Numerical analysis of parametric circuits

Wojciech Lipiński
West Pomerania University of Technology
71-126 Szczecin, ul. 26 Kwietnia 10, e-mail: lipinski70@wp.pl

The purpose of the paper is to present numerical calculations for circuits with switched capacitors and inductances. Computer simulations have been carried out for an exemplary circuit using PSpice software. Then the analysis has been performed for the circuit according to the numerical algorithm.

1. Introduction

Mathematical models of various systems used in electronics, physics, chemistry, mechanics, economics and biology can often be expressed as ordinary differential equations

$$\frac{d^n}{dt^n}x(t)=F\left[t,x(t),\frac{d}{dt}x(t),\frac{d^2}{dt^2}x(t),\cdots,\frac{d^{n-1}}{dt^{n-1}}x(t)\right] \tag{1}$$

The function x(t) is unknown. Using the following substitutions

$$x_0(t)=x(t),\ x_1(t)=\frac{d}{dt}x(t),\ x_2(t)=\frac{d^2}{dt^2}x(t),\ldots,x_{n-1}(t)=\frac{d^{n-1}}{dt^{n-1}}x(t)$$

we obtain a system of n first-order differential equations with unknowns $x_0,x_1,\ldots,x_{n-1}$.

$$\begin{gathered}\frac{d}{dt}x_0(t)=x_1(t),\quad \frac{d}{dt}x_1(t)=\frac{d^2}{dt^2}x_0(t)=x_2(t),\ldots,\frac{d}{dt}x_{n-1}(t)=F(t,x_0,x_1,\ldots,x_{n-1})\\ \frac{d^n}{dt^n}x(t)=\frac{d^{n-1}}{dt^{n-1}}x_1(t)=\frac{d^{n-2}}{dt^{n-2}}x_2(t)=\ldots=\frac{d}{dt}x_{n-1}(t)=F(t,x_0,x_1,\ldots,x_{n-1})\end{gathered} \tag{2}$$

$$\frac{d^n}{dt^n}x(t)=\frac{d}{dt}x_{n-1}(t)=F(t,x_0,\ldots,x_{n-1}),\qquad \frac{d}{dt}\begin{bmatrix}x_0(t)\\x_1(t)\\\cdots\\x_{n-1}(t)\end{bmatrix}=\begin{bmatrix}x_1(t)\\x_2(t)\\\cdots\\F(t,x_0,x_1,\ldots,x_{n-1})\end{bmatrix}$$

Parametric equation with time-varying coefficients:

$$a_n(t)\cdot\frac{d^n}{dt^n}x(t)+a_{n-1}(t)\cdot\frac{d^{n-1}}{dt^{n-1}}x(t)+\ldots+a_1(t)\cdot\frac{d}{dt}x(t)+a_0(t)\cdot x(t)=f(t)$$

In theory of parametric circuits we adopt coupled fluxes ψ (t) in coils and charges q (t) in capacitors as state variables. Charges q (t) in capacitors and coupled fluxes ψ (t) in coils are continuous functions of time.

$$i(t)=\frac{dq(t)}{dt}, \qquad u_L(t)=\frac{d\psi(t)}{dt}$$

Parametric circuits RL and RC:

$$u_L(t)=\frac{d\psi(t)}{dt}=\frac{d\psi(t)}{di(t)}\cdot\frac{di(t)}{dt}=L(t)\cdot\frac{di(t)}{dt},$$

$$\frac{d\psi(t)}{dt}+R(t)\cdot i(t)=e(t)\ , \quad \frac{d\psi(t)}{di(t)}=L(t). \tag{2a}$$

$$i(t)=\frac{dq(t)}{dt}=\frac{dq(t)}{du(t)}\cdot\frac{du(t)}{dt}=C(t)\cdot\frac{du(t)}{dt},$$

$$R(t)\cdot\frac{dq(t)}{dt}+u(t)=e(t)\ , \quad \frac{dq(t)}{du(t)}=C(t).$$

The equation for the first order:

$$\frac{d}{dt}x(t)+a(t)\cdot x(t)=f(t), \quad x(t)=\left[const+\int f(t)\cdot e^{\int a(t)\cdot dt}\cdot dt\right]\cdot\exp\left[-\int a(t)\cdot dt\right]$$

$$\frac{d}{dt}\left\{const\cdot\exp\left[-\int a(t)\cdot dt\right]\right\}+a(t)\cdot const\cdot\exp\left[-\int a(t)\cdot dt\right]=0$$

