Vibration of Simply Supported Plates in Contact with Liquid by Using Membrane Curvilinear Elements

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Abstract

This paper concerns the free vibrations of a simply supported plate in contact with liquid on one side. The plate is placed into a hole of an infinite rigid wall. The analysed problem is a coupled problem of the fluid-structure type. It is assumed that the fluid is inviscid and incompressible. The boundary integral equation is used for describing the hydrodynamic pressure. The plate equation is formulated in the form of two harmonic equations. The surface of the plate is discretized using triangular curvilinear 6-node elements of the membrane type. These elements are simultaneously the finite elements for the plate and the boundary elements for the liquid. Numerical examples of the free vibrations of circular and rectangular plates are considered and are compared with analytical and analytical-numerical solutions.

Keywords: fluid-structure interaction, hydroelastic vibrations, BEM and FEM coupling

1. Introduction

This paper presents the vibration analysis of a simply supported plate in contact with liquid on one side. The plate is placed into a hole of an infinite rigid wall. The vibrating plate induces vibrations of the surrounding fluid which in turn generate additional inertia forces due to the fluid mass. The analyzed problem is a coupled problem of the fluid – structure type. Such a problem arises for example in the dynamic analysis of localized vibration of ships, submarine hull vibrations, and vibration of liquid containers. This problem is solved by analytical and numerical methods, such as the finite element method (FEM) and the boundary element method (BEM).

The aim of this paper is to present a numerical method for the solution of the free vibration problem for a simply supported plate interacting with liquid by using membrane curvilinear elements. These elements are simultaneously the finite elements for the plate and the boundary elements for the liquid. The novelty in this paper is the analysis of a simply supported plate by means of curvilinear membrane elements. A similar problem like the one analyzed here was studied in papers [1, 2] in which free vibrations of circular and rectangular plates were considered by using analytical and analytical-numerical methods.

2. Problem formulation

The plate of any shape is supported in an infinite baffle and is in contact with liquid on one side (see Figure 1).

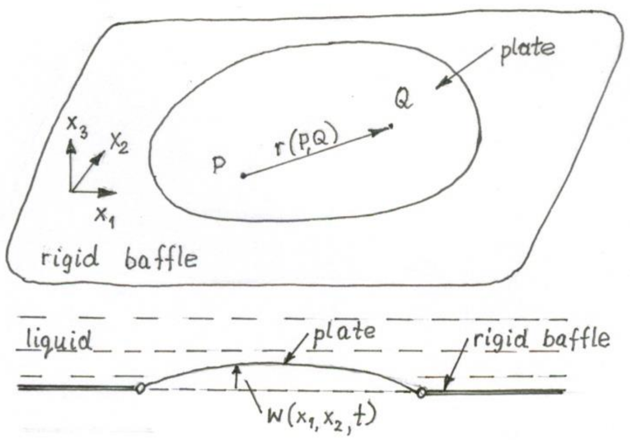

Figure 1. A plate in contact with liquid

The governing differential equation of the plate subjected to lateral loads p is

$$D\nabla^2\nabla^2 w = p, \tag{1}$$

where $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$, $D = Eh^3/12(1-\upsilon^2)$ is the flexural rigidity of the plate; υ and E are Poisson's ratio and Young's modulus, respectively.

In the case of simply supported plates, the boundary conditions permit the uncoupling of the equation (1) into two harmonic equations (see: e.g. [3]).

$$\nabla^2 M = -p, \tag{2}$$

$$\nabla^2 w = -\frac{M}{D}, \tag{3}$$

where $M = \frac{M_1+M_2}{1+\nu}$ is the so-called *moment-sum*, M_1 and M_2 are the bending moments in x_1 and x_2 directions, respectively.

It is interesting to note that equations (2) and (3) have a form similar to that of the differential equation of the membrane. In the analyzed problem, the lateral loads $p(x_1, x_1, t)$ take the following form

$$p(x_1, x_1, t) = -\mu \frac{\partial^2 w(x_1, x_1, t)}{\partial t^2} + p_h(x_1, x_1, t), \tag{4}$$

where μ is the plate unit mass, and $p_h(x_1, x_1, t)$ is the hydrodynamic pressure.

