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Necessary optimality conditions for robust nonsmooth multiobjective optimization problems*

by

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Abstract: This paper deals with a robust multiobjective optimization problem involving nonsmooth/nonconvex real-valued functions. Under an appropriate constraint qualification, we establish necessary optimality conditions for weakly robust efficient solutions of the considered problem. These optimality conditions are presented in terms of Karush-Kuhn-Tucker multipliers and convexificators of the related functions. Examples illustrating our findings are also given.

Keywords: convexificator; directional constraint qualification; efficient solution; optimality conditions; robust multiobjective optimization.

1. Introduction

In theoretical multiobjective optimization, the spotlight is placed on finding the global efficient solutions, representing the best possible objective values. Fortunately, from a practical point of view, one may not always be interested in finding the so-called global best solutions, in particular when these solutions are quite sensitive to the variable perturbations, which is generally often the case. In such situation, one would be attracted by finding robust solutions, which are not very sensitive to small perturbations in variables.

In order to find the sets of optimal policies that maintain feasibility (in the optimization sense) under a variety of operating conditions, researchers used to focus on the worst-case assumptions regarding uncertainty (see Ben-Tal and Nemirovski, 2000, or Bertsimas and Sim, 2004), named the pessimistic viewpoint, where the decision maker is powerless regarding the uncertainty, with no

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resources available to combat or mitigate unfavorable realizations of uncertain parameters. During the last decades, the optimistic viewpoint using best-case uncertainty has proven also to be a suitable approach for reducing the conservatism of the pessimistic viewpoint in intuitive ways that encode economically realistic modeling assumptions, such as the availability of resources that can be used to combat or mitigate uncertainty.

Robust optimization is an important technique for investigating optimization problems with uncertainties. It was introduced as a method to integrate uncertainties in mathematical programming models; the basic idea behind is to protect the solutions against worst-case realizations of the uncertain parameters. Recently, extensive work on theoretical and applied aspects in the area of robust optimization has been carried out (see Bertsimas and Sim, 2006; Bokrantz and Fredricksson, 2017; Chen, Köbis and Yao, 2019; Chuong, 2016; Craven and Islam, 2017; Jeyakumar, Li and Lee, 2012; Kuroiwa and Lee, 2012; Lee and Son, 2014; Saadati and Oveisha, 2021). For example, Chuong (2016) gave necessary and sufficient optimality conditions for robust (weakly) Pareto solutions of a robust nonsmooth multiobjective optimization problem in terms of multipliers and limiting subdifferentials of the related functions. Bokrantz and Fredriksson (2017) presented necessary and sufficient conditions for robust efficiency regarding multiobjective optimization problems that depend on uncertain parameters by using a scalarization method. Using a generalized alternative theorem, assuming concavity of the constraint-functions, Chen, Köbis and Yao (2019) established necessary optimality conditions for weakly robust efficient solutions and properly robust efficient solutions.

In this paper, we consider an uncertain multiobjective optimization problem of the form:

$$(UCP) : \min \ f(x) = (f_1(x),\ ...,\ f_k(x)) \qquad \text{subject to } g_i(x, v_i) \le 0,\ \forall i \in I,$$

where $f_s : \mathbb{R}^n \to \mathbb{R},\ s \in S := \{1,\ ...,\ k\}$, and $g_i : \mathbb{R}^n \times \mathbb{R}^{n_i} \to \mathbb{R},\ i \in I := \{1,\ ...,\ m\}$, are given continuous functions, $x \in \mathbb{R}^n$ is the vector of decision variables, $v_i \in V_i,\ i \in I := \{1, 2, \ldots, m\}$, are uncertain parameters, $V_i,\ i \in I$, is a nonempty, convex and compact subset of $\mathbb{R}^{n_i}$ and $k,\ n,\ m,\ n_i \in \mathbb{N}$. Here we suppose that we do not know the exact values of $v_i,\ i \in I$, but know that $v_i,\ i \in I$, belongs to some uncertainty sets $V_i,\ i \in I$.

