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INPUT RECONSTRUCTION BY FEEDBACK CONTROL FOR THE SCHLÖGL AND FITZHUGH–NAGUMO EQUATIONS

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Dynamical reconstruction of unknown time-varying controls from inexact measurements of the state function is investigated for a semilinear parabolic equation with memory. This system includes as particular cases the Schlögl model and the FitzHugh–Nagumo equations. A numerical method is suggested that is based on techniques of feedback control. An error analysis is performed. Numerical examples confirm the theoretical predictions.

Keywords: semilinear parabolic equation, input reconstruction.

1. Introduction

In this paper, we study the problem of reconstructing an unknown control in a parabolic equation with memory that is motivated by the following FitzHugh–Nagumo system:

$$
\begin{aligned}
\frac{\partial}{\partial t} y(x,t) - \Delta y(x,t) + R(y(x,t)) + \zeta z(x,t) & \\
= u(x,t) \qquad & \text{in } \Omega \times (0,T), \\
\partial_n y(x,t) = 0 \qquad & \text{on } \Gamma \times (0,T), \\
y(x,0) = y_0(x) \qquad & \text{in } \Omega, \\
\frac{\partial}{\partial t} z(x,t) + \beta\, z(x,t) - \gamma\, y(x,t) + \kappa & \\
= 0 \qquad & \text{in } \Omega \times (0,T), \\
z(x,0) = z_0(x) \qquad & \text{in } \Omega.
\end{aligned} \tag{1}
$$

Here, $\Omega \subset \mathbb{R}^n$, $n \in \{1,2,3\}$, is a bounded Lipschitz domain (hence open) with boundary Γ, $T > 0$ is a fixed terminal time, ζ, β, γ, κ are real numbers, $y_0 : \Omega \to \mathbb{R}$, $z_0 : \Omega \to \mathbb{R}$ are given initial data. The so-called reaction term R is the cubic polynomial

$$R(y) = k\,(y - y_1)(y - y_2)(y - y_3)$$

with given real numbers $k > 0$ and $y_1 \le y_2 \le y_3$. Notice that a real constant c_R exists such that

$$R'(y) \ge c_R, \quad \forall y \in \mathbb{R}. \tag{2}$$

By ∂_n we denote the outward normal derivative on Γ. In this system, the function y is often called the activator, while z is the inhibitor.

In a nutshell, the problem of control reconstruction can explained as follows: The system (1) is known and the activator y can be observed by measurements, while the control u is not accessible for observation, hence unknown. The aim is to reconstruct u from the measurements that are overlaid by certain errors, hence

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inexact. In this way, we consider a (dynamical) inverse problem.

The well-posedness of the system (1) and the regularity of its solutions was studied, e.g., by Jackson (1990). In the context of optimal control, the differentiability of the control-to-state mapping $u \mapsto (y, z)$ had to be additionally investigated. Here, we refer to Buchholz *et al*. (2013) for the Schlögl model (this is the equation for y with $a = 0$) and to the detailed study by Casas *et al*. (2013) for the full system (1). Following their approach, the inhibitor z is eliminated by Duhamel's formula

$$z(x,t) = e^{-\beta t} z_0(x) + \int_0^t e^{-\beta(t-s)} (\gamma\, y(x,s) - \kappa)\,\mathrm{d}s.$$

Moreover, the activator y is substituted by $y(x,t) = e^{\lambda t} v(x,t)$ with sufficiently large λ. This transformation leads to an equation with monotone nonlinearity. Eventually, renaming v again by y, we arrive at the following system with memory that we consider in our paper:

$$\begin{aligned} \frac{\partial}{\partial t} y(x,t) - \Delta y(x,t) + R_\lambda(t, y(x,t)) + (K_\lambda y(\cdot))(x,t) \\ = u(x,t) + f(x,t) \quad \text{in } Q_T, \\ \partial_n y(x,t) = 0 \qquad \text{in } \Sigma_T, \\ y(x,0) = y_0(x) \qquad \text{in } \Omega, \end{aligned} \tag{3}$$

where $Q_T = \Omega \times (0,T)$, $\Sigma_T = \Gamma \times (0,T)$, the function R_λ is defined by

$$R_\lambda(t,y) = e^{-\lambda t} R(e^{\lambda t} y) + \lambda y,$$

$K_\lambda : L^r(Q_T) \to L^r(Q_T)\ \forall r \in [1,\infty]$ is the linear Volterra-type integral operator

$$(K_\lambda y(\cdot))(x,t) = \int_0^t a\, e^{-(\beta+\lambda)(t-s)} y(x,s)\,\mathrm{d}s$$

with the real constant $a = \zeta\gamma$, and $f \in L_\infty(0,T; H^1(\Omega))$ is a given function that covers in particular the term related to the constant κ.

Let us now formulate the problem of control reconstruction in greater detail: in (3), the function (control) $u(\cdot)$ is unknown. At discrete times $\tau_i = i\,T/m$, $i \in [0 : m]$, the state $y(\tau_i)$ of Eqn. (3) is measured. Let $\delta = T/m$ denote the mesh size underlying the time grid $\mathcal{T} = \{\tau_i\}_{i=0}^m$. The results of these measurements are functions $\eta_i^h \in L_p(\Omega), i \in [1 : m]$, $p > 5/2$, satisfying the estimate

$$|y(\tau_i) - \eta_i^h|_{L_p(\Omega)} \le h. \tag{4}$$

Here, $h \in (0,1)$ stands for the level of informational noise.

Our main problem is the following: Find a method of approximate reconstruction of the unknown control u through the discrete measurements η_i^h, $i \in [1 : m]$. Below, we assume that

$$\eta_0^h \in H^1(\Omega), \quad |y_0 - \eta_0^h|_{H^1(\Omega)} \le h, \tag{5}$$

where $H^1(\Omega)$ is the Sobolev space of $L_2(\Omega)$-functions that have all first-order weak derivatives in $L_2(\Omega)$.

This problem belongs to the class of inverse problems of controlled dynamical systems. In a more general context, it is also included in the theory of ill-posed problems. Modifications of such *a posteriori* formulations were investigated by Tikhonov and Arsenin (1977), Lavrentiev *et al*. (1980), Banks and Kunisch (1989), Kabanikhin (2011), Barbu (1990), Shitao and Triggiani (2013), Lasiecka *et al*. (1999) or Avdonin and Bell (2015). The solution presented here follows the theory of stable dynamical inversion developed by Kryazhimskii and Osipov (1995), Maksimov (2002; 2009; 2016), Maksimov and Pandolfi (2002) or Maksimov and Mordukhovich (2017), who used a combination of methods from the theory of ill-posed problems (Tikhonov and Arsenin, 1977) and from the theory of positional control (Krasovskii and Subbotin, 1988). The essence of our approach is that an algorithm for input reconstruction is represented as a control algorithm for some artificial dynamical system (a model). Given current observations of the system, the control input in the model is chosen in such a way that its realization in time is obtained by some regularization principle that guarantees the stability of the numerical method.

In Fig. 1, a scheme of the solution method is shown that is stable with respect to informational noise and computational errors. According to this scheme, a given system $\mathcal{S}$ (in our case, Eqn. (3)) is accompanied by some artificial computer-modeled closed-loop control system (the model $\mathcal{M}$). This model, working on the time interval $[0,T]$, has an unknown control function $u^h(\cdot)$ and an output function $w^h(\cdot)$. The model $\mathcal{M}$ should be constructed. The process of synchronous feedback control of the systems S and M is running on the interval $[0,T]$. It is split into $m-1$ identical steps. In the i-th step that is carried out on the time interval $J_i = [\tau_i, \tau_{i+1})$, the following actions are performed. First, at time τ_i, according to some rule $\mathcal{U}$, the control $u^h(t) = u^h(\tau_i, \eta_i^h, w^h(\tau_i))$, $t \in J_i$ is calculated. Then (till the time τ_{i+1}), the control $u^h(t)$, $\tau_i \le t < \tau_{i+1}$, is fed to the system $\mathcal{M}$. The values $u^h(\tau_i)$ and $w^h(\tau_{i+1})$ are results of the algorithm at the i-th step. Thus, all the complexity of solving the problem is reduced to the appropriate choice of a model $\mathcal{M}$, computation of w^h and the construction of a function $u^h(\cdot)$.

In essence, the procedure of dynamical reconstruction is equivalent to solving the following two main problems:

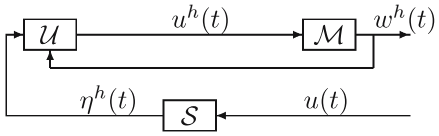

Fig. 1. Scheme of the control reconstruction method.

1. Define an appropriate model $\mathcal{M}$.

2. Set up a suitable rule $\mathcal{U}$ for forming a control $u^h(\cdot)$ in the model.

For some classes of partial differential equations, such a scheme was realized by Maksimov (2002; 2009; 2016), Maksimov and Pandolfi (2002) or Mordukhovich (2008). The main novelty of our paper is the investigation of the parabolic equation with memory (3). Moreover, we consider two cases.

In the first case, we assume that the controls are restricted by $u(t) \in U_{\text{ad}}$ for a.a. $t \in [0, T]$, where $U_{\text{ad}} \subset H^1(\Omega)$ is a given convex, bounded and closed set. In this way, we assume some *a-priori* knowledge on the location of the unknown controls $u(\cdot)$. In the second case, we do not assume such *a-priori* restrictions. Then the control $u(\cdot)$ on the right-hand side of Eqn. (3) is an unknown element of $L_p(Q_T)$ for all $p \leq 6$. The performance of control reconstruction is estimated by two criteria.

The first one is the deviation of the solution of Eqn. (3) corresponding to some real control $u(\cdot)$ from the solution of an auxiliary equation (the model) corresponding to a constructed approximation $u^h(\cdot)$ to this control. The value of this deviation will be denoted by the symbol $\varepsilon_h(\cdot)$.

The second one is the mean-square norm of the difference $u^h(\cdot) - u(\cdot)$. The choice of these two criteria is explained by the fact that, if they are small (under corresponding assumptions on the correlation between the parameters of the numerical method), then the approximation $u^h(\cdot)$ is close to the control $u(\cdot)$ in the mean-square norm (see Theorem 2 below).

2. Analysis of the state equation

Throughout the paper, we use the following notation. By $\langle \cdot, \cdot \rangle$, we denote the duality between the Sobolev spaces $H^1(\Omega)$ and $(H^1(\Omega))^*$, $\nabla y(x,t)$ is the gradient of the function $x \mapsto y(x,t)$. Moreover, we introduce the Hilbert spaces $H = L_2(\Omega)$ and $V = H^1(\Omega)$ equipped with the scalar products $(\cdot, \cdot)$ and $(\cdot, \cdot)_V$, respectively. The associated norms are denoted by $|\cdot|_H$ and $|\cdot|_V$. Moreover, $|\cdot|$ denotes the module of real numbers and $|\cdot|_{\mathbb{R}^n}$ stands for the norm in the Euclidean space $\mathbb{R}^n$. Next, we need the Sobolev space

$$W(0,T) = \{y \in L_2(0,T;V) : \dot{y} \in L_2(0,T;V^*)\}.$$

It is well known that $W(0,T)$ is continuously embedded in $C([0,T], H)$.

A function $y(\cdot) = y(\cdot; 0, y_0, u(\cdot)) \in W(0,T) \cap L_\infty(Q_T)$ is said to be a (weak) solution of Eqn. (3) if the equation

$$\begin{aligned} \int_0^T \langle \dot{y}(t), \varphi(t) \rangle \, \mathrm{d}t + \int_0^T \int_\Omega \{\nabla y(x,t) \cdot \nabla \varphi(x,t) \\ + R_\lambda(t, y(x,t)) \varphi(x,t)\} \, \mathrm{d}x \, \mathrm{d}t \\ + \int_0^T \int_\Omega (K_\lambda y(\cdot))(x,t) \varphi(x,t) \, \mathrm{d}x \, \mathrm{d}t \\ = \int_0^T \int_\Omega \{u(x,t) + f(x,t)\} \varphi(x,t) \, \mathrm{d}x \, \mathrm{d}t \end{aligned}$$

holds for all $\varphi \in W(0,T)$ and the initial condition $y(0) = y_0$ is fulfilled.