The liquid is assumed to be inviscid, incompressible and irrotational. Then, the liquid motion may be defined by a velocity potential function $\Phi(\mathbf{x},t)$, which satisfies the Laplace equation

$$\nabla^2 \Phi = 0. \tag{5}$$

The integral solution of this equation for the half-space ($x_3 \geq 0$) has the form of a boundary integral equation. This is the Rayleigh integral equation [4] for the incompressible air:

$$\Phi(P,t) = \frac{1}{2\pi}\int_S \frac{\partial \Phi(Q,t)}{\partial x_3}\frac{1}{r(P,Q)}dS_Q. \tag{6}$$

The boundary condition on the surface S is of the Neumann type and is the coupling condition between the plate and the liquid. It is expressed by the following equation:

$$\frac{\partial \Phi}{\partial x_3} = \frac{\partial w}{\partial t}. \tag{7}$$

The hydrodynamic pressure acting on the surface S is expressed by:

$$p_h = -\rho\frac{\partial \Phi}{\partial t}, \tag{8}$$

where $p_h = \tilde{p}_h e^{i\omega t}$, ω is the circular frequency, ρ is the liquid density.
Differentiating (6) with respect to time and using (7) and (8), we can rewrite (6) in the form of:

$$\tilde{p}_h(P) = \frac{\omega^2 \rho}{2\pi}\int_S \tilde{w}(Q)\frac{1}{r(P,Q)}dS_Q, \tag{9}$$

where $w(Q,t) = \tilde{w}(Q)e^{i\omega t}$.

3. Numerical solution of the problem

The analysed problem is described by one boundary integral equation (9) and differential equations (2), (3) and (4).

The finite element method is used to solve differential equations (2)-(4). The finite element equations are formulated by the method of weighted residuals with Galerkin's criterion [5]. The solution of the boundary integral equation (9) is determined by means of the boundary element method [6]. The surface of the plate is discretized by using six-node isoparametric curvilinear triangular elements. These elements are simultaneously the boundary elements for the liquid and the finite elements for the plate. Using the FEM to solve equations (2)-(4), one obtains the set of algebraic equations

$$(\mathbf{B}_2 + \mathbf{B}_3)\tilde{\mathbf{m}} = -\mathbf{B}_1\tilde{\mathbf{p}}, \tag{10}$$

$$(\mathbf{B}_2 + \mathbf{B}_3)\tilde{\mathbf{w}} = -\frac{1}{D}\mathbf{B}_1\tilde{\mathbf{m}}, \tag{11}$$

$$\mathbf{B}_1\tilde{\mathbf{p}} = \mathbf{B}_1(\omega^2\mu\tilde{\mathbf{w}} + \tilde{\mathbf{p}}_h), \tag{12}$$

where $\tilde{\mathbf{w}}, \tilde{\mathbf{m}}, \tilde{\mathbf{p}}$ and $\tilde{\mathbf{p}}_h$ are the quantities in the element nodes forming the vectors. The matrices $\mathbf{B}_1$, $\mathbf{B}_2$ and $\mathbf{B}_3$ are constructed from the finite element matrices $\mathbf{B}_1^e, \mathbf{B}_2^e$ and $\mathbf{B}_3^e$ which are expressed by:

$$\mathbf{B}_1^e = \int_{S_e} \mathbf{N}^{\mathrm{T}}\, \mathbf{N}\, dS_e, \qquad \mathbf{B}_2^e = -\int_{S_e} \frac{\partial \mathbf{N}^{\mathrm{T}}}{\partial x_1}\frac{\partial \mathbf{N}}{\partial x_1}\, dS_e,$$
$$\mathbf{B}_3^e = -\int_{S_e} \frac{\partial \mathbf{N}^{\mathrm{T}}}{\partial x_2}\frac{\partial \mathbf{N}}{\partial x_2}\, dS_e, \tag{13}$$

where $\mathbf{N} = [N_1, ..., N_6]$ is the matrix of element shape functions.
The integrals (13) are computed numerically by using 7-Gauss integration points.