The robust counterpart of (UCP) is given by

$$(RCP) : \min \ f(x) \ \text{ subject to } x \in F$$

where

$$F := \{x \in \mathbb{R}^n : g_i(x, v_i) \le 0,\ \forall v_i \in V_i,\ \forall i \in I\}$$

is the set of all robust feasible solutions of (RCP). A vector $\overline{x}$ is called a robust feasible solution of (UCP) if it is a feasible solution of (RCP). A vector $\overline{x} \in F$

is called a weakly robust efficient solution of (UCP) (see Chen, Köbis and Yao, 2019) if and only if

$$f(x) - f(\overline{x}) \notin -int\mathbb{R}^k_+, \ \forall x \in F.$$

This paper is devoted to investigation of the necessary optimality conditions for weakly robust efficient solutions of problem (UCP), formulated with nondifferentiable/nonconvex functions. Without having recourse to the concavity of $g_i(x,.)$ at the reference point x, our approach consists of using a scalarization technique together with a directional constraint qualification (see Kabagani and Soleimani-damaneh, 2017), corresponding to the robust feasible solutions set F. These optimality conditions are presented in terms of Karush-Kuhn-Tucker multipliers and convexificators of the related functions. It should be emphasized that convexificators (Jeyakumar and Luc, 1999) are closed sets that are not necessarily bounded or convex; they were recently introduced to unify the existing results and to refine others in non-smooth analysis and optimization (see Babahadda and Gadhi, 2006; Demianov and Jeyakumar, 1997; Dutta and Chandra, 2002, 2004; Jeyakumar and Luc, 1999; Li and Zhang, 2006). To the best of our knowledge, there is no work that has been published dealing with optimality conditions using convexificators in the above-defined optimization problems involving nonsmooth/nonconvex functions. It is worth pointing out that for a locally Lipschitz function, the most known subdifferentials such as the subdifferential of Clarke are convexificators (see Jeyakumar and Luc, 1999, and the references therein).

The rest of the paper is organized as follows. Section 2 contains some basic definitions and some auxiliary results. In Section 3, we establish necessary optimality conditions for weakly robust efficient solutions of problem (UCP).

2. Preliminaries

Throughout this paper, $\mathbb{R}^n$ is the usual n-dimensional Euclidean space. We denote by $\langle .,. \rangle$ and $\mathbb{R}^n_+$ the inner product and the non-negative orthant of $\mathbb{R}^n$, defined by

$$\mathbb{R}^n_+ = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_i \geq 0\}.$$

For a subset Ω of $\mathbb{R}^n$, the sets $int\ \Omega$, $cl\ \Omega$ and $conv\ \Omega$ stand for the interior of Ω, the closure of Ω and the convex hull of Ω, respectively. The negative polar cone of Ω is defined by

$$\Omega^\circ := \{v \in \mathbb{R}^n : \langle v, x \rangle \leq 0, \ \forall x \in \Omega\}.$$

A function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is said to be locally Lipschitzian around $x \in domf := \{z \in \mathbb{R}^n \mid f(z) \in \mathbb{R}\}$ if there exist a neighbourhood $U \subseteq domf$ of x and $k \geq 0$ such that

$$|f(z) - f(y)| \leq k\|z - y\| \ \forall z, y \in U,$$

where $\|.\|$ denotes the Euclidean norm in $\mathbb{R}^n$. The expressions

$$f_d^-(x,v) = \liminf_{t \searrow 0} \left[f(x+tv) - f(x) \right]/t$$

and

$$f_d^+(x,v) = \limsup_{t \searrow 0} \left[f(x+tv) - f(x) \right]/t$$

signify, respectively, the lower and upper Dini directional derivatives of f at x in the direction v. In the case of $f_d^-(x,v) = f_d^+(x,v)$, their common value is denoted by $f'(x,v)$, which is called Dini derivative of f at x in the direction v. The function f is called Dini differentiable at x iff its Dini derivative at x exists in all directions.

Definition 1 (Jeyakumar and Luc, 1999; Dutta and Chandra, 2002) *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a given function and take $x \in \mathbb{R}^n$ at which f is finite.*

- *f is said to have an upper convexificator (UCF) $\partial^* f(x)$ at x if $\partial^* f(x) \subset \mathbb{R}^n$ is closed and for each $v \in \mathbb{R}^n$,*

$$f_d^-(x,v) \leq \sup_{x^* \in \partial^* f(x)} \langle x^*, v \rangle ;$$

- *f is said to have a lower convexificator (LCF) $\partial_* f(x)$ at x if $\partial_* f(x) \subset \mathbb{R}^n$ is closed and for each $v \in \mathbb{R}^n$,*

$$f_d^+(x,v) \geq \inf_{x^* \in \partial_* f(x)} \langle x^*, v \rangle .$$

A closed set $\partial^* f(x) \subset \mathbb{R}^n$ is said to be a convexificator of f at x if it is an upper and lower convexificator of f at x.