Notice that, for all times $t \in [0,T]$, the state $y(t)$ of Eqn. (3) at the time t is a well-defined element of H. It follows from the work of Casas *et al.* (2013, Thm. 2), that under Assumption 1 presented below, for any $y_0 \in L_\infty(\Omega)$, $u \in L_p(Q_T)$, and $p > 5/2$ (in particular, for $u \in L_\infty(0,T;V)$), there exists a unique (weak) solution of Eqn. (3).

An equation similar to (3) was introduced by Buchholz *et al.* (2013). The Schlögl and FitzHugh–Nagumo equations can be reduced to this type. The FitzHugh–Nagumo equations (1) play an important role in mathematical biology and medicine. For instance, they serve as a very simplified mathematical model for some electrical processes in the human heart and can be used to understand associated control strategies. Moreover, they describe the transport of impulses in human nerve cells. The Schlögl model that is obtained for $\zeta = 0$ is well known in physical chemistry. Invoking (2), we are able to transform our equation to one with monotone nonlinearity; see the substitution $y = e^\lambda v$ below.

Note that control problems for such equations were discussed, for example, by Casas *et al.* (2013), Buchholz *et al.* (2013), Ryll *et al.* (2016), Breiten and Kunisch (2014), or Maksimov (2017); see also the bibliography therein. Moreover, we mention Gugat and Tröltzsch (2015), who studied the problem of boundary feedback stabilization. A study of this paper reveals that an extension of our method to boundary control should be possible.

Let the following assumption be fulfilled, which after the substitution $y = e^{\lambda t} v$ leads to an equation of a monotone type.

Assumption 1. The parameter λ in Eqn. (3) satisfies the inequality

$$\max\{3|c_R|, 3|a|T^{1/2}, 0.5-\beta\} \le \lambda,$$

with c_R defined in (2).

The following result was proved by Casas *et al.* (2013). Here, $\overline{\Omega}$ denotes the closure of the set Ω.

Lemma 1. *Let Assumption* 1 *be fulfilled. If λ is sufficiently large, $u(\cdot) \in L_p(Q_T), p > 5/2$, and $y_0 \in L_\infty(\Omega)$, then there exists a unique solution $y(\cdot) \in W(0,T) \cap L_\infty(Q_T) \cap C(\overline{\Omega} \times (0,T])$ of Eqn.* (3). *If even $y_0 \in C(\overline{\Omega})$, then the inclusion $y(\cdot) \in C(\overline{\Omega} \times [0,T])$ takes place.*

Lemma 2. *Let the assumptions of Lemma* 1 *be fulfilled and $y_0 \in V$. Then, $\dot{y}(\cdot) \in L_2(0,T;H)$. Moreover, for each $M > 0$, there exists a constant $c_0 = c_0(M) > 0$ such that the inequality*

$$\operatorname*{vrai\ sup}_{t\in(0,T)} |y(t)|_H^2 + \int_0^T |\dot{y}(t)|_H^2 \, \mathrm{d}t \le C_0 \left\{ |y_0|_V^2 + \int_0^T \left(|u(t)|_H^2 + |f(t)|_H^2\right) \mathrm{d}t + 1 \right\} \quad (6)$$

holds for all $u(\cdot) \in L_p(Q_T)$ with $|u(\cdot)|_{L_p(Q_T)} \le M$.

Proof. Thanks to Lemma 1, the function $y(\cdot)$ belongs to $L_\infty(Q_T)$. Therefore, the function $(x,t) \mapsto F_1(x,t) = e^{-\lambda t} R(e^{\lambda t} y(x,t)) + \lambda y(x,t)$ is an element of the space $L_\infty(Q_T)$. Also, from the results of Casas *et al.* (2013), the function $(x,t) \mapsto F_2(x,t) = (K_\lambda y(\cdot))(x,t)$ is an element of $L_\infty(Q_T)$. Consequently, the function $y(\cdot)$ is a solution of the parabolic equation

$$\begin{aligned} \frac{\partial}{\partial t} y(x,t) - \Delta y(x,t) &= F(x,t) && \text{in } Q_T, \\ \partial_n y(x,t) &= 0 && \text{in } \Sigma_T, \\ y(x,0) &= y_0(x) && \text{in } \Omega, \end{aligned}$$

where the function $(x,t) \mapsto F(x,t) = -F_1(x,t) - F_2(x,t) + u(x,t) + f(x,t)$ belongs to $L_2(0,T;H)$.

The validity of the inclusion $\dot{y}(\cdot) \in L_2(0,T;H) \cong L_2(Q_T)$ follows from Theorem 5 of Evans (1998, Chapter 7.1). This theorem is formulated for homogeneous Dirichlet boundary conditions, but an inspection of the proof shows that it can be extended to homogeneous Neumann boundary conditions. Theorem 5 of Evans (1998, Chapter 7.1) also ensures that

$$y \in L_2(0,T;H^2(\Omega)) \cap L_\infty(0,T;V),$$

and that the estimate

$$\operatorname*{vrai\ sup}_{t\in(0,T)} |y(t)|_V + |y(\cdot)|_{L_2(0,T;H^2(\Omega))} + |\dot{y}(\cdot)|_{L_2(Q_T)} \le c_1(M)(|F(\cdot)|_{L_2(Q_T)} + |y_0|_V)$$

holds true with some constant $c_1(M) > 0$.

The integral operator $K_\lambda : L_2(Q_T) \to L_2(Q_T)$ is linear and continuous. Estimating now

$$|R_\lambda(\cdot, y(\cdot))|_{L_\infty(Q_T)} \le c_2(M), \quad |K_\lambda y(\cdot)|_{L_2(Q_T)} \le c_3(M)$$

for all $|u(\cdot)|_{L_p(Q_T)} \le M$, we get

$$|F(\cdot)|_{L_2(Q_T)} \le |u(\cdot)|_{L_2(Q_T)} + |f(\cdot)|_{L_2(Q_T)} + c_4(M).$$

Now we obtain the result of the lemma in the form

$$\operatorname*{vrai\ sup}_{t\in(0,T)} |y(t)|_V + |y(\cdot)|_{L_2(0,T;H^2(\Omega))} + |\dot{y}(\cdot)|_{L_2(Q_T)} \le c_5(M)\{|y_0|_V + |u(\cdot)|_{L_2(Q_T)} + |f(\cdot)|_{L_2(Q_T)} + 1\}.$$

The final statement of the lemma follows by squaring the above inequality and employing Young's inequality. ■

3. Reconstruction of restricted controls

Let us now turn over to the reconstruction problem. First, we consider the case of restricted controls.

Definition 1. The symbol $\boldsymbol{\eta}_{\mathcal{T}}^h(y(\cdot))$ denotes the set of all functions $\eta^h(\cdot) \in L_p(Q_T)$, $p > 5/2$, which is associated to a uniform partition $\mathcal{T}$ of $[0,T]$, such that

$$\eta^h(t) = \eta_i^h \in L_p(\Omega) \quad \text{for a.a. } t \in [\tau_i, \tau_{i+1}),$$

$i \in [0 : m-1]$. Moreover, the values η_i^h are assumed to obey the properties (4) and (5). We call such functions η^h *admissible measurements.*

As model $\mathcal{M}$, we introduce the parabolic equation

$$\begin{aligned} \frac{\partial}{\partial t} w(x,t) - \Delta w(x,t) + R_\lambda(t, w(x,t)) + (K_\lambda \eta^h(\cdot))(x,t) \\ = u^h(x,t) + f(x,t) \quad \text{in } Q_T, \\ \partial_n w(x,t) = 0 \quad \text{in } \Sigma_T, \\ w(x,0) = \eta_0^h(x) \quad \text{in } \Omega. \end{aligned} \quad (7)$$

Let the symbol $w^h(\cdot) = w(\cdot; 0, \eta_0^h, u^h(\cdot), \eta^h(\cdot))$ denote the solution of the differential equation (7). Note that, for any admissible measurement $\eta^h(\cdot)$, the function $(x,t) \mapsto (K_\lambda \eta^h(\cdot))(x,t)$ is an element of the space $L_p(Q_T)$. Therefore, as follows from results of Casas *et al.* (2013), for every admissible measurements $\eta^h(\cdot) \in \boldsymbol{\eta}_{\mathcal{T}}^h(y(\cdot))$ and $u^h(\cdot) \in L_p(Q_T)$, there exists a unique solution of Eqn. (7), i.e., a function $w^h(\cdot) \in W(0,T) \cap L_\infty(Q_T)$.

Remark 1. If $\eta_0^h \in V$, then $\dot{w}^h(\cdot) \in L_2(0,T;H)$. This fact is easily verified along the lines of Lemma 2.

Let us now introduce the *set of admissible controls* by

$$\mathcal{U}_{\text{ad}} = \{u(\cdot) \in L_2(0,T;V) : u(t) \in U_{\text{ad}} \text{ a.e. in}[0,T]\}.$$

The proof of the next lemma is completely analogous to the proof of Lemma 2.

Lemma 3. *Let $w^h(\cdot)$ be the unique weak solution of* (7) *associated with given $u^h(\cdot) \in \mathcal{U}_{ad}$, and initial data η_0^h satisfying* (5) *for some $h \in (0,1)$. Then the inequality*

$$\operatorname{vrai}\sup_{t\in(0,T)} |w^h(t)|_H^2 + \int_0^T |\dot{w}^h(t)|_H^2 \,\mathrm{d}t \le C_1 \left\{ |y_0|_V^2 + \int_0^T \{|u^h(t)|_H^2 + |f(t)|_H^2\} \,\mathrm{d}t + 1 \right\} \tag{8}$$

holds uniformly with respect to all partitions $\mathcal{T}$ of the time interval $[0,T]$ with mesh size $\delta = \delta(\mathcal{T}) \in (0,1)$, controls $u(\cdot) \in \mathcal{U}_{ad}$, solutions $y(\cdot) = y(\cdot;0,y_0,u(\cdot))$ of Eqn. (3)*, and admissible measurements $\eta^h(\cdot) \in \boldsymbol{\eta}^h_{\mathcal{T}}(y(\cdot)), h \in (0,1)$. Here, C_1 is a constant independent of $h, \mathcal{T}, u(\cdot), y(\cdot), \eta^h(\cdot)$, and η_0^h.*

Proof. We already know from Lemma 1 that a unique weak solution $w^h(\cdot) \in L_\infty(Q_T)$ of (7) exists (we apply the lemma for $a = 0$ with an appropriate right-hand side including $K_\lambda(w^h)$). Now we move $R_\lambda(t,w^h)$ and $K_\lambda(w^h)$ to the right-hand side and define

$$F^h(x,t) = u^h(x,t) + f(x,t) - R_\lambda(t, w^h(x,t)) - (K_\lambda \eta^h(\cdot))(x,t).$$

This function belongs to $L_2(Q_T)$. Clearly, $w^h(\cdot)$ solves the linear parabolic equation

$$\frac{\partial}{\partial t} w(x,t) - \Delta w(x,t) = F^h(x,t) \quad \text{in } Q_T,$$
$$\partial_n w(x,t) = 0 \quad \text{in } \Sigma_T,$$
$$w(x,0) = \eta_0^h(x) \quad \text{in } \Omega.$$

Invoking again Theorem 5 of Evans (1998, Chapter 7.1), we know that

$$w^h(\cdot) \in L_2(0,T;H^2(\Omega)) \cap L_\infty(0,T;V),$$
$$\dot{w}^h(\cdot) \in L_2(Q_T),$$

and

$$\operatorname{vrai}\sup_{t\in(0,T)} (|w^h(t)|_V + |w^h(\cdot)|_{L_2(0,T;H^2(\Omega))} + |\dot{w}^h(\cdot)|_{L_2(Q_T)}) \le c_1(M)(|F^h(\cdot)|_{L_2(Q_T)} + |\eta_0^h|_V)$$

holds true with some constant $c_1(M) > 0$, where M is the $L^6(\Omega)$-bound for all functions of the bounded set U_{ad}.