For example, the equation $\frac{d}{dt}x(t)+2\cdot t\cdot x(t)=t^3$ has a solution:

$$x(t)=\left[const+\int t^3\cdot e^{\int 2\cdot t\cdot dt}\cdot dt\right]\cdot\exp\left[-\int 2\cdot t\cdot dt\right]=const\cdot\exp\left(-t^2\right)+0.5\cdot\left(t^2-1\right)$$

$$\frac{d}{dt}\left[0.5\cdot\left(t^2-1\right)\right]+2\cdot t\cdot 0.5\cdot\left(t^2-1\right)=t+t\cdot\left(t^2-1\right)=t^3$$

In the case of linear systems with coefficients constant in time we receive:

$$a_n\cdot\frac{d^n}{dt^n}x(t)+a_{n-1}\cdot\frac{d^{n-1}}{dt^{n-1}}x(t)+\ldots+a_2\cdot\frac{d^2}{dt^2}x(t)+a_1\cdot\frac{d}{dt}x(t)+a_0\cdot x(t)=f(t)$$

$$\frac{1}{2}\cdot L\cdot i(t)^2 \ , \quad p_L(t)=\frac{d}{dt}\left[\frac{1}{2}\cdot L\cdot i(t)^2\right]=i(t)\cdot L\cdot\frac{d}{dt}\left[i(t)\right]=i(t)\cdot u_L(t)$$

$$\frac{1}{2}\cdot C\cdot u(t)^2 \ , \quad p_C(t)=\frac{d}{dt}\left[\frac{1}{2}\cdot C\cdot u(t)^2\right]=u(t)\cdot C\cdot\frac{d}{dt}\left[u(t)\right]=u(t)\cdot i(t)$$

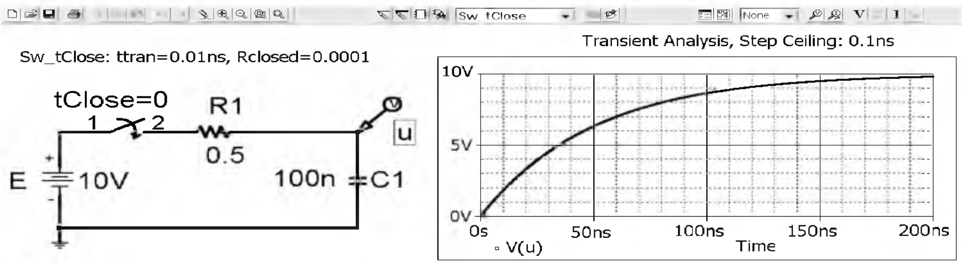

Fig. 1. The simple example in Pspice

2. Circuit simulation using PSpice software

An example of a circuit with switched capacitors and inductances in PSpice:

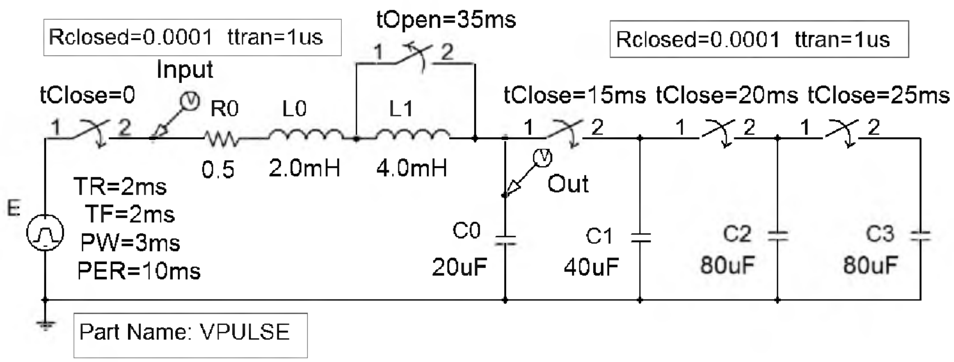

Fig. 2. Circuit with switched capacitors and inductances in PSpice

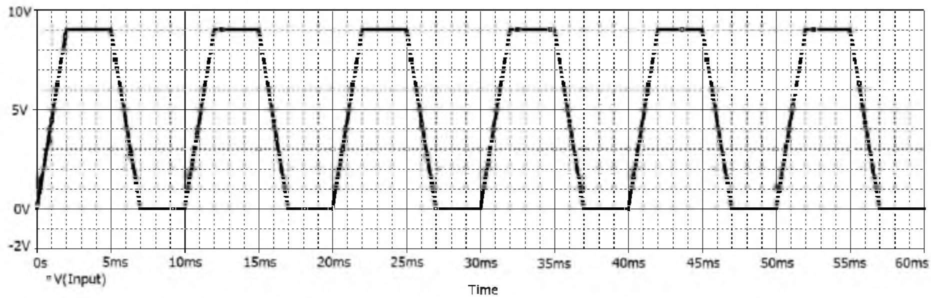

Fig. 3. Input voltage

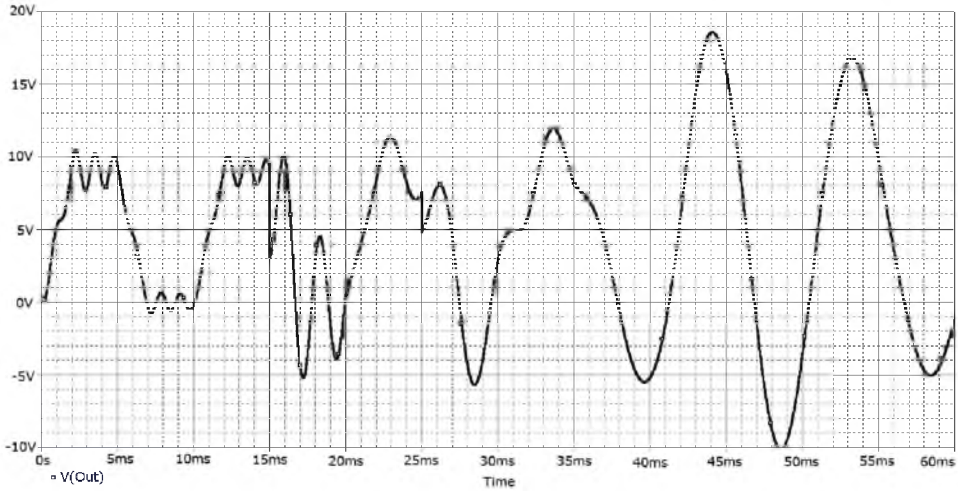

Fig. 4. Voltage across the capacitor C0. Result of computer simulation using PSpice