The boundary element discretization of equation (9) results in the matrix equation

$$\tilde{\mathbf{p}}_h = \omega^2 \rho \mathbf{A} \tilde{\mathbf{w}}. \tag{14}$$

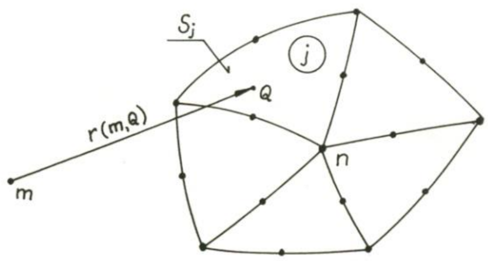

Figure 2. The notation for equation (15)

The elements A_{nm} of the matrix $\mathbf{A}$ are given by (see Figure 2)

$$A_{nm} = \frac{1}{2\pi} \sum_{j=1}^{e} \int_{S_j} N_n^{(j)} \frac{1}{r(m,Q)} dS_Q\,, \tag{15}$$

where $N_n^{(j)}$ is the interpolation function for element j specified at the nth node and e denotes the number of elements coincident with node n. The integrals (15) are computed numerically by using 25-Gauss integration points. For $m=n$ a singularity of $1/r$ type occurs and special analytical – numerical integrations are adapted [7].
Inserting equation (14) into (12) and then into (10), we can obtain

$$\mathbf{B}\tilde{\mathbf{m}} = -\omega^2 \left(\mathbf{M}_p + \mathbf{M}_w\right) \tilde{\mathbf{w}}, \tag{16}$$

where $\mathbf{B} = \mathbf{B}_2 + \mathbf{B}_3$, $\mathbf{M}_p = \mu \mathbf{B}_1$ is the mass matrix of the plate, $\mathbf{M}_w = \rho \mathbf{B}_1 \mathbf{A}$ is the liquid mass matrix.

Now using equations (16) and (11), we can obtain

$$(\mathbf{K} - \lambda\mathbf{M})\widetilde{\mathbf{w}} = \mathbf{0}, \tag{17}$$

where $\lambda = \omega^{\mathbf{2}}$, $\mathbf{K} = D\mathbf{B}\mathbf{B}_1^{-1}\mathbf{B}$ is the stiffness matrix of the plate, $\mathbf{M} = \mathbf{M}_p + \mathbf{M}_w$ is the mass matrix of the coupled system plate and liquid.
Equation (17) represents a generalized eigenvalue problem. The eigenvectors express the vibration modes of the plate and the eigenvalues λ enable calculation of the natural circular frequencies ω of the coupled system plate-liquid.

4. Numerical results

Based on the problem formulation given in sections 2 and 3, computer programs were developed. The calculations were performed for two types of plates: circular and rectangular. The results are compared with references [1] and [2]. In the analysis of the plate vibration in contact with liquid, the nondimensional added virtual mass increment, Γ, (NAVMI factor) is used. This factor is described by the following equation

$$\Gamma = \frac{1}{\beta}\left(\frac{f_a^2}{f_w^2} - 1\right), \tag{18}$$

where f_w is the natural frequency in liquid, f_a is the natural frequency in vacuum, $\beta = \rho a/\mu$ is a nondimensional parameter in which ρ is the liquid density, μ is the plate unit mass and a is the width for rectangular plates and the radius for circular plates.
The NAVMI factors idea based on the hypothesis that wet made shapes are equal to dry made shapes. Then, the natural frequencies of free vibration in liquid, f_w, can be related to natural frequencies vacuum, f_a, by the following formula

$$f_w = \frac{f_a}{\sqrt{1 + \beta\Gamma}}. \tag{19}$$

Example 1. The circular plate

The plate is discretized by 18 elements (Figure 3). The system has 31 degrees of freedom. The values of the NAVMI factors for the lowest four frequencies in comparison with the analytical solutions (Ref. [2]) are presented in Table 1. The agreement between the results is good.