Notice that our assumptions guarantee $H^1(\Omega) \subset L^6(\Omega)$ hence U_{ad} is bounded in some space $L^p(\Omega)$ with $p > 5/2$. The integral operator $K_\lambda : L_2(Q_T) \to L_2(Q_T)$ is linear and continuous. Estimating now

$$|R_\lambda(\cdot, w^h(\cdot))|_{L_\infty(Q_T)} \le c_2(M),$$
$$|K_\lambda \eta^h(\cdot)|_{L_2(Q_T)} \le c_3(M),$$

we find

$$|F^h(\cdot)|_{L_2(Q_T)} \le |u^h(\cdot)|_{L_2(Q_T)} + |f(\cdot)|_{L_2(Q_T)} + c_4(M).$$

Notice that, by the definition of the set of admissible measurements and by the estimate (4), η^h is contained in a bounded set of $L^2(Q_T)$. Moreover, the estimate (5) yields

$$|\eta_0^h|_V \le |y_0|_V + h \le |y_0|_V + 1.$$

Now it is easy to see that

$$\operatorname{vrai}\sup_{t\in(0,T)} |w^h(t)|_V + |w^h(\cdot)|_{L_2(0,T;H^2(\Omega))} + |\dot{w}^h(\cdot)|_{L_2(Q_T)} \le c_5(M)\{|u^h(\cdot)|_{L_2(Q_T)} + |f(\cdot)|_{L_2(Q_T)} + |y_0|_V + 1\}$$

holds. The claim of the lemma follows by squaring the above inequality and employing Young's inequality. ■

Let us fix now a family of uniform partitions $\mathcal{T}_h$ of the interval $[0,T]$ with mesh size $\delta(h) > 0$ such that

$$\mathcal{T}_h = \{\tau_{h,i}\}_{i=0}^{m_h}, \quad \tau_{h,0} = 0, \quad \tau_{h,m_h} = T,$$
$$\tau_{h,i+1} = \tau_{h,i} + \delta(h), \quad \delta(h) \in (0,1),$$

and a function $\alpha(h) : (0,1) \to (0,1)$.

Definition 2. The rule $\mathcal{U}$ of forming the control $u^h(\cdot)$ in the model (7) is defined as a set-valued mapping $\mathcal{U}_h$ that assigns to $\eta \in L_p(\Omega)$ and $w \in H$ the set

$$\mathcal{U}_h(\eta, w) = \Big\{ u^h \in U_{\text{ad}} : \langle w - \eta, u^h \rangle + \alpha(h)|u^h|_V^2 \le \inf_{u\in U_{\text{ad}}} (\langle w - \eta, u\rangle + \alpha(h)|u|_V^2) + h \Big\}. \tag{9}$$

Let us now describe the numerical method for solving the problem. At the beginning, we fix values $h \in (0,1)$, $\alpha = \alpha(h)$, and a uniform partition $\mathcal{T}_h = \{\tau_i\}_{i=0}^{m_h}$ of $[0,T]$ with diameter $\delta(h) > 0$. The work of the algorithm is decomposed into $m-1$ identical steps.

On the half-open interval $[0,\tau_1)$, we first select some $u_0^h \in \mathcal{U}_h(\eta_0^h, \eta_0^h)$. Then we define

$$u^h(t) = u_0^h, \quad \forall t \in [0, \tau_1). \tag{10}$$

Under the action of this control as well as of the unknown control on the interval $[0, \tau_1)$ that we denote by $u_{0,\tau_1}(\cdot)$, the solutions $y(\cdot) = y(\cdot; 0, y_0, u_{0,\tau_1}(\cdot))$ and $w^h(\cdot) = w(\cdot; 0, \eta_0^h, u_{0,\tau_1}^h(\cdot), \eta_{0,\tau_1}^h(\cdot))$ of Eqns. (3) and (7) on the interval $[0, \tau_1]$ are realized. Notice that we cannot compute $y(\cdot)$, since $u_{0,\tau_1}(\cdot)$ is unknown. However, we have a measurement $\eta_1^h \in L_p(\Omega)$ of $y(\tau_1)$ that obeys the estimate

$$|\eta_1^h - y(\tau_1)|_{L_p(\Omega)} \le h, \tag{11}$$

where $u_{0,\tau_1}^h(\cdot)$ is the function $u^h(t)$ defined above on the half-open interval $[0, \tau_1)$.

At the time $t = \tau_1$, we select a u_1^h that obeys the condition

$$u_1^h \in \mathcal{U}_h(\eta_1^h, w^h(\tau_1)). \tag{12}$$

In the interval $[\tau_1, \tau_2)$, we fix the control u^h by

$$u^h(t) = u_1^h \quad \text{for} \quad t \in [\tau_1, \tau_2).$$

Let us denote by $u_{\tau_1,\tau_2}^h(\cdot)$ this part of the control function defined on $[\tau_1, \tau_2)$. In the same way, we denote by $u_{\tau_i,\tau_{i+1}}^h(\cdot)$ and $\eta_{\tau_i,\tau_{i+1}}^h(\cdot)$ the restrictions of $u^h(\cdot)$ and $\eta^h(\cdot)$ to $[\tau_i, \tau_{i+1})$, respectively. Here, the mapping $\mathcal{U}_h$ is defined by the relation (9). Then, we calculate $w^h(\cdot) = w(\cdot; \tau_1, w^h(\tau_1), u_{\tau_1,\tau_2}^h(\cdot), \eta_{\tau_1,\tau_2}^h(\cdot))$, i.e., the solutions of Eqn. (7) on the interval $[\tau_1, \tau_2)$.

Let the solutions $y(\cdot)$ (of Eqn. (3)) and $w^h(\cdot)$ (of Eqn. (7)) be defined on the interval $[0, \tau_i]$. At the time $t = \tau_i$, we calculate

$$\begin{aligned} u_i^h &\in \mathcal{U}_h(\eta_i^h, w^h(\tau_i)), \\ |\eta_i^h - y(\tau_i)|_{L_p(\Omega)} &\le h, \quad \eta_i^h \in L_p(\Omega), \end{aligned} \tag{13}$$

and we set

$$u^h(t) = u_i^h \quad \text{for} \quad t \in [\tau_i, \tau_{i+1}).$$

As a result of the action of this control and of the unknown control $u_{\tau_i,\tau_{i+1}}(\cdot)$, the solutions

$$y(\cdot) = y(\cdot; \tau_i, y(\tau_i), u_{\tau_i,\tau_{i+1}}(\cdot))$$

and

$$w^h(\cdot) = w(\cdot; \tau_i, w^h(\tau_i), u_{\tau_i,\tau_{i+1}}^h(\cdot), \eta_{\tau_i,\tau_{i+1}}^h(\cdot))$$

of Eqns. (3) and (7) on the time interval $[\tau_i, \tau_{i+1}]$ are obtained. The above procedure stops at the moment T.

Theorem 1. *Let, for given $h \in (0,1)$, the control $u^h(\cdot)$ be defined by the formulas* (9)–(13)*, and let $y(\cdot)$ and $w^h(\cdot)$ be the solutions of Eqns.* (3) *and* (7)*, respectively; set $\mu^h(\cdot) = w^h(\cdot) - y(\cdot)$. Assume that $\delta(h)$ is the diameter of a uniform partition $\mathcal{T}_h$ of $[0, T]$. Define a function $\varepsilon_h : [0, T] \to [0, \infty)$ by*

$$\begin{aligned} \varepsilon_h(t) = |\mu^h(t)|_H^2 + 2 \int_0^t \Big\{ \int_\Omega |\nabla \mu^h(x,s)|_{\mathbb{R}^n}^2 \, \mathrm{d}x \\ + \frac{2}{3} \lambda \, |\mu^h(s)|_H^2 \Big\} \, \mathrm{d}s. \end{aligned}$$

Then there are constants $C_2 = C_2(y_0)$ and $C_3 = C_3(y_0)$ not depending on h, $u(\cdot), u^h(\cdot), y(\cdot)$, and $w^h(\cdot)$, such that the estimates

$$\varepsilon_h(t) \le C_2(\alpha(h) + \delta(h)^{1/2} + h), \quad \forall t \in [0, T], \tag{14}$$

$$\int_0^T |u^h(s)|_V^2 \, \mathrm{d}s \le \int_0^T |u(s)|_V^2 \, \mathrm{d}s + C_3(\delta(h)^{1/2} + h)\alpha(h)^{-1} \tag{15}$$

are satisfied.

Proof. To prove the theorem, we estimate the variation of the function $\varepsilon_h(t)$. Define $\hat{R}_\lambda(t, v) = e^{-\lambda t} R(e^{\lambda t} v) + \lambda/3 \, v$. Then Eqns. (3) and (7) take the following form:

$$\begin{aligned} \frac{\partial}{\partial t} y(x,t) - \Delta y(x,t) + \hat{R}_\lambda(t, y(x,t)) \\ + \frac{2\lambda}{3} \, y(x,t) + (K_\lambda y(\cdot))(x,t) \\ = u(x,t) + f(x,t) \quad \text{in } Q_T, \\ \partial_n y(x,t) = 0 \qquad \text{in } \Sigma_T, \\ y(0) = y_0(x) \quad \text{in } \Omega, \end{aligned} \tag{16}$$

and

$$\begin{aligned} \frac{\partial}{\partial t} w^h(x,t) - \Delta w^h(x,t) + \hat{R}_\lambda(t, w^h(x,t)) \\ + \frac{2\lambda}{3} w^h(x,t) + (K_\lambda \eta^h(\cdot))(x,t) \\ = u^h(x,t) + f(x,t) \quad \text{in } Q_T, \\ \partial_n w^h(x,t) = 0 \qquad \text{in } \Sigma_T, \\ w^h(x,0) = \eta_0^h(x) \quad \text{in } \Omega. \end{aligned} \tag{17}$$

Subtracting (17) from (16), multiplying scalarly (in H) the difference obtained by $\mu^h(t)$, and taking into account the monotonicity of the mapping $v \to \hat{R}_\lambda(t, v)$ for all sufficiently large λ, we find

$$\begin{aligned} \rho^h(t) + ((K_\lambda(\eta^h(\cdot) - y(\cdot)))(t), \mu^h(t)) \\ \le (\mu^h(t), u^h(t) - u(t))), \end{aligned} \tag{18}$$

where

$$\begin{aligned}\rho^h(t) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\varepsilon_h(t)\\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mu^h(t)|_H^2 + \int_\Omega |\nabla\mu^h(x,t)|_{\mathbb{R}^n}^2\,\mathrm{d}x\\ &\quad + \frac{2\lambda}{3}\,|\mu^h(t)|_H^2.\end{aligned}$$