- *f is said to have an upper semi-regular convexificator (USRCF) $\partial^* f(x)$ at x if $\partial^* f(x)$ is a closed set and for each $v \in \mathbb{R}^n$,*

$$f_d^+(x,v) \leq \sup_{x^* \in \partial^* f(x)} \langle x^*, v \rangle .$$

Remark 1 *For locally Lipschitz functions, one may find upper semi-regular convexificators, which are smaller than the Clarke subdifferential (Clarke, 1983) and the Mordukhovich subdifferential (Mordukhovich and Shao, 1995), as Example 1 shows. In addition, an upper semi-regular convexificator may just contain only a finite number of elements.*

Example 1 Take $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$f(x,y) = 3|x| - 2|y|, \ \forall (x,y) \in \mathbb{R}^2.$$

- The function f admits

$$\partial^* f(0,0) := \{(3,-2), (-3,2)\}$$

as an upper semi-regular convexificator at $(0,0)$; whereas the Mordukhovich subdifferential of f at $(0,0)$ and the Clarke subdifferential of f at $(0,0)$ are, respectively, the sets

$$\partial^m f(0,0) = \left\{(t,2) \in \mathbb{R}^2 : -3 \leq t \leq 3\right\} \cup \left\{(t,-2) \in \mathbb{R}^2 : -3 \leq t \leq 3\right\}$$

and

$$\partial^c f\,(0,0) = conv\ \{(3,-2)\,,(-3,2)\,,(3,2)\,,(-3,-2)\}\,.$$

- Observe that the upper semi-regular convexificator $\partial^* f(0,0)$ is strictly included in the Mordukhovich subdifferential $\partial^m f(0,0)$. More than that, the convex hull of $\partial^* f(0,0)$ is a proper subset of both $\partial^c f(0,0)$ and $conv\ \partial^m f(0,0)$.

REMARK 2 *Example 1 shows that the necessary optimality conditions that are expressed in terms of USRCFs may provide sharp conditions even for locally Lipschitz functions.*

In the following result, we give an upper semi-regular convexificator for a max function.

PROPOSITION 1 *Let $f_s : \mathbb{R}^n \to \mathbb{R}$, $s \in S$, be continuous given functions. Let*

$$h\,(x) := \max\,(f_1\,(x)\,,\ ...,\ f_k\,(x)) \ \text{ and } \mathcal{S}\,(\overline{x}) := \{s \in S \text{ such that } h\,(\overline{x}) = f_s\,(\overline{x})\}\,.$$

Suppose that for each $s \in \mathcal{S}\,(\overline{x})$, the function f_s admits an upper semi-regular convexificator $\partial^* f_s\,(\overline{x})$ at $\overline{x}$. Then, $\Lambda\,(\overline{x}) := \bigcup_{s\in\mathcal{S}(\overline{x})} \partial^* f_s\,(\overline{x})$ is an upper semi-regular convexificator of h at $\overline{x}$.

PROOF Let $v \in \mathbb{R}^n$ and let $(t_n)_{n\in\mathbb{N}}$ be a positive sequence such that $\lim_{n\mapsto+\infty} t_n = 0$. Since $f_s : \mathbb{R}^n \to \mathbb{R}$, $s \in S$, are continuous function, for n large enough, we have

$$\mathcal{S}\,(\overline{x} + t_n v) \subseteq \mathcal{S}\,(\overline{x})\,.$$

Notice that h is continuous at $\overline{x}$ (Apostol, 2004, Theorem 4.20) and that

$$h^+\,(\overline{x}, v) = \lim_{n\to+\infty} \frac{h\,(\overline{x} + t_n v) - h\,(\overline{x})}{t_n}.$$

Since $\mathcal{S}\,(\overline{x})$ is finite, we can find an index $s \in S$ and a subsequence $(t_{s_n})_n$ such that

$$h\,(\overline{x} + t_{s_n} v) = f_i\,(\overline{x} + t_{s_n} v)\,.$$

Since

$$h_d^+ (\overline{x}, v) = \lim_{n \to +\infty} \frac{f_s (\overline{x} + t_{s_n} v) - f_s (\overline{x})}{t_{s_n}}$$
$$\leq (f_s)_d^+ (\overline{x}, v) \leq \sup_{x^* \in \partial^* f_s (\overline{x})} \langle x^*, v \rangle \leq \sup_{x^* \in \Lambda(\overline{x})} \langle x^*, v \rangle, \ \forall v \in \mathbb{R}^n$$

one deduces that $\Lambda (\overline{x})$ is an upper semi-regular convexificator of h at $\overline{x}$. ∎