3. Input voltage and changes of L(t) and C(t)

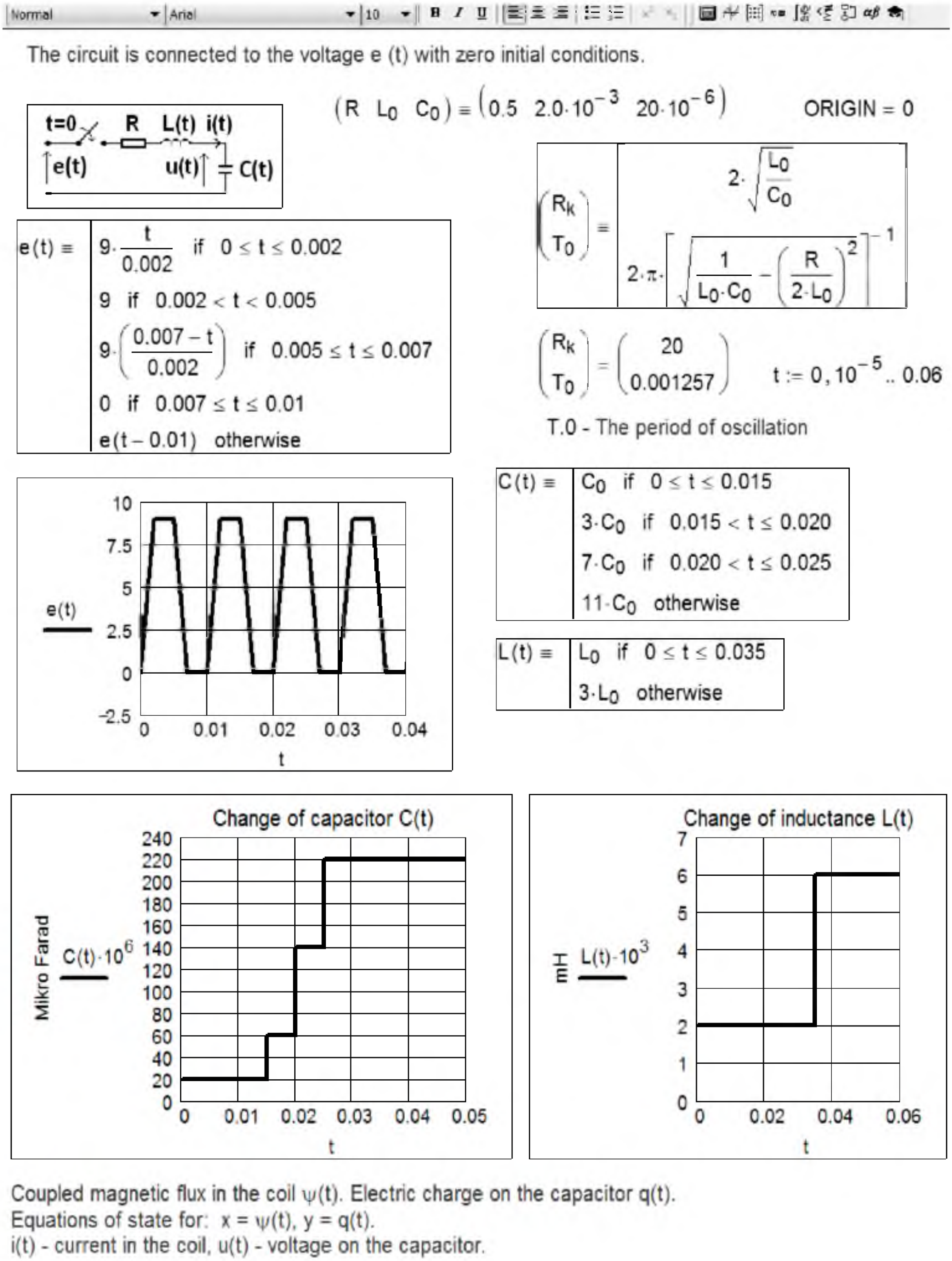

Fig. 5. Input voltage and changes of C(t) and L(t) obtained using Mathcad

4. Runge-Kutta algorithm in MATHCAD

Numerical methods for initial value problems are very easily adapted for systems of differential equations of first order. The fourth order Runge-Kutta algorithm is one of the most accurate one-step methods and the most widely used. The RK4 algorithm has local truncation error of order h^5, so it is more accurate with larger steps than Euler's method. Let an initial value problem be specified as follows. At the start, time is t_0 and y is y_0.

$$y' = f(t, y), \quad y(t_0) = y_0$$

$$y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(t, y) \cdot dt \approx y_n + f(t_n, y_n) \cdot h = y_n + k_1$$

The RK4 method for this problem is given by the following equations:

$$\begin{aligned} y_{n+1} &= y_n + (k_1 + 2 \cdot k_2 + 2 \cdot k_3 + k_4), \quad t_{n+1} = t_n + h \\ k_1 &= f(t_n, y_n) \cdot h \\ k_2 &= f(t_n + 0.5 \cdot h, y_n + 0.5 \cdot k_1) \cdot h \\ k_3 &= f(t_n + 0.5 \cdot h, y_n + 0.5 \cdot k_2) \cdot h \\ k_4 &= f(t_n + h, y_n + k_3) \cdot h \end{aligned} \tag{3}$$