Notice that, owing to Lemmas 2 and 3, the function $\dot\mu_h(\cdot)$ belongs to $L_2(0,T;H)$. Since $L_p(\Omega)$, $p > 5/2$, is continuously embedded in H, there exists a number $c^* > 0$ such that

$$|y|_H \le c^*|y|_{L_p(\Omega)}, \quad \forall y \in L_p(\Omega).$$

In view of this inequality, and (4), (5), for all $s \in [\tau_i, \tau_{i+1}]$, $i \in [0 : m-1]$, the estimate

$$\begin{aligned}&|y(s) - \eta^h(s)|_H^2\\ &\quad = |y(s) - \eta_i^h|_H^2\\ &\quad \le \Big|y(\tau_i) - \eta_i^h + \int_{\tau_i}^{s} \dot y(t)\,\mathrm{d}t\Big|_H^2\\ &\quad \le 2\Big\{|y(\tau_i) - \eta_i^h|_H^2 + \Big|\int_{\tau_i}^{s} \dot y(t)\,\mathrm{d}t\Big|_H^2\Big\}\\ &\quad \le 2\Big\{|y(\tau_i) - \eta_i^h|_H^2 + (s-\tau_i)\int_{\tau_i}^{s} |\dot y(t)|_H^2\,\mathrm{d}t\Big\}\\ &\quad \le d_0(\delta + h^2)\end{aligned} \tag{19}$$

is fulfilled with some constant $d_0 > 0$. Here and below we skip the dependence of δ and α on ε. Now, from (19), with some other constants d_1, d_2, we get for all $t \in [0,T]$

$$\begin{aligned}&\int_0^t \big|(K_\lambda(y(\cdot) - \eta^h(\cdot)))(s)\big|_H^2\,\mathrm{d}s\\ &\quad \le d_1 \int_0^t\int_0^\tau |y(s) - \eta^h(s)|_H^2\,\mathrm{d}s\,\mathrm{d}\tau \le d_2(\delta + h^2).\end{aligned}$$

By the Cauchy–Bunyakovsky inequality and Lemmas 2 and 3, the last inequality implies

$$\begin{aligned}&\int_0^t \big|(K_\lambda(y(\cdot) - \eta^h(\cdot)))(s), \mu^h(s))\big|\,\mathrm{d}s\\ &\quad \le \Big(\int_0^t |(K_\lambda(y(\cdot) - \eta^h(\cdot)))(s)|_H^2\,\mathrm{d}s\Big)^{1/2}\\ &\qquad \times \Big(\int_0^t |\mu^h(s)|_H^2\,\mathrm{d}s\Big)^{1/2}\\ &\quad \le d_3(\delta^{1/2} + h)\end{aligned} \tag{20}$$

with another constant d_3. By virtue of Lemmas 3 and 4, we have $|\mu^h(t)|_H \le \mathrm{const}$. Note that, if $u^h(t), u(t) \in V$ and $\mu^h(t) \in H$, then the duality pairing on $(V)^* \times V$ is equivalent to the scalar product in H:

$$(\mu^h(t), u^h(t) - u(t)) = \langle \mu^h(t), u^h(t) - u(t)\rangle.$$

Moreover, for a.a. $t \in [\tau_i, \tau_{i+1}]$ and each $i \in [0 : m-1]$, owing to (9)–(13), we derive the inequalities

$$\begin{aligned}&\langle \mu^h(t), u^h(t) - u(t)\rangle + \frac{1}{2}\alpha\left\{|u^h(t)|_V^2 - |u(t)|_V^2\right\}\\ &= \langle \mu^h(\tau_i), u^h(t) - u(t)\rangle\\ &\quad + \Big\langle \int_{\tau_i}^{t} \dot\mu^h(s)\,\mathrm{d}s, u^h(t) - u(t)\Big\rangle\\ &\quad + \frac{1}{2}\alpha\left\{|u^h(t)|_V^2 - |u(t)|_V^2\right\}\\ &\le \langle w^h(\tau_i) - \eta_i^h, u^h(t) - u(t)\rangle\\ &\quad + \frac{1}{2}\alpha\left\{|u^h(t)|_V^2 - |u(t)|_V^2\right\}\\ &\quad + d_4 h + d_5\int_{\tau_i}^{\tau_{i+1}} \big|\dot y(s) - \dot w^h(s)\big|_{V^*}\,\mathrm{d}s\\ &\le d_4 h + d_5\int_{\tau_i}^{\tau_{i+1}} \big|\dot y(s) - \dot w^h(s)\big|_{V^*}\,\mathrm{d}s.\end{aligned} \tag{21}$$

Notice that, owing to (4), we have

$$\begin{aligned}&\langle \mu^h(\tau_i), u^h(t) - u(t)\rangle\\ &\quad = \langle w^h(\tau_i) - y(\tau_i), u^h(t) - u(t)\rangle\\ &\quad \le \langle w^h(\tau_i) - \eta_i^h, u^h(t) - u(t)\rangle + d_4 h.\end{aligned}$$

For the last inequality of (21), we invoked Definition 2 of $\mathcal{U}_h$. By the continuous embedding of H in V^*, there exists a number $c_* > 0$ such that

$$|y|_{V^*} \le c_*|y|_H \quad \text{for any } y \in H.$$

Since u^h is bounded in $L^6(\Omega)$, independently of h, by the boundedness of U_{ad} in $H^1(\Omega)$, in view of Lemmas 2 and 3, we get

$$\int_0^T \big|\dot y(t) - \dot w^h(t)\big|_{V^*}\,\mathrm{d}t \le d_6. \tag{22}$$

In turn, from (18) and (20)–(22), we derive the inequality

$$\begin{aligned}
&\int_0^t \rho^h(s)\,\mathrm{d}s + \frac{\alpha}{2}\int_0^t \left\{ \left|u^h(s)\right|_V^2 - |u(s)|_V^2 \right\} \mathrm{d}s \\
&\le \int_0^t \left|(K_\lambda(y(\cdot) - \eta^h(\cdot)))(s), \mu^h(s))\right| \mathrm{d}s \\
&\quad + \int_0^t (\mu^h(s), u^h(s) - u(s))\,\mathrm{d}s \\
&\quad + \frac{\alpha}{2}\int_0^t \left\{ \left|u^h(s)\right|_V^2 - |u(s)|_V^2 \right\} \mathrm{d}s \\
&\le d_7(\delta^{1/2} + h) + d_8\delta \int_0^T \left|\dot y(s) - \dot w^h(s)\right|_{V^*} \mathrm{d}s \\
&\le d_9(\delta^{1/2} + h).
\end{aligned} \tag{23}$$

The term related to $d_8\delta$ is obtained as follows: We consider the step function $\varphi_\delta : [0,T] \to \mathbb{R}$ defined by

$$\varphi_\delta(t) = \int_{\tau_i}^{\tau_{i+1}} |\dot y(s) - \dot w^h(s)|_{(H^1(\Omega))^*}\,\mathrm{d}s, \qquad t \in [\tau_i, \tau_{i+1}).$$

Then

$$\begin{aligned}
\int_0^t \varphi_\delta(s)\,\mathrm{d}s \le \int_0^T \varphi_\delta(s)\,\mathrm{d}s
&= \sum_{i=0}^{m_h-1} \int_{\tau_i}^{\tau_{i+1}} \varphi_\delta(s)\,\mathrm{d}s \\
&= \sum_{i=0}^{m_h-1} \delta \int_{\tau_i}^{\tau_{i+1}} |\dot y(s) - \dot w^h(s)|_{(H^1(\Omega)))^*}\,\mathrm{d}s \\
&= \delta \int_0^T |\dot y(s) - \dot w^h(s)|_{(H^1(\Omega))^*}\,\mathrm{d}s.
\end{aligned}$$

This inequality is valid for all $\delta \in (0,1)$, $h \in (0,1)$, and $t \in [0,T]$. Then, by virtue of (23) and $\rho^h(t) = 0.5\,\dot{\varepsilon_h}(t)$, the relation

$$\begin{aligned}
\varepsilon_h(t) &= 2\int_0^t \rho^h(\tau)\,\mathrm{d}\tau + \varepsilon_h(0) \\
&\le 2d_9(\delta^{1/2} + h) + d_{10}\alpha + |y(0) - \eta_0^h|_H^2
\end{aligned} \tag{24}$$

is found. Therefore, from (23), (24), and (5), we obtain the inequalities (14) and (15). ■

Theorem 2. *Let $\alpha(h) \to 0, \delta(h) \to 0, (\delta^{1/2}(h) + h)\alpha^{-1}(h) \to 0$ as $h \to 0$. Then we have*

$$\lim_{h\to 0} u^h(\cdot) = u(\cdot) \quad \text{in } L_2(0,T;H^1(\Omega)).$$

Proof. We are going to show that, for any sequence $h_j \to 0+$ as $j \to \infty$, any family $\{\mathcal{T}_{h_j}\} = \{\tau_{h_j,i}\}_{i=0}^{m_{h_j}}$ of uniform partitions of $[0,T]$, and any sequence of admissible measurements $\eta^{h_j}(\cdot) \in \boldsymbol{\eta}^{h_j}_{\mathcal{T}_{h_j}}(y(\cdot))$, we get

$$u^{h_j}(\cdot) \to u(\cdot) \quad \text{in } L_2(0,T;H^1(\Omega)) \text{ as } j \to \infty.$$

Here and below, the controls $u^{h_j}(\cdot)$ are defined by the rule (9) and the steps (10)–(13) of the reconstruction method for $h = h_j$. Assuming the contrary, we conclude that this convergence does not happen. By virtue of the boundedness of set $\{u^{h_j}(\cdot)\}_{j=1}^\infty$ in the space $L_2(0,T;H^1(\Omega))$, there exists a weakly convergent subsequence of $\{u^{h_j}(\cdot)\}_{j=1}^\infty$, denoted for simplicity by $\{u^{h_j}(\cdot)\}_{j=1}^\infty$ again, such that

$$u^{h_j}(\cdot) \rightharpoonup u_0(\cdot) \quad \text{in } L_2(0,T;H^1(\Omega)) \text{ as } j \to \infty, \tag{25}$$

where

$$u_0(\cdot) \ne u(\cdot). \tag{26}$$

Notice that the set U_{ad} was assumed to be bounded; hence the sequence $\{u^{h_j}(\cdot)\}$ is bounded in $L_2(0,T;H^1(\Omega))$.

Let $y^{h_j}(\cdot) = w^{h_j}(\cdot) - y^0(\cdot)$, where $w^{h_j}(\cdot) = w^{h_j}(\cdot; 0, \eta_0^{h_j}, u^{h_j}(\cdot), \eta^{h_j}(\cdot))$ is the solution of Eqn. (7) for $h = h_j$ and $y^0(\cdot)$ is the solution of

$$\begin{aligned}
&\frac{\partial}{\partial t} y^0(x,t) - \Delta y^0(x,t) + \hat R_\lambda(t, y^0(x,t)) \\
&\qquad + \frac{2}{3}\lambda y^0(x,t) + (K_\lambda y^0(\cdot))(x,t) \\
&\qquad = u_0(x,t) + f(x,t) \quad \text{in } Q_T, \\
&\partial_n y^0(x,t) = 0 \qquad \text{in } \Sigma_T, \\
&y^0(x,0) = y_0(x) \quad \text{in } \Omega.
\end{aligned} \tag{27}$$

In other words, $y^0(\cdot)$ is the state function associated with the weak limit control $u_0(\cdot)$. Subtracting (27) from (7) (in (7) we set $h = h_j$) and multiplying the obtained difference scalarly (in H) by y^{h_j}, after integration we conclude that

$$\begin{aligned}
&\left|y^{h_j}(t)\right|_H^2 \\
&\quad + 2\int_0^t \left\{ \int_\Omega \left|\nabla y^{h_j}(x,s)\right|_{\mathbb{R}^n}^2 \mathrm{d}x + \frac{2}{3}\lambda \left|y^{h_j}(s)\right|_H^2 \right\} \mathrm{d}s \\
&\quad + \int_0^t ((K_\lambda(\eta^{h_j}(\cdot) - y^0(\cdot))(s), y^{h_j}(s))\,\mathrm{d}s \\
&\le \int_0^t \left(y^{h_j}(s), u^{h_j}(s) - u_0(s)\right) \mathrm{d}s + |y_0 - \eta_0^{h_j}|_H^2.
\end{aligned} \tag{28}$$

Note that

$$|y^{h_j}(t)|_H^2 \le 2|w^{h_j}(t) - y(t)|_H^2 + 2\nu^0(t),$$

where we have set $\nu^0(t) = |y^0(t) - y(t)|_H^2$ for brevity.