The contingent and normal cones to Ω at $x \in cl\ \Omega$ are given by

$$T_\Omega (x) := \{v \in \mathbb{R}^n : \exists t_n \ \downarrow 0 \text{ and } \exists v_n \to v \text{ such that } x + t_n v_n \in \Omega, \ \forall n \in \mathbb{N}\}$$

and

$$N_\Omega (x) := \{\xi \in \mathbb{R}^n : \langle \xi, v \rangle \leq 0, \ \forall v \in T_\Omega (x)\} = T_\Omega (x)^\circ .$$

The cone of feasible directions to Ω at x is defined by

$$D_\Omega (x) := \{d \in \mathbb{R}^n : \exists \delta \ > 0 \text{ s.t } \forall \lambda \in (0, \delta) \ x + \lambda d \in \Omega\} .$$

The cone $D_\Omega (x)$ is neither closed nor convex necessarily, while $T_\Omega (x)$ is closed but not necessarily convex (see Li and Zhang, 2006). In general, $D_\Omega (x) \subseteq T_\Omega (x)$; however, if Ω is convex, we have $cl\ D_\Omega (x) = T_\Omega (x)$ (Tung, 2020, Remark 1). The cone and the convex cone generated by $\Omega \subseteq \mathbb{R}^n$ are, respectively, defined by

$$\text{cone}(\Omega) := \{\lambda y : \lambda \geq 0, \ y \in \Omega\}$$

and

$$pos\ \Omega := \left\{ y \in \mathbb{R}^n : \exists l \in \mathbb{N} \text{ s.t. } y = \sum_{i=1}^{l} \lambda_i y_i, \ \lambda_i \geq 0, \ y_i \in \Omega, \ i = 1, 2, \dots, l \right\}.$$

Now, we recall the notion of locally star-shaped set, introduced by Ewing (1977).

Definition 2 (Ewing, 1977) *A nonempty set $\Omega \subseteq \mathbb{R}^n$ is said to be locally star-shaped at $\overline{x} \in \Omega$, if corresponding to $\overline{x}$ and each $x \in \Omega$, there exists some scalar $a (\overline{x}, x) \in (0, 1]$ such that*

$$\overline{x} + \lambda (x - \overline{x}) \in \Omega, \ \textit{for all } \lambda \in (0, a (\overline{x}, x)) .$$

If $a (\overline{x}, x) = 1$ for each $x \in \Omega$, then Ω is said to be star-shaped at $\overline{x}$ (Ewing, 1977).

As examples of locally star-shaped sets, convex sets are locally star-shaped at each of their elements, and cones are locally star-shaped at the origin (Kabgani and Soleimani-damaneh, 2019). If Ω is closed and is locally star-shaped at each $\overline{x} \in \Omega$, then Ω is convex (Kaur, 1983). Notice that there exist locally star-shaped sets (at some $\overline{x}$) that are not star-shaped (at $\overline{x}$). For example,

$$\Omega = \left\{(x, y) \in \mathbb{R}^2 : |x| < y\right\} \cup \left\{(x, y) \in \mathbb{R}^2 : |x| > y\right\} \cup \{(0,0)\}$$

is locally star-shaped at $\overline{x} = (0,0)$ and is not star-shaped at $\overline{x}$.

Let $\mathcal{L}(\overline{x})$ be the set of all locally star-shaped sets $\Omega \subseteq \mathbb{R}^n$ at $\overline{x}$. According to Kabgani and Soleimani-damaneh (2021, Theorem 3.1), for any $\Omega \in \mathcal{L}(\overline{x})$, we have

$$T_\Omega(\overline{x}) = \mathrm{cl}\left(\mathrm{cone}\left(\Omega - \overline{x}\right)\right) = \mathrm{cl}\left(D_\Omega(\overline{x})\right) \tag{1}$$

and

$$N_\Omega(\overline{x}) = \left(pos\ \left(\Omega - \overline{x}\right)\right)^\circ = \left\{d : \langle d, x - \overline{x}\rangle \leq 0, \forall x \in \Omega\right\}.$$

We shall need the following lemmas.