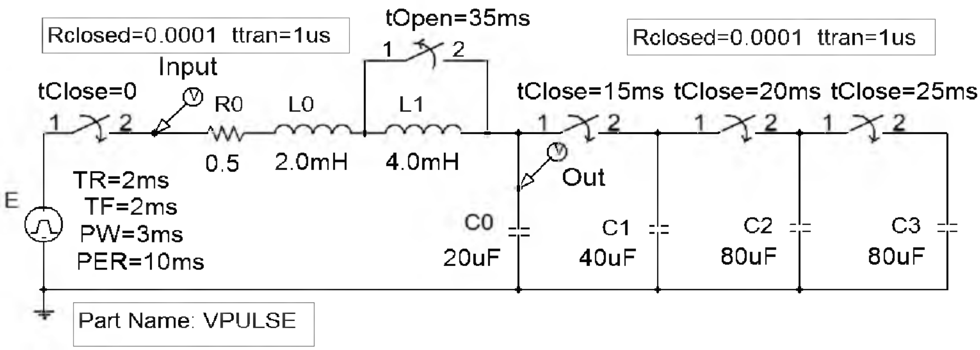

Fig. 6. Circuit with switched capacitors and inductances

$$\begin{bmatrix} \frac{d}{dt}\psi(t) \\ \frac{d}{dt}q(t) \end{bmatrix} = \begin{bmatrix} a_1(t) & a_2(t) \\ a_3(t) & a_4(t) \end{bmatrix} \cdot \begin{bmatrix} \psi(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} e(t) \\ 0 \end{bmatrix} \tag{4}$$

Coupled magnetic flux in the coil is $\psi(t)$. Electric charge on the capacitor is $q(t)$.

Equations of state for: $x = \psi(t)$, $y = q(t)$, $i(t)$ - current flowing through the coil, $u(t)$ - voltage across the capacitor.

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$$\frac{d}{dt}\begin{pmatrix}\psi(t)\\ q(t)\end{pmatrix} = \begin{pmatrix}-\frac{R}{L(t)} & -\frac{1}{C(t)}\\ \frac{1}{L(t)} & 0\end{pmatrix}\cdot\begin{pmatrix}\psi(t)\\ q(t)\end{pmatrix} + \begin{pmatrix}e(t)\\ 0\end{pmatrix}$$

time discretization,
h - time step h=Δt,
N - number of time steps

$N := 60000$

$\Delta t := \frac{0.06}{N}$

$n := 0..N$

$t_n := n\cdot\Delta t$

$$\frac{d}{dt}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} = \begin{pmatrix}F(t,x,y)\\ G(t,x,y)\end{pmatrix}$$

This procedure RK depends on h and N, it declares functions F(t,x,y) and G(t,x,y)

