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On Undecidability of Non-monotonic Logic

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Abstract. The degree of undecidability of nonmonotonic logic is investigated. A proof is provided that arithmetical but not recursively enumerable sets of sentences definable by nonmonotonic default logic are elements of Δ_{n+1} but not Σ_n nor Π_n for some $n \geq 1$ in Kleene-Mostowski hierarchy of arithmetical sets.

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1 Introduction

While first-order logic is often thought of as the "correct" (whatever it means) logic for classical mathematics, nonmonotonic logic seems to have gained more acceptance in Artificial Intelligence. First-order provability relation is semi-decidable but, in general, undecidable, that is, except for monadic languages, it is in class $\Sigma_1 \setminus \Delta_1$. It turns out that similar relation in nonmonotonic logic that, in addition of deriving consequences of asserted axioms, is able to derive conclusions from a non-provability of certain sentences is more undecidable than the first-order logic is.

For instance, the monadic case of logic of minimal entailment (think of it as a $\forall$-fragment of monadic first-order logic with semantics restricted to models that are relation-minimal) has a nonmonotonic consequence relation that is not even semi-decidable, or, more specifically, it is in class $\Pi_1 \setminus \Sigma_1$ (see [3] page 382 for a proof). Its prioritized (and more adequate for AI applications) variant is even more undecidable; its relation of satisfaction in a finite model, clearly a decidable (in Δ_1, that is) kind of relation for any first-order logic, may fall into class $\Pi_1 \setminus \Sigma_1$ (see [4] page 277 for a proof).

In this paper, we will prove that arithmetical non-r.e. sets (not in Σ_1, that is) of sentences definable by nonmonotonic default logic are elements of Δ_{n+1} but not Σ_n nor Π_n for some $n \geq 1$.

2 The Kleene - Mostowski hierarchy

We will follow notation from [1] and [2].

The Kleene-Mostowski hierarchy of arithmetical sets is defined as usual:

Definition 2.1

$\Sigma_0 = \Pi_0$ = {all recursive relations}.
Σ_{n+1} = {all projections of elements of Π_n}.
Π_{n+1} = {all complements of elements of Σ_{n+1}}.
Finally, $\Delta_{n+1} = \Sigma_{n+1} \cap \Pi_{n+1}$. □

In particular, Δ_1 is the set of all recursive relations (sometimes referred to as decidable relations) Σ_1 is the set of all r.e. relations (sometimes referred to as semi-recursive relations), and Π_1 is the set of all co-r.e. relations (sometimes referred to as co-semirecursive relations).

Definition 2.2

A k-ary relation X is an *upper limit* of a k+1-ary relation R (notation: $X = \overline{\lim}_{n \to \infty} R(n)$) if, and only if, $x \in X \equiv (\exists n \in \omega)(\forall m \geq n)\ x \in R(m)$.

A k-ary relation X is a *total limit* of a k+1-ary relation R (notation: $X = \lim_{n \to \infty} R(n)$) if, and only if, both $X = \lim_{n \to \infty} \overline{R(n)}$ and $\overline{X} = \overline{\lim}_{n \to \infty} \overline{R(n)}$.

A relation X is asymptotically decidable if, and only if, X is a total limit of some recursive relation.

Any such a recursive relation is called an asymptotic computation of X. □

Theorem 2.3 (due to Shoenfield and Kleene)

The following are equivalent:

1. X is asymptotically decidable
2. $X \leq_T K$ (that is, X is Turing-reducible to the halting set $K = \{e \mid \varphi_e(e){\downarrow}\}$)
3. $X \in \Delta_2$. □

Example 2.4

$K \bowtie \overline{K}$ is asymptotically decidable but not r.e. nor co-r.e. (that is, $K \bowtie \overline{K} \in \Delta_2 \setminus (\Sigma_1 \cup \Pi_1)$), where $2x \in (A \bowtie B)$ if, and only if, $x \in A$ and $2x + 1 \in (A \bowtie B)$ if, and only if, $x \in B$. □

Fact 2.5

Δ_n is closed under set-theoretic operations. □

3 Nonmonotonic Logics

In this section we will focus on the undecidability of nonmonotonic logics that are based on the concept of default, the so-called *default logics*, whose relations of consequence may fall outside of $\Pi_1 \cup \Sigma_1$ even in the purely propositional case. In what follows, we will use some standard terminology and definitions from default logic, a brief account of which can be found in [5].