Lemma 1 (Gadhi, 2021) *Let $B \subseteq \mathbb{R}^n$ be a nonempty, convex and compact set and $A \subseteq \mathbb{R}^n$ be a convex cone. If*

$$\sup_{v \in B}\langle v, d\rangle \geq 0, \quad \textit{for all } d \in A^\circ$$

then

$$0 \in B + cl\ A.$$

Lemma 2 (Li and Zhang, 2006) *Let Ω_1 and Ω_2 be two nonempty subsets of $\mathbb{R}^n$. Then,*

$$conv\left(\Omega_1 + \Omega_2\right) = conv\ \Omega_1 + conv\ \Omega_2.$$

3. Robust optimality conditions of (UCP)

For $i \in I$, we define a family of real-valued functions $\varphi_i : \mathbb{R}^n \to \mathbb{R}$ as follows:

$$\varphi_i(x) := \sup_{v_i \in V_i} g_i(x, v_i). \tag{2}$$

Using (2), the set of robust feasible solutions F can be equivalently described as follows:

$$F = \left\{x \in \mathbb{R}^n : \varphi_i(x) \leq 0,\ i \in I\right\}.$$

The following assumption is needed in the sequel.

ASSUMPTION 1

- *For each* $v_i \in V_i(\overline{x})$, *the function* $x \mapsto g_i(x, v_i)$ *admits* $\partial_x^* g_i(\overline{x}, v_i)$ *as a bounded UCF at* $\overline{x}$.
- *The function* $\varphi_i,\ i \in I(\overline{x})$, *admits a bounded UCF* $\partial^* \varphi_i(\overline{x}),\ i \in I(\overline{x})$, *at* $\overline{x} \in F$ *such that*

$$\partial^* \varphi_i(\overline{x}) \subseteq conv \left\{ \bigcup_{v_i \in V_i(\overline{x})} \partial_x^* g_i(\overline{x}, v_i) \right\} \tag{3}$$

where

$$V_i(\overline{x}) := \{v_i \in V_i \text{ such that } g_i(\overline{x}, v_i) = \varphi_i(\overline{x})\}.$$

Inclusion (3) has been frequently used in the literature (see, for instance, Kabagani and Soleimani-damaneh, 2017; Li, Ng and Pong, 2008; Kanzi, 2011; Li, Nahak and Singer, 2000; or Gadhi and Ichatouhane, 2021) and it is worth pointing out that sometimes it is called Pshenichniy-Levin-Valadier property. Let

$$\Gamma(\overline{x}) := \bigcup_{i \in I(\overline{x})} \partial^* \varphi_i(\overline{x})$$

where

$$I(\overline{x}) := \{i \in I : \varphi_i(\overline{x}) = 0\}.$$

We shall need the following constraint qualifications. They will be used to obtain the necessary optimality conditions.

DEFINITION 3 (KABGANI AND SOLEIMANI-DAMENEH, 2017) *Let* $\overline{x} \in F$. *We say that*

- *Directional Constraint Qualification* (DCQ) *holds at* $\overline{x} \in F$ *iff there exist some* $y \in F$ *and* $\varepsilon > 0$ *such that* $y + \varepsilon \frac{d}{\|d\|} \in F$, *for each* $d \in \Gamma(\overline{x})$.
- *Abadie Constraint Qualification* (ACQ) *holds at* $\overline{x} \in F$ *iff* $\Gamma(\overline{x})^\circ \subseteq T_F(\overline{x})$.

PROPOSITION 2 *Suppose that* $F \in \mathcal{L}(\overline{x})$ *and that Assumption 1 is fulfilled. If DCQ holds at* $\overline{x}$ *and if* $\Gamma(\overline{x})$ *is compact, then pos* $\Gamma(\overline{x})$ *is closed.*

PROOF The result is a direct consequence of Kabgani and Soleimani-dameneh (2017), Theorem 3.3 (*ii*) and Lemma 3.1 (*iii*)-(*iv*). ■