$h := \Delta t$

$h = 1\times10^{-6}$

$$F(t,x,y) \equiv -\frac{R}{L(t)}\cdot x - \frac{1}{C(t)}\cdot y + e(t) \qquad G(t,x,y) \equiv \frac{1}{L(t)}\cdot x$$

```
RK(h,N) := | x0 ← 0
           | y0 ← 0
           | t ← 0
           | for i ∈ 0..2
           |   A0,i ← 0
           | for n ∈ 0..N
           |   | k1 ← F(t, xn, yn)·h
           |   | K1 ← G(t, xn, yn)·h
           |   | k2 ← F(t + 0.5·h, xn + 0.5·k1, yn + 0.5·K1)·h
           |   | K2 ← G(t + 0.5·h, xn + 0.5·k1, yn + 0.5·K1)·h
           |   | k3 ← F(t + 0.5·h, xn + 0.5·k2, yn + 0.5·K2)·h
           |   | K3 ← G(t + 0.5·h, xn + 0.5·k2, yn + 0.5·K2)·h
           |   | k4 ← F(t + h, xn + k3, yn + K3)·h
           |   | K4 ← G(t + h, xn + k3, yn + K3)·h
           |   | t ← t + h
           |   | xn+1 ← 1/6·(k1 + 2·k2 + 2·k3 + k4) + xn
           |   | yn+1 ← 1/6·(K1 + 2·K2 + 2·K3 + K4) + yn
           |   | An+1,0 ← t
           |   | An+1,1 ← xn+1
           |   | An+1,2 ← yn+1
           | A
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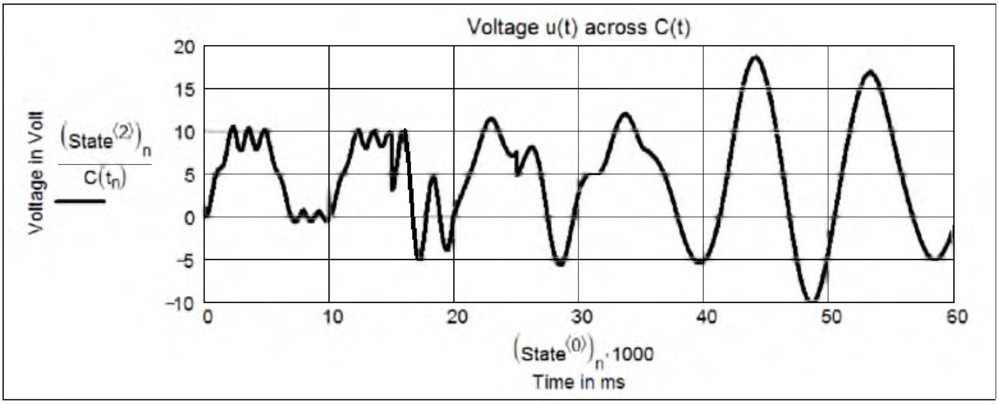
A fourth-order Runge-Kutta method is used. It returns the matrix A in which the zero column contains the time points at which the solution is valuated. First and second column contain the solution.