Table 1. NAVMI factors for simply supported circular plates (ν = 0.3)

Γ_{in}	Γ_{00}	Γ_{01}	Γ_{10}	Γ_{02}
Present	0.7540	0.3349	0.2432	0.2277
Ref. [1]	0.7554	0.3322	0.2568	0.2268

i = number of nodal circles
n = numbers of nodal diameters

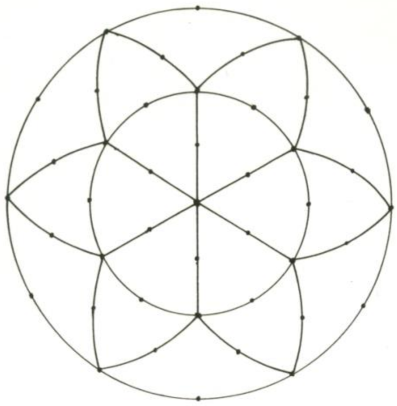

Figure 3. The discretization of a circular plate

Example 2. The rectangular plate

The plate is discretized by 32 elements (Figure 4) for the length – to – width ratio a/b=1.0, 1.5 or 2.0. The system has 49 degrees of freedom. The relation of the NAVMI factors for the lowest three frequencies *vs* the ratio b/a in comparison with Ref. [1] is given in Figure 5. The agreement of the results is good.

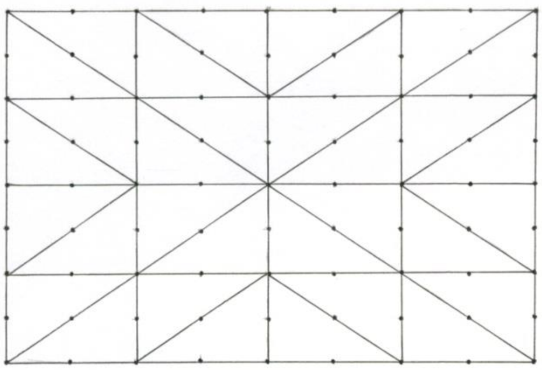

Figure 4. The discretization of a rectangular plate

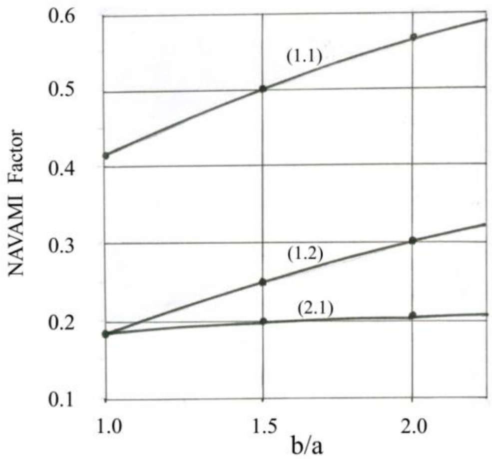

Figure 5. NAVMI factors for simply supported rectangular plates (ν=0.3): solid line-Ref. [1]; points – present work

5. Conclusions

A numerical method has been proposed for the determination of natural frequencies of a simply supported plate interacting with liquid. The hydrodynamic pressure associated with the plate deformation is described by the boundary integral equation. The most important in the boundary integral equation formulation is the reduction of computational dimension by one. Three-dimensional problems are solved as the two-dimensional ones. The plate equation is formulated in the form of two harmonic equations, so the membrane type elements are used for calculating the plate mass and stiffness matrices. These membrane triangular curvilinear 6-node elements are also the boundary elements for the liquid. Typically, for a plate in bending each finite element nodal point has three degrees of freedom, but for a membrane transversally loading it has only one degree of freedom. Thus, the presented formulation leads to three times smaller numbers of unknowns.

The effect of the liquid on the natural frequencies of the plate is significant. For the higher frequencies, the influence of the liquid is less pronounced. Numerical examples of the free vibration of circular and rectangular plates gives good agreement with solutions in Ref. [1,2].

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