THEOREM 1 *Let* $\overline{x}$ *be a weakly efficient solution of* (UCP), *for which DCQ holds. Assume that Assumption 1 is fulfilled, that* $F \in \mathcal{L}(\overline{x})$, *that* $\Gamma(\overline{x})$ *is* *compact and that* $f_s,\ s \in S$, *admits a bounded USRCF* $\partial^* f_s(\overline{x}),\ s \in S$, *at* $\overline{x}$. *Then, there exist* $\lambda_s^* \geq 0,\ s \in S$, *and* $\tau_i \geq 0,\ i \in I(\overline{x})$, *such that*

$$\sum_{s \in S} \lambda_s^* = 1 \tag{4}$$

and

$$0 \in \sum_{s \in S} \lambda_s^* \ conv\ \partial^* f_s(\overline{x}) + \sum_{i \in I(\overline{x})} \tau_i \ conv\ \left\{ \bigcup_{v_i \in V_i(\overline{x})} \partial_x^* g_i(\overline{x}, v_i) \right\}. \tag{5}$$

Proof Suppose that $\overline{x}$ is a weakly robust efficient solution of (UCP). Then, we can find $s_0 \in S$ such that

$$f_{s_0}(x) - f_{s_0}(\overline{x}) \geq 0,\ \forall x \in F.$$

Consequently,

$$\max_{s \in S} (f_s(x) - f_s(\overline{x})) \geq 0,\ \forall x \in F.$$

The preceding inequality implies that $\overline{x}$ minimizes $\psi(x) := \max_{s \in S} (f_s(x) - f_s(\overline{x}))$ over the set F. Notice that $\psi(\overline{x}) = 0$.

- We claim

$$\sup_{\eta \in \partial^* \psi(\overline{x})} \langle \eta, d \rangle \geq 0,\ \forall d \in D_F(\overline{x}) \tag{6}$$

where $\partial^* \psi(\overline{x})$ stands for a bounded USRCF of ψ at $\overline{x}$. Now, to the contrary, suppose that there exists $d_0 \in D_F(\overline{x})$ such that

$$\sup_{\eta \in \partial^* \psi(\overline{x})} \langle \eta, d_0 \rangle < 0.$$

Since

$$\psi^+(\overline{x}; d_0) = \sup_{\eta \in \partial^* \psi(\overline{x})} \langle \eta, d_0 \rangle,$$

we can find $t > 0$ satisfying $\overline{x} + td_0 \in F$ and $\psi(\overline{x} + td_0) < \psi(\overline{x})$, which contradicts the optimality of $\overline{x}$.

- We claim that there exists $\tau_i \geq 0,\ i \in I(\overline{x})$, such that

$$0 \in conv\ \partial^* \psi(\overline{x}) + \sum_{i \in I(\overline{x})} \tau_i \ conv\ \left\{ \bigcup_{v_i \in V_i(\overline{x})} \partial_x^* g_i(\overline{x}, v_i) \right\}. \tag{7}$$

Indeed, by combining (1) and (6), we obtain

$$\sup_{\eta\in\partial^*\psi(\overline{x})} \langle \eta, d\rangle \geq 0,\ \forall d \in T_F(\overline{x}).$$

Consequently,

$$\sup_{\eta\in conv\ \partial^*\psi(\overline{x})} \langle \eta, d\rangle \geq 0,\ \forall d \in T_F(\overline{x}).$$

By Kabgani and Soleimani-dameneh (2017, Theorem 3.3 (i)) and Lemma 3.1 (ii), we deduce that ACQ holds at $\overline{x}$ and that $T_F(\overline{x}) = \Gamma(\overline{x})^\circ$. Then,

$$\sup_{\eta\in conv\ \partial^*\psi(\overline{x})} \langle \eta, d\rangle \geq 0,\ \text{for all } d \in \Gamma(\overline{x})^\circ.$$

Therefore,

$$\sup_{\eta\in conv\ \partial^*\psi(\overline{x})} \langle \eta, d\rangle \geq 0,\ \text{for all } d \in (pos\ \Gamma(\overline{x}))^\circ.$$

Since $\partial^*\psi(\overline{x},\overline{y})$ is also a closed set, $conv\ \partial^*\psi(\overline{x},\overline{y})$ is a compact set (see Hiriart-Urruty and Lemarechal, 2001, Theorem 1.4.3). By Lemma 1 we get

$$0 \in conv\ \partial^*\psi(\overline{x}) + cl\ pos\ \Gamma(\overline{x}).$$

By Proposition 2, since $\Gamma(\overline{x})$ is compact and since DCQ holds at $\overline{x}$, we deduce that $pos\ \Gamma(\overline{x})$ is closed. Consequently,

$$0 \in conv\ \partial^*\psi(\overline{x}) + pos\ \Gamma(\overline{x}).$$

By Kabgani and Soleimani-dameneh (2019, Theorem 5.2), we can find $\tau_i \geq 0,\ i \in I(\overline{x})$, such that

$$0 \in conv\ \partial^*\psi(\overline{x}) + \sum_{i\in I(\overline{x})} \tau_i\ conv\ \partial^*\varphi_i(\overline{x}). \tag{8}$$

By combining (3) and (8), we obtain the desired inclusion.