Now, (19) and (20) imply

$$\begin{aligned}
&\int_0^t \big(((K_\lambda(\eta^{h_j}(\cdot) - y^0(\cdot))(s)), y^{h_j}(s)\big)\,\mathrm{d}s \\
&\le c_0\Big(\int_0^t\int_0^s \nu^0(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\Big)^{1/2}\Big(\int_0^t |y^{h_j}(s)|_H^2\,\mathrm{d}s\Big)^{1/2} \\
&\quad + \Big(\int_0^t |K_\lambda(\eta^{h_j}(\cdot) - y^0(\cdot))(s)|_H^2\,\mathrm{d}s\Big)^{1/2} \\
&\quad \times \Big(\int_0^t |y^{h_j}(s)|_H^2\,\mathrm{d}s\Big)^{1/2} \\
&\le c_1\int_0^t \nu^0(s)\,\mathrm{d}s + \int_0^t |y^{h_j}(s)|_H^2\,\mathrm{d}s \\
&\quad + [d_2(\delta(h_j) + h_j^2)]^{1/2}\Big(\int_0^t |y^{h_j}(s)|_H^2\,\mathrm{d}s\Big)^{1/2} \\
&\le \nu^{h_j}(t) + c_4\int_0^t \nu^0(s)ds + c_2(\delta(h_j) + h_j^2),
\end{aligned} \tag{29}$$

where

$$\nu^{h_j}(t) = c_3\int_0^t |y(s) - w^{h_j}(s)|_H^2\,\mathrm{d}s.$$

Moreover, we mention the simple inequality

$$\begin{aligned}\nu^0(t) &\le 2|y^{h_j}(t)|_H^2 \\ &\quad + 2|y(t) - w^{h_j}(t)|_H^2 \quad \text{for } t \in (0,T).\end{aligned} \tag{30}$$

Next, we rewrite this inequality, use the inequality (28) for y^{h_j} (notice that the second term in (28) is nonnegative), and estimate the integral term with K_λ by (29). We arrive at

$$\begin{aligned}
&\frac{1}{2}\nu^0(t) - |y(t) - w^{h_j}(t)|_H^2 \\
&\le |y^{h_j}(t)|_H^2 \le \nu^{h_j}(t) + |y_0 - \eta_0^{h_j}|_H^2 \\
&\quad + \int_0^t \big\langle w^{h_j}(s) - y(s), u^{h_j}(s) - u_0(s)\big\rangle\,\mathrm{d}s \\
&\quad + \int_0^t \big\langle y(s) - y^0(s), u^{h_j}(s) - u_0(s)\big\rangle\,\mathrm{d}s \\
&\quad + c_4\int_0^t \nu^0(s)\,\mathrm{d}s + c_2(\delta(h_j) + h_j^2).
\end{aligned} \tag{31}$$

The first term on the right-hand side of the inequality (31) tends to zero as $j \to \infty$. This follows from Theorem 1 and the inclusions $\eta^{h_j}(\cdot) \in \boldsymbol{\eta}_{T_j}^{h_j}(y(\cdot))$; notice that $w^{h_j}(t) - y(t) = \mu^{h_j}(t)$. Thanks to $u^{h_j}(\cdot) \rightharpoonup u_0(\cdot)$ (see (25)), the sequence $\{u^{h_j}(\cdot) - u_0(\cdot)\}$ is bounded. Moreover, Theorem 1 also yields that $\sup_{t\in(0,T)}|y(t) - w^{h_j}(t)|_H$ tends to zero. Therefore, also the third term converges to zero. The convergence to zero of the fourth term follows again from $u^{h_j}(\cdot) \rightharpoonup u_0(\cdot)$. Also, from (5) we deduce $|y_0 - \eta_0^{h_j}|_H^2 \to 0$ as $j \to \infty$. In view of all these convergence result, from (31) we obtain

$$\nu^0(t) \le \lim_{j\to\infty}\nu_*^{h_j}(t) + 2c_4\int_0^t \nu^0(s)\,\mathrm{d}s, \quad t \in [0,T], \tag{32}$$

where

$$\lim_{j\to\infty}\nu_*^{h_j}(t) = 0 \quad \text{as} \quad j \to \infty. \tag{33}$$

Therefore, by virtue of (32), (33) and the Gronwall lemma, we deduce that

$$\sup_{t\in T}\nu^0(t) = 0.$$

Consequently, it follows that

$$y^0(t) = y(t), \quad t \in [0,T],$$

and hence

$$u_0(\cdot) = u(\cdot).$$

We have the contradiction with (25) and (26). Thus,

$$u^{h_j}(\cdot) \rightharpoonup u(\cdot) \quad \text{in} \quad \mathrm{L}_2(0,T;H^1(\Omega)) \quad \text{as} \quad \mathrm{j} \to \infty. \tag{34}$$

In the sequel, by $|\cdot|_{L_2}$ we denote the norm of $L_2(0,T;H^1(\Omega))$. Since norms are weakly lower semicontinuous, from (34) we derive

$$\underline{\lim}_{j\to\infty}|u^{h_j}(\cdot)|_{L_2} \ge |u(\cdot)|_{L_2}. \tag{35}$$

From (15), we obtain the inequality

$$|u^{h_j}(\cdot)|_{L_2}^2 \le |u(\cdot)|_{L_2}^2 + C_3(\delta^{1/2}(h_j) + h_j)\alpha^{-1}(h_j).$$

Owing to the assumptions of the theorem, this implies

$$\overline{\lim}_{j\to\infty}|u^{h_j}(\cdot)|_{L_2} \le |u(\cdot)|_{L_2}, \tag{36}$$

and, invoking (35) and (36),

$$\overline{\lim_{j\to\infty}}|u^{h_j}(\cdot)|_{L_2} \le |u(\cdot)|_{L_2} \le \underline{\lim_{j\to\infty}}|u^{h_j}(\cdot)|_{L_2}.$$

Therefore, we have the convergence of norms

$$|u^{h_j}(\cdot)|_{L_2} \to |u(\cdot)|_{L_2} \quad \text{as} \quad j\to\infty. \tag{37}$$

In view of the weak convergence (34), we conclude that

$$u^{h_j}(\cdot) \to u(\cdot) \text{ in } L_2 \text{ as } j\to\infty.$$

The theorem is proved. ■

Under some additional conditions, we are able to derive a convergence rate of the reconstruction algorithm (see Theorem 3 below).

In what follows, we need the lemma below.

Lemma 4. (Maksimov, 2002, p. 47) *Let $u(\cdot)\in L_\infty(a,b)$ and $v(\cdot)\in BV(a,b)$, $-\infty<a<b<+\infty$, satisfy*

$$\Big|\int_a^t u(\tau)\,\mathrm{d}\tau\Big| \le \varepsilon_*,$$
$$|v(t)|\le C, \quad \forall\, t\in[a,b].$$

Then, for all $t\in[a,b]$, the inequality

$$\Big|\int_a^t u(\tau)v(\tau)\,\mathrm{d}\tau\Big| \le \varepsilon_*(C+\mathrm{var}([a,b];v(\cdot)))$$

is valid.

Here, the symbol $\mathrm{var}([a,b];v(\cdot))$ denotes the total variation of the function $v(\cdot)$ in the interval $[a,b]$, and $BV(a,b)$ is the space of all real functions $v:[a,b]\to\mathbb{R}$ of bounded variation.

Assumption 2. The set U_{ad} has the special form

$$U_{\text{ad}} = \{u\in H^1(\Omega): u = u_1\,\omega(\cdot), u_1\in[a,b]\}.$$

Here, $\omega\in H^1(\Omega)$ is a given function with $|\omega|_{H^1(\Omega)}\ne 0$, $-\infty<a<b<+\infty$. Thus the role of the control to be reconstructed is now played by the Lebesgue measurable function $u_1:[0,T]\to[a,b]$. Now the problem of reconstruction of $u(\cdot)$ is equivalent to that of finding a function $u_1(\cdot)\in L_2(0,T)$. The result of the algorithm will be a scalar function $u_1^h(\cdot)\in L_2(0,T)$. Then, the rule $\mathcal{U}_h$ (see (9)) of forming the control $u(\cdot)$ (or $u_1(\cdot)$) in the model (7) takes the form

$$\begin{aligned}\mathcal{U}_h(\eta,w^h) = \Big\{u_1^h : \langle w^h-\eta,\omega\rangle u_1^h + \alpha(h)(u_1^h)^2|\omega|_V^2\Big\}\\ \le \inf\Big\{\langle w^h-\eta,\omega\rangle u_1 + \alpha(h)(u_1)^2|\omega|_V^2 :\\ u_1\in[a,b]\Big\}+h\Big\}.\end{aligned}$$

By redefining α, we are justified to simplify the rule in the form

$$\begin{aligned}\mathcal{U}_h(\eta,w^h) = \Big\{u_1^h : \langle w^h-\eta,\omega\rangle u_1^h + \alpha(h)(u_1^h)^2\Big\}\\ \le \inf\Big\{\langle w^h-\eta,\omega\rangle u_1 + \alpha(h)(u_1)^2 :\\ u_1\in[a,b]\Big\}+h\Big\}.\end{aligned} \tag{38}$$

Notice that we allow for an inexact minimization by the appearance of the term $+h$ in (38). Thanks to the unrestricted minimization, we are able to determine the explicit solution of the exact minimization in (38). The solution is

$$u_1^h = \mathbb{P}_{[a,b]}\left((2\alpha(h))^{-1}\langle\eta-w^h,\omega\rangle\right), \tag{39}$$

where $\mathbb{P}_{[a,b]}$ means pointwise projection onto the interval $[a,b]$, namely

$$\mathbb{P}_{[a,b]}(s) = \max\{a,\min\{b,s\}\}.$$

In turn, the inequality (15) may be rewritten in the form

$$\begin{aligned}&\int_0^T (u_1^h(s))^2\,\mathrm{d}s\\ &\le \int_0^T (u_1(s))^2\,\mathrm{d}s + C_3(\delta^{1/2}(h)+h)\alpha^{-1}(h).\end{aligned} \tag{40}$$

Theorem 2 implies the convergence

$$u_1^h(\cdot)\to u_1(\cdot) \quad \text{in} \quad L_2(0,T) \text{ as } h\to 0.$$

Assumption 3. The initial value y_0 and the measurements η_i^h satisfying (4) and (5) are elements of the space $L_\infty(\Omega)$ such that

$$|\eta_i^h - y(\tau_i)|_{L_\infty(\Omega)} \le h. \tag{41}$$

Note that, for all $p\le 6$, $H^1(\Omega)$ is continuously embedded in $L_p(\Omega)$, but $H^1(\Omega)\subset L_\infty(\Omega)$ only holds for $n=1$. From the work of Casas *et al.* (2013, Lemma 2.2), we obtain the following result.

Lemma 5. *Let $p>5/2$ and Assumptions 2 and 3 be fulfilled. Then there exists a number $C_4>0$ such that the inequality*

$$|y(\cdot)|_{L_\infty(Q_T)} \le C_4(1+|y_0|_{L_\infty(\Omega)})$$

is fulfilled uniformly with respect to $y_0\in L_\infty(\Omega), u(\cdot)\in U_{\text{ad}}$.

Here, $y(\cdot) = y(\cdot; 0, y_0, u(\cdot))$ is the weak solution of Eqn. (3), and $U_{\rm ad}$ is defined in Assumption 2

The integral operator K_λ is continuous in $L_\infty(Q_T)$. Therefore, using the previous lemma, we conclude the next result.