Fig. 7. The fourth order Runge-Kutta algorithm in Mathcad

State := RK(Δt, N)

State =

	0	1	2
0	0	0	0
1	$1\cdot10^{-6}$	$2.24981\cdot10^{-9}$	$3.74977\cdot10^{-13}$
2	$2\cdot10^{-6}$	$8.99843\cdot10^{-9}$	$2.99961\cdot10^{-12}$
3	$3\cdot10^{-6}$	$2.02446\cdot10^{-8}$	$1.0123\cdot10^{-11}$
4	$4\cdot10^{-6}$	$3.59868\cdot10^{-8}$	$2.39935\cdot10^{-11}$
5	$5\cdot10^{-6}$	$5.62236\cdot10^{-8}$	$4.68589\cdot10^{-11}$
6	$6\cdot10^{-6}$	$8.09534\cdot10^{-8}$	$8.0966\cdot10^{-11}$
7	$7\cdot10^{-6}$	$1.10174\cdot10^{-7}$	$1.28561\cdot10^{-10}$



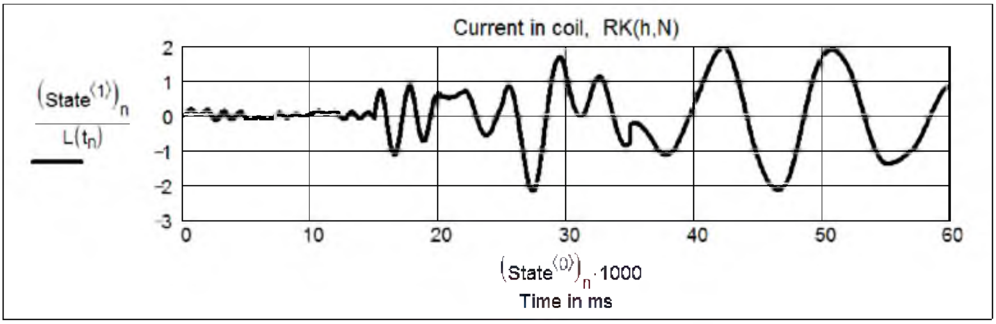
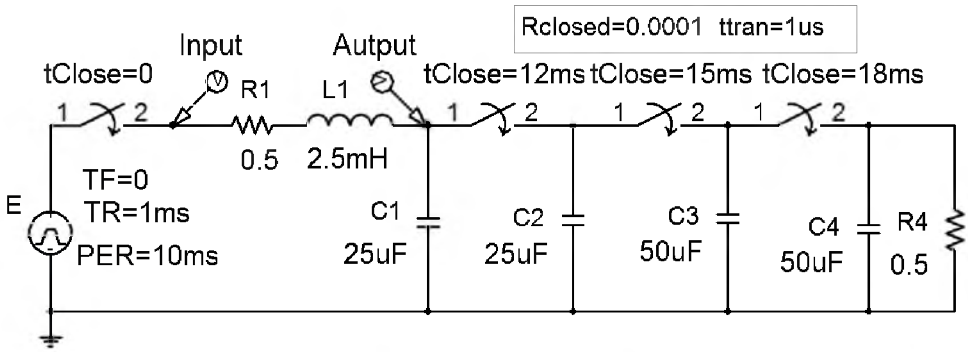


Fig. 8. Voltage across the capacitor. Current in the coil. Calculations using Mathcad

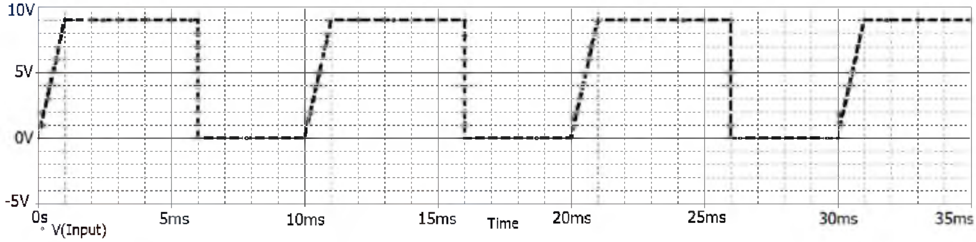
The results given in Fig. 8 have been compared with those shown in Fig. 4. A consistency of results is found.

5. Circuit simulation in PSpice

In Fig.8. shows a simulation of the RLC circuit switching to periodic voltage e(t). In moments of time t = 12ms, 15ms, 20ms connected capacitors C2, C3, C4.



Periodic alternating voltage supply VPULSE, PER = 10 ms period



Voltage V across the capacitor C1

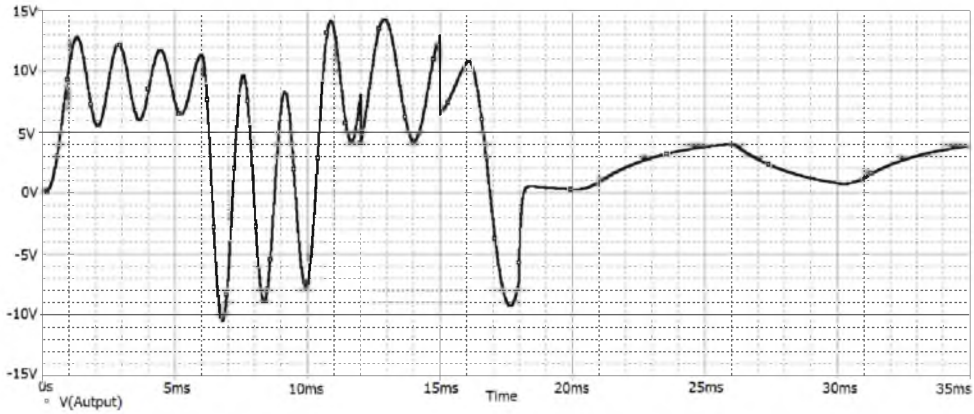


Fig. 9. RLC circuit with keying capacitors, PSpice program

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