- From (8), using Proposition 1, we get

$$0 \in conv\left(\bigcup_{s\in S} \partial^* f_s(\overline{x})\right) + \sum_{i\in I(\overline{x})} \tau_i\ conv\ \left\{\bigcup_{v_i\in V_i(\overline{x})} \partial_x^* g_i(\overline{x}, v_i)\right\}. \tag{9}$$

Consequently, by Lemma 2, we can find $\lambda_s^* \geq 0,\ s \in S$, such that

$$\sum_{s\in S} \lambda_s^* = 1$$

and

$$0 \in \sum_{s \in S} \lambda_s^* \ conv \ \partial^* f_s(\overline{x}) + \sum_{i \in I(\overline{x})} \tau_i \ conv \left\{ \bigcup_{v_i \in V_i(\overline{x})} \partial_x^* g_i(\overline{x}, v_i) \right\}.$$

■

The following Theorem 2 makes use of ACQ instead of DCQ. The argument is similar to that of Theorem 1, since the closeness of *pos* $\Gamma(\overline{x})$ is assumed directly, instead of the compactness of $\Gamma(\overline{x})$.

THEOREM 2 *Let $\overline{x}$ be a weakly efficient solution of (UCP), for which ACQ holds. Assume that Assumption 1 is fulfilled, that $F \in \mathcal{L}(\overline{x})$, that pos $\Gamma(\overline{x})$ is closed and that f_s, $s \in S$, admits a bounded USRCF $\partial^* f_s(\overline{x})$, $s \in S$, at $\overline{x}$. Then, there exist $\lambda_s^* \geq 0$, $s \in S$, and $\tau_i \geq 0$, $i \in I(\overline{x})$, such that (4) and (5) hold.*

The following example illustrates Theorem 1.

EXAMPLE 2 *Let $\mathbb{R}^n = \mathbb{R}^2$, $\mathbb{R}^k = \mathbb{R}^2$, $\mathbb{R}^m = \mathbb{R}^2$, $V_1 = \left[\frac{-\pi}{2},\ \pi\right]$ and $V_2 = \left[-\frac{1}{2},\ 0\right]$. Let $f_1 : \mathbb{R}^2 \to \mathbb{R}$ and $f_2 : \mathbb{R}^2 \to \mathbb{R}$ be defined by*

$$f_1(x) = 4|x_1| - \frac{1}{2}x_2, \ f_2(x) = x_1 + x_2, \ \forall x = (x_1, x_2) \in \mathbb{R}^2.$$

We consider the problem (UCP) with the constraint functions $g_1 : \mathbb{R}^2 \times V_1 \to \mathbb{R}$ and $g_2 : \mathbb{R}^2 \times V_2 \to \mathbb{R}$, given by

$$g_1(x, v_1) = x_1^2 |\cos v_1| - 1, \ \forall v_1 \in V_1,$$

and

$$g_2(x, v_2) = |x_2| - 2 + \ln(1 + v_2), \ \forall v_2 \in V_2.$$

By computation, we obtain

$$\varphi_1(x) = \max_{v_1 \in V_1} g_1(x, v_1) = x_1^2 - 1 \ \text{and} \ \varphi_2(x) = \max_{v_2 \in V_2} g_2(x, v_2) = |x_2| - 2.$$

Therefore,

$$F = [-1,\ 1] \times [-2,\ 2].$$

$$f_1(x) - f_1(\overline{x}) = 4|x_1| - \frac{1}{2}x_2 - 3, \ f_2(x) - f_2(\overline{x}) = x_1 + x_2 - 3, \ \forall x = (x_1, x_2) \in \mathbb{R}^2.$$