Lemma 6. *Let the conditions of Lemma 5 and the inequalities* (41) *hold. Then there exists a number* $C_5 > 0$ *such that the estimate*

$$|w^h(\cdot)|_{L_\infty(Q_T)} \le C_5(1 + |y_0|_{L_\infty(\Omega)})$$

holds uniformly with respect to $h \in (0,1), u^h(\cdot) \in U_{ad}$, $y_0 \in L_\infty(\Omega)$, *the partitions* $\mathcal{T}_h$ *of the interval* $[0,T]$ *and the values* η_i^h *satisfying the inequalities* (41). *Here,* $w^h(\cdot) = w(\cdot; 0, \eta_0^h, u^h(\cdot), \eta^h(\cdot))$ *is the solution of Eqn.* (7).

Theorem 3. *Suppose that Assumptions* 1–3 *are satisfied. Let also* $u(t) = \omega\, u_1(t)$ *and* $u_1(\cdot) \in W(0,T)$. *Then the reconstruction algorithm has the following convergence rate:*

$$\begin{aligned} &|u_1(\cdot) - u^h(\cdot)|^2_{L_2(0,T)} \\ &\quad \le C_6\,\{h^{1/2} + \alpha^{1/2}(h) + \delta^{1/4}(h) \\ &\qquad + (h + \delta^{1/2}(h))\alpha^{-1}(h)\}, \end{aligned}$$

where C_6 *is some constant not depending on* h, *and* $|\cdot|_{L_2(0,T)}$ *is the norm of the space* $L_2(0,T)$.

Proof. Subtracting (16) from (17) and multiplying the difference scalarly in H by $v \in H^1(\Omega)$, integrating over $[0,t]$, and recalling $\mu^h = w^h - y$, we derive

$$\Big|\Big(\int_0^t \omega\,(u_1^h(\tau) - u_1(\tau))\,\mathrm{d}\tau, v\Big)\Big|$$

$$= \Big|(\omega, v)\int_0^t (u_1^h(\tau) - u_1(\tau))\,\mathrm{d}\tau\Big| \le \sum_{j=1}^5 I_t^{(j)}, \quad (42)$$

where

$$\begin{aligned} I_t^{(1)}(v) &= |(\mu^h(t), v) - (\mu^h(0), v)|, \\ I_t^{(2)}(v) &= \frac{2}{3}|\lambda|\,\Big|\Big(\int_0^t \mu^h(\tau)\,\mathrm{d}\tau, v\Big)\Big|, \\ I_t^{(3)}(v) &= \Big|\Big(\int_0^t (K_\lambda(y(\cdot) - \eta^h(\cdot))(\tau)\,\mathrm{d}\tau, v\Big)\Big|, \\ I_t^{(4)}(v) &= \Big|\int_0^t\int_\Omega (R_\lambda(\tau, w^h(x,\tau)) \\ &\qquad - R_\lambda(\tau, y(x,\tau)))v(x)\,\mathrm{d}x\,\mathrm{d}\tau\Big|, \\ I_t^{(5)}(v) &= \Big|\int_0^t\int_\Omega (\nabla\mu^h(x,\tau), \nabla v(x))_{\mathbb{R}^n}\,\mathrm{d}x\,\mathrm{d}\tau\Big|. \end{aligned}$$

The symbol $(\cdot,\cdot)_{\mathbb{R}^n}$ stands for the scalar product in the space $\mathbb{R}^n$. From Lemmas 5 and 6, we find

$$\begin{aligned} &|R_\lambda(\tau, w^h(x,\tau)) - R_\lambda(\tau, y(x,\tau))| \\ &\qquad\qquad \le \mathrm{const}|w^h(x,\tau) - y(x,\tau)|. \end{aligned}$$

Therefore,

$$\sup_{t\in(0,T)}\ \sup_{v\in H^1(\Omega), |v|_{H^1(\Omega)}\le 1} I_t^{(4)}(v) \le c_1 \int_0^t |\mu^h(\tau)|_H\,\mathrm{d}\tau. \quad (43)$$

Moreover,

$$\sup_{v\in H^1(\Omega), |v|_{H^1(\Omega)}\le 1} I_t^{(2)}(v) \le c_2 \int_0^t |\mu^h(\tau)|_H\,\mathrm{d}\tau, \quad (44)$$

$$\begin{aligned} &\sup_{v\in H^1(\Omega), |v|_{H^1(\Omega)}\le 1} I_t^{(5)}(v) \\ &\qquad \le c_3 \int_0^t \Big(\int_\Omega |\nabla\mu^h(x,\tau)|^2_{\mathbb{R}^n}\,\mathrm{d}x\Big)^{1/2}\mathrm{d}\tau. \quad (45) \end{aligned}$$

Analogously to (20), we get

$$\begin{aligned} &\sup_{v\in H^1(\Omega), |v|_{H^1(\Omega)}\le 1} I_t^{(3)}(v) \\ &\qquad \le \Big(\int_0^t |K_\lambda(y(\cdot) - \eta^h(\cdot))|^2_H\,\mathrm{d}\tau\Big)^{1/2} \quad (46) \\ &\qquad \le c_4(h + \delta^{1/2}(h)). \end{aligned}$$

Let $b_1 = |\omega|_{H^1(\Omega)}$, $b_2 = |\omega|_H$, $v_1 = \omega/b_1$. Then $|v_1|_{H^1(\Omega)} = 1$, $(\omega, v_1) = b_2^2/b_1$. Therefore,

$$\begin{aligned} &\sup_{v\in H^1(\Omega), |v|_{H^1(\Omega)}\le 1} \Big|(\omega, v)\int_0^t (u_1^h(\tau) - u_1(\tau))\,\mathrm{d}\tau\Big| \\ &\qquad \ge \Big|(\omega, v_1)\int_0^t (u_1^h(\tau) - u_1(\tau))\,\mathrm{d}\tau\Big| \quad (47) \\ &\qquad = b_2^2/b_1\Big|\int_0^t (u_1^h(\tau) - u_1(\tau))\,\mathrm{d}\tau\Big|. \end{aligned}$$

From (42), (43)–(47) and the inequalities (14), we derive

$$\begin{aligned} &\sup_{t\in(0,T)}\Big|\int_0^t (u_1^h(\tau) - u_1(\tau))\,\mathrm{d}\tau\Big| \\ &\qquad \le c_5(h^{1/2} + \alpha^{1/2}(h) + \delta^{1/4}(h)). \quad (48) \end{aligned}$$

In turn, from (40), we have

$$\begin{aligned} &|u_1^h(\cdot) - u_1(\cdot)|^2_{L_2(0,T)} \\ &\quad \le 2|u_1(\cdot)|^2_{L_2(0,T)} \\ &\qquad - 2\int_0^T u_1^h(\tau)u_1(\tau)\,\mathrm{d}\tau + C_2(h+\delta^{1/2}(h))\alpha^{-1}(h) \\ &\quad = 2\int_0^T (u_1(\tau) - u_1^h(\tau))u_1(\tau)\,\mathrm{d}\tau \\ &\qquad + C_2(h+\delta^{1/2}(h))\alpha^{-1}(h). \end{aligned} \tag{49}$$

Using (48) and Lemma 5, we get

$$\begin{aligned} |u^h(\cdot) - u(\cdot)|^2_{L_2(0,T)} &\le c_6(h^{1/2} + \alpha(h)^{1/2} + \delta^{1/4})(h) \\ &\quad + C_2(h+\delta^{1/2}(h))\alpha^{-1}(h). \end{aligned}$$

The theorem is proved. ■

4. Reconstruction of unrestricted controls

Let us now discuss a method of solving the reconstruction problem in the case of unrestricted controls.

As model $\mathcal{M}$, we consider again Eqn. (7). The rule of forming the control $u^h(\cdot)$ in the model is now given by the formula

$$\mathcal{U}_h(\eta, w) = \alpha^{-1}(h)(\eta - w),$$

with fixed $\alpha(h) > 0$. The right-hand side is the explicit solution of the problem

$$\min_{u\in H}\Big\{(w-\eta, u) + \frac{1}{2}\alpha(h)|u|_H^2\Big\};$$

cf. (38).

Let again a family of partitions of $[0,T]$ be given:

$$\begin{aligned} \mathcal{T}_h &= \{\tau_{h,i}\}_{i=0}^{m_h}, & \tau_{h,0} &= 0, \\ \tau_{h,m_h} &= T, & \tau_{h,i+1} &= \tau_{h,i} + \delta(h), \end{aligned} \tag{50}$$

and the function $\alpha(h) : (0,1) \to (0,1)$ be fixed.

The solution algorithm is analogous to the one for restricted inputs with one exception: the mapping $\mathcal{U}_h$ is now defined by

$$\mathcal{U}_h(\eta_i^h, w^h(\tau_i)) = \alpha^{-1}(h)(\eta_i^h - w^h(\tau_i)). \tag{51}$$

The proof of the following result is analogous to that of Lemma 2.

Lemma 7. *Let $|u^h(\cdot)|_{L_p(Q_T)} \le M_0$ hold for all $h \in (0,1)$. Then there exists a number $C_7 = C_7(y(\cdot), M_0) > 0$ such that the inequality*

$$\begin{aligned} &\operatorname{vrai}\sup_{t\in(0,T)} |w^h(t)|_H^2 + \int_0^T |\dot{w}^h(t)|_H^2\,\mathrm{d}t \\ &\qquad \le C_7\Big\{|y_0|_V^2 + \int_0^T \{|u^h(t)|_H^2 + |f(t)|_H^2\}\,\mathrm{d}t + 1\Big\} \end{aligned}$$

holds for all partitions $\mathcal{T}$ of the interval $[0,T]$ with diameter $\delta = \delta(h) \in (0,1)$ and all functions $\eta^h(\cdot) \in \boldsymbol{\eta}^h_{\mathcal{T}}(y(\cdot)), h \in (0,1)$.

Note that for every fixed h the solution $w^h(\cdot)$ of Eqn. (7) is bounded in $L_p(Q_T)$.

Moreover, we need the following lemma.

Lemma 8. (Discrete Gronwall inequality) (cf. Samarskii, 1971) *Let constants $c_0 > 0$ and $\kappa > 0$ be given. Assume that numbers $0 \le \phi_j$, $0 \le f_j$, $j \in [0 : m]$, are given with $\phi_1 \le f_0$ and $f_j \le f_{j+1}$, $j \in [0 : m-1]$. Then the inequalities*

$$\phi_{j+1} \le c_0\kappa\sum_{i=1}^{j}\phi_i + f_j, \quad j \in [1 : m-1],$$

imply

$$\phi_{j+1} \le f_j\,\exp(c_0 j\kappa), \quad j \in [0 : m-1].$$

Assumption 4. The functions $h \mapsto \delta(h) \in (0,1)$ and $h \mapsto \alpha(h) \in (0,1)$ are such that $\alpha(h) \to 0$, $\delta(h) \to 0$,

$$h\alpha^{-1}(h) \to 0, \tag{52}$$

$$h^2\delta^{-1}(h)\alpha^{-1}(h) \to 0, \tag{53}$$

$$\delta(h)\alpha^{-2}(h) \to 0 \tag{54}$$

as $h \to +0$.

