$\sqrt{\tau_k^{(j)}} = \frac{\pi\left(k-\frac{1}{2}\right)}{a}, \quad j=1,2,\ldots,g-2,\ k\in\mathbb{N}, \tag{4.14}$$

$$\sqrt{\tau_k^{(g-1)}} = \frac{\pi(k-1)}{a} \pm \frac{1}{a}\arcsin\frac{1}{\sqrt{g}}, \quad k\in\mathbb{N}; \tag{4.15}$$

(2) *the spectrum of problem* (4.10)–(4.11) *and the spectrum* $\{\lambda_k\}_1^\infty$ *of each of the problems* (4.12)–(4.13) *is such that*

$$\sqrt{\lambda_k} = \frac{\pi\left(k-\frac{1}{2}\right)}{a}.$$

Then $q_j \overset{a.e.}{=} 0$ *for all* j.

Proof. Equations (4.14), (4.15) imply that $\bigcup_{j=1}^{q-1}\{\tau_k^{(j)}\}_{k=1}^\infty$ is the set of zeros of the function

$$\phi(\lambda) = \cos^{g-2}(\sqrt{\lambda}a)(1-g\sin^2(\sqrt{\lambda}a)) = P_1(\cos(\sqrt{\lambda}a)).$$

which is the characteristic function of problem (4.5)–(4.9) in the case of $q_j=0$ for all js and $P_1(z)=z^{g-2}(gz^2-g+1)$. The functions $\tilde{\phi}^{(r)}(\lambda)=\cos(\sqrt{\lambda}a)=\tilde{\phi}_0^{(r)}(\lambda)$ $(r=2,3,\ldots,g)$. Thus, the conditions of Theorem 3.1 are fulfilled and the statement of Theorem 4.2 follows. □

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