Let $\overline{x} = (1,\ 2) \in F$. In this case,

$$S = \{1, 2\}, \ I(\overline{x}) = I = \{1, 2\}, \ V_1(\overline{x}) = \{0, \pi\}, \ V_2(\overline{x}) = \{0\}$$

and

$$\Gamma(\overline{x}) = \{(2,0),\ (0,1),\ (0,-1)\}.$$

The functions f_1 *and* f_2 *admit*

$$\partial^* f_1(\overline{x}) = \left\{ \left(-4, -\frac{1}{2}\right),\ \left(4, -\frac{1}{2}\right) \right\} \text{ and } \partial^* f_2(\overline{x}) = \{(1,1)\}$$

as bounded USRCF at $\overline{x}$. *Notice that Assumption 1 is fulfilled, that* $\Gamma(\overline{x})$ *is compact and that* $\overline{x}$ *is a weak robust efficient solution of the problem (UCP).*

- *F is locally star-shaped at* $\overline{x} = (1,\ 2) \in F$ *due to the convexity of* F.
- *DCQ holds at* $\overline{x}$. *Indeed, for* $y = (0,1)$ *and* $\varepsilon = \frac{1}{2}$, *we get* $y + \varepsilon \frac{d}{\|d\|} \in F$, *for each* $d \in \Gamma(\overline{x})$.
- *Upon noticing that* $\left(-\frac{2}{3}, -\frac{1}{2}\right) \in conv\ \partial^* f_1(\overline{x})$, *inclusion* (5) *is satisfied for* $\lambda_1^* = \frac{3}{4},\ \lambda_2^* = \frac{1}{4},\ \tau_1 = \frac{1}{8}$ *and* $\tau_2 = \frac{1}{8}$.

The following example explains how to employ Theorem 1.

Example 3 *Let* $\mathbb{R}^n = \mathbb{R}^2,\ \mathbb{R}^k = \mathbb{R}^2,\ \mathbb{R}^m = \mathbb{R}^2,\ V_1 = \left[\frac{-\pi}{2},\ \pi\right]$ *and* $V_2 = [-2,-1]$. *Let* $f_1 : \mathbb{R}^2 \to \mathbb{R}$ *and* $f_2 : \mathbb{R}^2 \to \mathbb{R}$ *be defined by*

$$f_1(x) = x_2^2 + 2x_2(1 - x_1),\ f_2(x) = 2x_2 + x_1^3,\ \forall x = (x_1, x_2) \in \mathbb{R}^2.$$

We consider the problem (UCP) with a constraint function $g_1 : \mathbb{R}^2 \times V_1 \to \mathbb{R}$ *and* $g_2 : \mathbb{R}^2 \times V_2 \to \mathbb{R}$, *given by*

$$g_1(x, v_1) = x_1^2 \sin v_1,\ g_2(x, v_2) = x_2^2 - v_2^2,\ \forall v_1 \in V_1,\ \forall v_2 \in V_2.$$

By computation, we obtain

$$\varphi_1(x) = \max_{v_1 \in V_1} g_1(x, v_1) = x_1^2 \text{ and } \varphi_2(x) = \max_{v_2 \in V_2} g_2(x, v_2) = x_2^2 - 1.$$

Therefore,

$$F = \{0\} \times [-1,\ 1].$$

Let $\overline{x} = (0,\ 1) \in F$. *In this case,*

$$S = \{1,2\},\ I(\overline{x}) = I = \{1,2\},\ V_1(\overline{x}) = \left\{\frac{\pi}{2}\right\},$$

$$V_2(\overline{x}) = \{-1\},\ \partial^* f_1(\overline{x}) = \{(-2,4)\},\ \partial^* f_2(\overline{x}) = \{(0,2)\}$$

and

$$\Gamma(\overline{x}) = \{(0,0),(0,2)\}.$$

Remark that Assumption 1 is fulfilled and that $\Gamma(\overline{x})$ *is compact.*

- *F is locally star-shaped at* $\overline{x} \in F$ *due to the convexity of* F.
- *DCQ holds at* $\overline{x}$. *Indeed, for* $y = (0, \frac{1}{4})$ *and* $\varepsilon = \frac{1}{4}$, *we get* $y + \varepsilon \frac{d}{\|d\|} \in F$, *for each* $d \in \Gamma(\overline{x})$.
- *Notice that for all* $\lambda_s^* \geq 0,\ s \in S$, *and* $\tau_i \geq 0,\ i \in I(\overline{x})$, *inclusions* $(4)-(5)$ *are never satisfied. Taking into account Theorem 1, one sees that* $\overline{x} \in F$ *is not a weakly efficient solution of* (UCP).

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