Theorem 4. *Let Assumption 4 be satisfied and let there exist $h_* \in (0,1)$, $M > 0$ such that the inequality $|u^h(\cdot)|_{L_p(Q_T)} \le M$ holds for all $h \in (0,h_*)$ and all admissible measurements $\eta^h(\cdot) \in \boldsymbol{\eta}^h_{\mathcal{T}_h}(y(\cdot))$. Then, uniformly with respect to $h \in (0,1)$, the inequalities*

$$\varepsilon_h(t) \le C_8\alpha(h), \quad t \in [0,T], \tag{55}$$

$$\int_0^T |u^h(t)|_H^2\,\mathrm{d}t \le \int_0^T |u(t)|_H^2\,\mathrm{d}t + C_9\rho(h)\alpha^{-1}(h) \tag{56}$$

are fulfilled. Here, $C_8 > 0$ and $C_9 > 0$ are constants that do not depend on h, $\alpha(h)$ and $\delta(h)$, the function $\varepsilon_h(\cdot)$ is defined as in Theorem 1, and

$$\rho(h) = h^2\delta^{-1}(h) + \delta^{1/2}(h) + h + \delta(h)\alpha^{-1}(h).$$

Proof. We follow the proof of Theorem 1 and confirm the inequality (18). It is easily seen that the inequality

$$(\mu^h(t), u^h(t) - u(t)) \le (u^h(t) - u(t), w^h(\tau_i) - \eta_i^h) + \varrho_i(t,h)$$

is fulfilled for a.a. $t \in J_i$, where

$$\varrho_i(t,h) = c_1(|u^h(t)|_H + |u(t)|_H)(h + \int_{\tau_i}^{t} \{|\dot w^h(\tau)|_H + |\dot y(\tau)|_H\}\, \mathrm{d}\tau).$$

Therefore, for a.a. $t \in J_i$, the inequality

$$\frac{1}{2}\frac{\mathrm{d}\varepsilon_h(t)}{\mathrm{d}t} \le (u^h(t) - u(t), w^h(\tau_i) - \eta_i^h) + \phi_t + \varrho_i(t,h) \tag{57}$$

holds, where

$$\phi_t = \left|(K_\lambda(y(\cdot) - \eta^h(\cdot)))(t), \mu^h(t))\right|.$$

Consequently, the relation (57) implies the inequality

$$\begin{aligned}\frac{\mathrm{d}\varepsilon_h(t)}{\mathrm{d}t} &+ \alpha(h)\{|u^h(t)|_H^2 - |u(t)|_H^2\} \\ &\le 2(u^h(t), w^h(\tau_i) - \eta_i^h) + \alpha(h)|u^h(t)|_H^2 \\ &\quad - 2(u(t), w^h(\tau_i) - \eta_i^h) - \alpha(h)|u(t)|_H^2 \\ &\quad + 2\varrho_i(t,h) + 2\phi_t \quad \text{for a.a.} \quad t \in J_i.\end{aligned} \tag{58}$$

Let

$$\varepsilon^h(t) = \varepsilon_h(t) + \alpha(h)\int_0^t \{|u^h(\tau)|_H^2 - |u(\tau)|_H^2\}\, \mathrm{d}\tau.$$

Therefore, by the rule (51) of forming the control $u^h(\cdot)$, we conclude from (58) that, for $t \in J_i, i \in [0 : m-1]$,

$$\begin{aligned}\varepsilon^h(t) &\le \varepsilon^h(\tau_i) + 2c_1\int_{\tau_i}^t \{|u^h(\tau)|_H + |u(\tau)|_H\}\, \mathrm{d}\tau \\ &\quad \times \Big(h + \int_{\tau_i}^t \{|\dot w^h(\tau)|_H + |\dot y(\tau)|_H\}\, \mathrm{d}\tau\Big) \\ &\quad + 2\int_{\tau_i}^t \phi_\tau\, \mathrm{d}\tau \\ &\le \varepsilon^h(\tau_i) + c_2h^2 \\ &\quad + c_3\delta(h)\int_{\tau_i}^t \{|u^h(\tau)|_H^2 + |u(\tau)|_H^2\}\, \mathrm{d}\tau \\ &\quad + c_4\delta(h)\int_{\tau_i}^t \{|\dot w^h(\tau)|_H^2 + |\dot y(\tau)|_H^2\}\, \mathrm{d}\tau \\ &\quad + 2\int_{\tau_i}^t \phi_\tau\, \mathrm{d}\tau.\end{aligned} \tag{59}$$

Summing the right-hand and left-hand parts of (59) over i and taking into account Lemmas 2 and 7, for $t \in (0,T)$, we obtain

$$\begin{aligned}\varepsilon^h(t) &\le \varepsilon^h(0) + c_5h^2\delta(h)^{-1} \\ &\quad + c_6\delta(h)\Big\{1 + \int_0^t \{|u^h(\tau)|_H^2 + |u(\tau)|_H^2\Big\}\, \mathrm{d}\tau\} \\ &\quad + 2\int_0^t \phi_\tau\, \mathrm{d}\tau.\end{aligned} \tag{60}$$

Now, we use the inclusion $u(\cdot) \in L_2(0,T;H)$ and the relation

$$\begin{aligned}\int_0^t &|u^h(\tau)|_H^2\, \mathrm{d}\tau \\ &= \sum_{j=0}^{i_h(t)-1}\int_{\tau_j}^{\tau_{j+1}} |u^h(\tau)|_H^2\, \mathrm{d}\tau + \int_{\tau_{i_h(t)}}^t |u^h(\tau)|_H^2\, \mathrm{d}\tau \\ &\le \delta\sum_{j=0}^{i_h(t)} |u_j^h|_H^2.\end{aligned}$$

Here the symbol $i_h(t)$ stands for the integer part of the value $t\delta^{-1}(h)$.

By virtue of (60), we deduce that

$$\begin{aligned}\varepsilon^h(t) &\le \varepsilon^h(0) + c_7h^2\delta(h)^{-1} + c_8\delta(h) \\ &\quad + c_9\gamma_{h,\delta(h)}(t) + 2\int_0^t \phi_\tau\, \mathrm{d}\tau,\end{aligned} \tag{61}$$

where

$$\gamma_{h,\delta(h)}(t) = \delta(h)^2\sum_{j=0}^{i_h(t)} |u_j^h|_H^2.$$

Note that, in view of (5), we have

$$\varepsilon^h(0) \le c_{10}h^2. \tag{62}$$

Moreover, from (20), it follows that

$$\int_0^T \phi_\tau \,\mathrm{d}\tau \le c_{11}(h + \delta^{1/2}(h)). \tag{63}$$

Thus, from (61)–(63), we derive, for $\delta \in (0,1)$,

$$\begin{aligned}\varepsilon_h(t) \le c_{12}(&\delta^{1/2}(h) + h^2\delta^{-1}(h)\\ &+ h + \alpha(h) + \gamma_{h,\delta(h)}(t)).\end{aligned} \tag{64}$$

By (4), (5) and the rule (51) of forming u_i^h, we deduce that

$$|u_i^h|_H^2 \le c_{13}(\varrho_i^h + h^2)\alpha^{-2}(h), \tag{65}$$

where $\varrho_i^h = |y(\tau_i) - w^h(\tau_i)|_H^2$. In addition, we have

$$\varrho_i^h \le \varepsilon_i, \tag{66}$$

where $\varepsilon_i = \varepsilon_h(\tau_i)$. For $t \in [\tau_i, \tau_{i+1}]$, using (65) and (66), we conclude that

$$\gamma_{h,\delta(h)}(t) \le c_{13}\delta^2(h) \sum_{j=0}^{i_h(t)} (\varepsilon_j + h^2)\alpha(h)^{-2}.$$

Therefore,

$$\gamma_{h,\delta(h)}(\tau_i) \le c_{13}\delta^2(h) \sum_{j=0}^{i} (\varepsilon_j + h^2)\alpha^{-2}(h). \tag{67}$$

From (64) and (67), we deduce

$$\begin{aligned}\varepsilon_{i+1} \le\ & c_{12}(\delta^{1/2}(h) + h^2\delta^{-1}(h) + h + \alpha(h))\\ &+ c_{14}\delta(h)h^2\alpha^{-2}(h)\\ &+ c_{14}\delta^2(h)\alpha^{-2}(h)\sum_{j=0}^{i}\varepsilon_j.\end{aligned} \tag{68}$$

In view of Lemma 8, from (68) we get

$$\begin{aligned}\varepsilon_i \le\ & c_{15}(\alpha(h) + \delta^{1/2}(h) + h^2\delta^{-1}(h) + h\\ &+ \delta(h)h^2\alpha^{-2}(h))\exp\{c_{14}T\delta(h)\alpha^{-2}(h)\}.\end{aligned} \tag{69}$$

From the relations

$$h^2\delta^{-1}(h)\alpha^{-1}(h) \le \text{const},$$

$$\delta\alpha^{-2}(h) \le \text{const} \quad \text{as } h \to 0$$

(see Assumption 4) and (69), we derive

$$\varepsilon_i \le c_{16}\alpha(h), \quad i \in [0:m].$$

This inequality and (67) imply the estimate

$$\begin{aligned}\gamma_{h,\delta(h)}(\tau_i) &\le c_{17}(\delta(h)\alpha^{-1}(h) + \delta(h)h^2\alpha^{-2}(h))\\ &\le c_{18}\delta(h)\alpha^{-1}(h),\end{aligned} \tag{70}$$

$$i \in [0:m], \quad h \in (0,1), \quad \delta(h) \in (0,1).$$

The function $t \to \gamma_{h,\delta(h)}(t)$ is nondecreasing; therefore, using (64) and (70), we deduce that

$$\begin{aligned}\varepsilon_h(t) &\le c_{12}(\delta^{1/2}(h) + h^2\delta^{-1}(h)\\ &\qquad + h + \alpha(h) + \gamma_{h,\delta(h)}(T))\\ &\le c_{19}(\alpha(h) + \delta(h)\alpha^{-1}(h))\\ &\le c_{20}\alpha(h), \quad t \in [0,T].\end{aligned}$$

The estimate (55) follows from these inequalities. Let us prove (56). From (61), by virtue of (62), we obtain the inequality

$$\begin{aligned}\alpha(h)\int_0^T |u^h(t)|_H^2\,\mathrm{d}t \le\ & \alpha(h)\int_0^T |u(t)|_H^2\,\mathrm{d}t\\ &+ c_{21}(h^2\delta^{-1}(h) + \delta(h))\\ &+ c_9\gamma_{h,\delta(h)}(T) + 2\int_0^T \phi_\tau\,\mathrm{d}\tau,\end{aligned}$$

which, by means of (63) and (70), implies

$$\alpha(h)\int_0^T |u^h(t)|_H^2\,\mathrm{d}t \le \alpha(h)\int_0^T |u(t)|_H^2\,\mathrm{d}t + c_{22}\rho(h).$$

The relation (56) follows from this inequality. The theorem is proved. ∎

The proof of the next theorem is analogous to that of Theorem 2.

Theorem 5. *Let the conditions of Theorem 4 hold. Assume that the sequence $Z = \{u^{h_j}(\cdot)\}$ of controls of the form* (51) *with $h = h_j$ converges weakly in $L_2(0,T;H)$ to a limit $u_Z(\cdot)$ that generates a unique associated solution $y(\cdot;0,y_0,u_Z(\cdot))$ of Eqn.* (3). *Then we have*

$$u^h(\cdot) \to u_Z(\cdot) \quad \text{in } L_2(0,T;H) \text{ as } h \to 0.$$

Theorem 4 implies the following.

Corollary 1. *Let the assumptions of Theorem* 4 *hold. Let also $\delta(h) = h$, $\alpha(h) = h^{1/2-\nu}$, and fix some number $\nu \in (0,1/2)$. Then the inequalities*

$$\varepsilon_h(t) \le C_{10}h^{1/2-\nu}, \quad t \in (0,T),$$

$$\int_0^T |u^h(t)|_H^2\,\mathrm{d}t \le \int_0^T |u(t)|_H^2\,\mathrm{d}t + C_{11}h^\nu$$

are fulfilled. Here, C_{10} and $C_{11} > 0$ are constants independent of $h \in (0,1)$ and $\nu \in (0,1/2)$.

Remark 2. Comparing the formulas (9) and (51), we conclude that the reconstruction algorithm described in this section (in essence, the algorithm for forming the control in model (7)) is simpler than that described in Section 3. However, its convergence requires the additional correlation condition (53) on the parameters: $h^2/(\delta(h)\alpha(h)) \to 0$ as $h \to 0$. Moreover, this algorithm guarantees the convergence of the functions $u^h(\cdot)$ to the real control in the norm of the space $L_2(0,T;L_2(\Omega))$, whereas the algorithm from Section 3 provides the convergence in the stronger norm of the space $L_2(0,T;H^1(\Omega))$.

5. Numerical examples

In this section, we present two numerical examples. For simplicity, we consider the case of the pure Schlögl model. For this purpose, in both examples we set $a = 0$ so that the integral operator K_λ vanishes.

Example 1. The given quantities in Eqn. (3) are as follows:

$$\Omega = [0,1], \quad T = 2, \quad y_1 = 0.1, \quad y_2 = 0.2,$$
$$y_3 = 0.3, \quad k = 1, \quad \lambda = 0, \quad f(t,x) = 0.$$

As the initial state of (3), we take the function $y_0(x) = 0.25\,(1-x)$, $x \in \Omega$. We assume

$$U_{\text{ad}} = \{u \in H^1(\Omega) : u(\cdot) = u_1\,\omega(\cdot),\ u_1 \in [-2,2]\};$$

cf. Assumption 2 The input in the right-hand part of Eqn. (3) is $u(x,t) = u_1(t)\,\omega(x)$ with $\omega(x) = 0.5\,x(1-x)$ for $x \in \Omega$, and

$$u_1(t) = \begin{cases} 1, & t \in [0,1/4), \\ \sin t, & t \in [1/4,2]. \end{cases}$$

This function u plays the role of the unknown exact control generating the unknown state function y. According to the rule (39), the control $u^h(x,t) = u_1^h(t)\omega(x)$ in the right-hand part of Eqn. (7) is calculated by the formulas

$$u^h(x,t) = u_{1i}^h\,\omega(x) \quad \text{for } x \in \Omega,\ t \in [\tau_i, \tau_{i+1}),$$

$$u_{1,i}^h = \arg\min\{\big(w^h(\tau_i) - \eta_i^h, \omega\big)_{L_2(0,1)}\ u_1 + \alpha(h)u_1^2 : u_1 \in [-2,2]\},$$

i.e.,

$$u_{1,i}^h = \mathbb{P}_{[-2,2]}\left\{(2\alpha(h))^{-1}(\eta_i^h - w^h(\tau_i), \omega)_{L_2(0,1)}\right\}.$$

Equations (3) and (7) are solved by a finite difference method with mesh size Δx in the domain Ω and time step δ in the time interval $[0,T]$. The associated results are presented in Figs. 2–4 for the following mesh sizes: $\Delta x = 1/200$, $\delta = 1/1000$. The regularization parameter α is fixed by $\alpha = 0.00005$, hence independent of h. In the numerical tests, we assume

$$\eta_i^h = \eta^h(\tau_i, x_j) = y(\tau_i, x_j) + h,$$

where $x_j = j\Delta x$, $j = 0,\ldots,1/\Delta x$. The results for $h = 0.3$, $h = 0.1$, and $h = 0.01$ are displayed in Figs. 2–4. The solid line represents the reconstructed function $u_1^h(\cdot)$, the dashed line shows the exact control function $u_1(\cdot)$. For $h = 0.01$, the corresponding curves show a fairly good coincidence. The norms $d(h) = \|u_1^h - u\|_{L_2(0,T)}$ are as follows: $d(h) = 0.1766$, 0.10363, 0.10255, 0.10224, 0.10185 for $h = 0.3$, 0.2, 0.1, 0.05, 0.01, respectively. This fits the estimate of Theorem 3: since $\alpha = 0.00005$ is very small, there should approximately hold

$$d^2(h) \le c_6(h + \sqrt{\delta})/\alpha = c_6\,(20000\,h + 632).$$

This is indeed satisfied, but $d^2(h)$ becomes essentially smaller than the right-hand side for decreasing h. The estimate of Theorem 3 appears to be too pessimistic. ♦

Example 2. We can present this example here by courtesy of Minh Huyen Ly Le (Technische Universität Berlin); it is adopted from her MSc thesis (Le, 2019, pp. 64–67), and allows for the more general ansatz

$$u^h(x,t) = \sum_{i=1}^{m} u_i^h(t)\omega_i(x), \tag{71}$$

where the functions $\omega_i : \Omega \to \mathbb{R}$, $i = 1,\ldots m$, are standard piecewise linear and continuous finite element functions (“hat functions”) associated with a uniform partition $0 = x_1 < \cdots < x_m = 20$ of $\Omega = (0,20)$. The functions ω_i satisfy $\omega_i(x_j) = \delta_{ij}$. Though the ansatz (71) is not covered by our theory, the example shows that our method of dynamical reconstruction works also for this more general setting.

Here, the given quantities in Eqn. (3) are

$$\Omega = (0,20), \quad T = 5, \quad y_1 = -2.5,$$
$$y_2 = 0, \quad y_3 = \sqrt{3}, \quad k = 1/3,$$
$$\lambda = 0, \quad f(t,x) = 0.$$

As the initial state of (3), we take the function $y_0 = 1.2\sqrt{3}\,\chi_{[8,12]}$. We impose the restrictions

$$-2 \le u_i^h(t) \le 2, \quad i = 1,\ldots,m,$$

on the controls to be reconstructed; cf. Assumption 2.

The example is related to Example 5.2 of Buchholz *et al.* (2013), where a control is found to achieve a desired

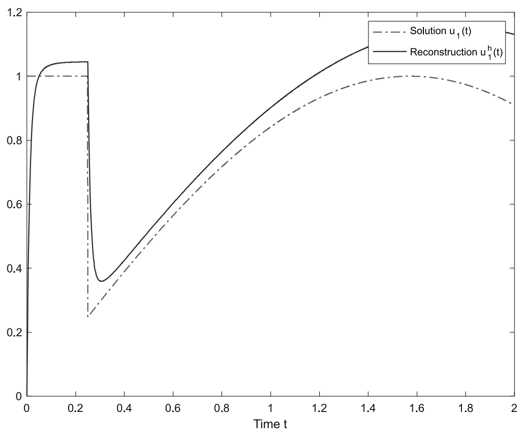

Fig. 2. Example 1: reconstruction for $h = 0.3$.

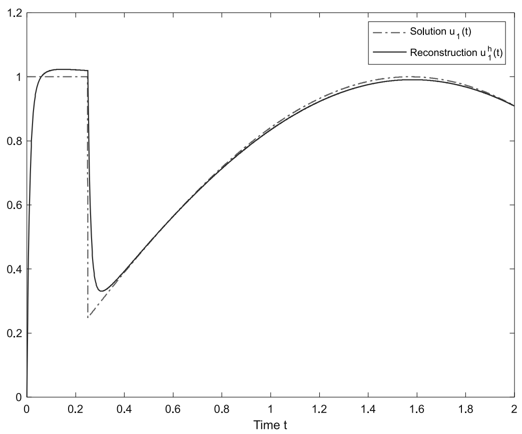

Fig. 3. Example 1: reconstruction for $h = 0.1$.

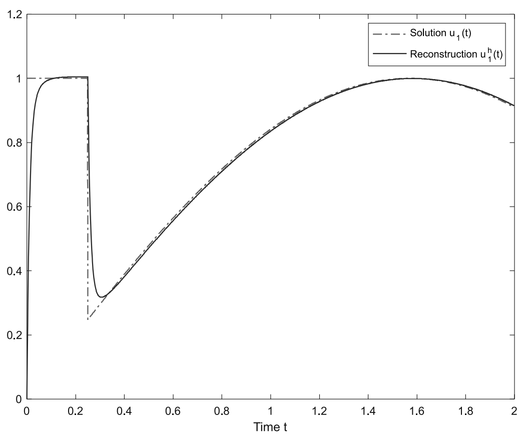

Fig. 4. Example 1: reconstruction for $h = 0.01$.

solution y_Q. For $0 \le t \le 2.5$, y_Q is the solution of the uncontrolled Schlögl model, i.e., for the control $u = 0$ with initial data $y_0 = 1.2\sqrt{3}\,\chi_{[8,12]}$ that is denoted by y_{nat}. From $2.5 < t \le 5$, it is shifted to the left,

$$y_Q(x,t) = \begin{cases} y_{\text{nat}}(x,t), & t \in [0, 2.5] \\ y_{\text{nat}}(x + 1.62\,(t-2.5), 2.5), & t \in (2.5, 5]; \end{cases}$$

cf. Fig. 5.1 of Buchholz *et al*. (2013). In the work of Buchholz *et al*. (2013), this control was determined by an optimization technique. Here, the associated control was found by the application of our reconstruction method to the given perturbed state observation $y_Q + \tilde{h}$ with $\tilde{h} = 0.0001$ for $\tilde{\alpha} = 0.1$ and $m = 50$. This control, denoted by u, is taken as a reference control; see Fig. 6. The associated reference state is denoted by y; see Fig. 5.

With our method, u^h was reconstructed in the form (71) for different values of h and m. The approximation of the associated functions $u_i^h(t)$ is displayed in Fig. 7 for $\alpha = 0.005$ and $h = 0.001$. Graphically, the associated function u^h is hard to be distinguished from the control u in Fig. 6. For further details, we refer to the thesis of Le (2019). ♦

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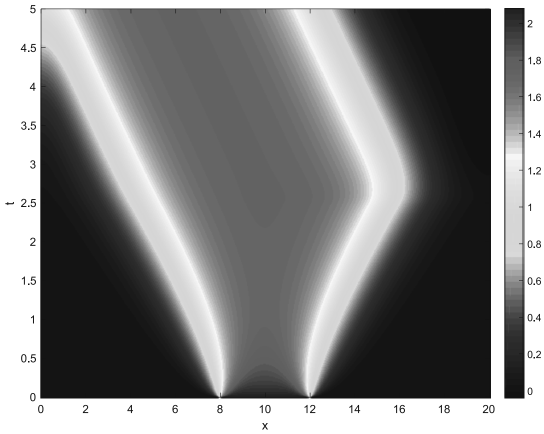

Fig. 5. Example 2: reference state y.

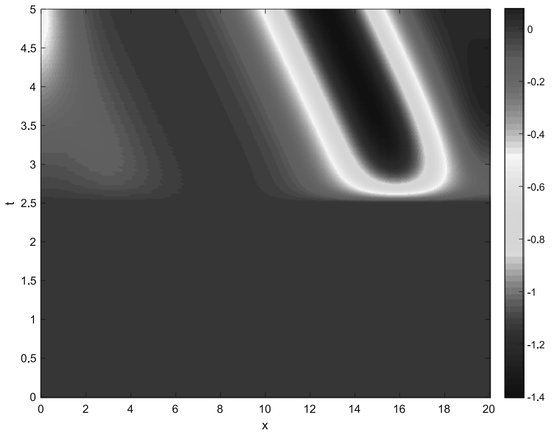

Fig. 6. Example 2: reference control u.

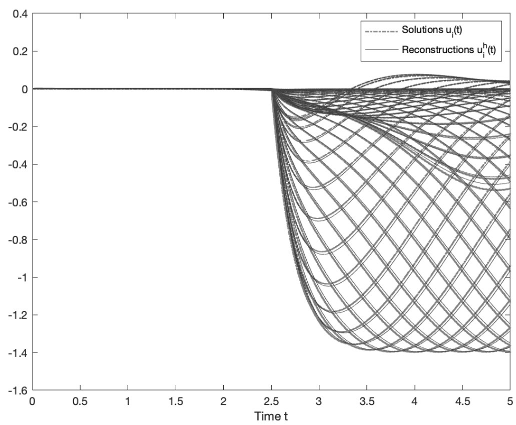

Fig. 7. Example 2: reconstructed controls $u_i^h(t)$, $i = 1, \ldots, 50$.

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