Let T be a (recursive) set of first-order sentences, $\vdash$ - the first-order provability relation, and $Cn(T)$ - the set of first-order consequences of T.

3.1 Nonmonotonic rules of inference

The rules of nonmonotonic inference allow for deriving conclusion from non-provability of some sentences. They, typically, have a form of:

$$\frac{T \nvdash \varphi \mid \ldots}{T \vdash \psi}$$

The intentional meaning of the above rule is: if φ is not provable from T and ... then infer ψ. While the set $Cn(T)$ of first-order consequences of T is r.e. in T, the set of first-order nonmonotonic consequences of T is usually not, for a similar reason the set $K \bowtie \overline{K}$ K is not r.e.; it may need an oracle for $\overline{\mathrm{Cn}(T)}$.

In the case of default logics, the nonmonotonic consequence operation is usually defined in terms of fixed-points of a continuous consequence operator.

Let D be a (recursive) set of the following nonmonotonic rules of inference, referred to as *defaults*:

$$\frac{\varphi \mid \Diamond\psi_1 \mid \ldots \mid \Diamond\psi_n}{\chi} .$$

Let the consequence operator $\Phi_D(T, E)$ of T under the first-order consequences and rules from D relative to E be defined by:

$$\frac{T \vdash \varphi \mid \neg\psi_1 \notin E \mid \ldots \mid \psi_n \notin E}{\chi \in \Phi_D(T, E)} .$$

Definition 3.1.1
The nonmonotonic closure of T relative to Φ_D is a set E that

1. contains T
2. is closed under first-order (propositional, modal, etc.) consequence
3. is a solution of the equation

$$\Phi_D(T, E) = E.$$

□

Fact 3.1.2
Operator $\Phi_D(T, E)$ is:

1. monotone w.r.t. T (that is, for $T \subseteq T'$, $\Phi_D(T, E) \subseteq \Phi_D(T', E)$)
2. non-monotone w.r.t. E (but monotone w.r.t. $\overline{E}$)
3. continuous w.r.t. both arguments(because all defaults rules of inference are finitary). □

Since one can express completeness using a recursive set of defaults, despite its seemingly simplicity the degree of undecidability of nonmonotonic logic with a recursive set of axioms may be enormously high.

Example 3.1.3
Let D consist of all rules of the form

$$\frac{true \mid \Diamond\psi}{\psi}$$

where ψ is a first-order sentence. If E is a consistent solution of the equation

$$\Phi_D(PA, E) = E$$

(where PA is the set of axioms of Peano Arithmetic) then E is not arithmetical (a classic result due to Gödel). □

Theorem 3.1.4
For any recursive T, recursive set of defaults D, and every nonmonotonic closure E of T relative to D, if E is arithmetical and not in Σ_1 then

$$E \in \Delta_{n+1} \setminus (\Sigma_n \cup \Pi_n) \text{ for some } n \geq 1.$$

Proof is based on an observation that since all operations involved in the definition of Φ can be reduced to intersections of E with r.e. sets, the set $\overline{E}$ defined by the above fixed-point equation, unless a member of Σ_1, cannot be more undecidable than $\overline{E}$.

Indeed, let E be arithmetical. Let n be the smallest number such that $E \in \Sigma_n \cup \Pi_n$. We have:

$\chi \in \Phi_D(T, E)$ if, and only if, $\exists\psi_1...\exists\psi_n\ \exists\theta_1...\exists\theta_m\ \exists\varphi <\psi_1,...,\psi_n> \notin E^*$ and $<\theta_1,...,\theta_m>$ is a proof of φ from *T* and

$$\frac{\varphi \mid \Diamond\psi_1 \mid ... \mid \Diamond\psi_n}{\chi} \in D$$

if, and only if, $\exists x\ \exists y\ x \in Cn(T)$ and $y \in \overline{E}^*$ & $f(x, y, \chi) \in D$,
where *f* is a recursive function. Hence, by the recursive eumerability of *Cn*(*T*) and the recursiveness of *D*, $\Phi_D(T, E)$, and, therefore, *E*, is the intersection of an r.e. set with $\overline{E}^*$ and with a recursive set.

Assume $E \in \Sigma_n \setminus \Pi_n$, where $n \geq 2$, that is, $\overline{E} \in \Pi_n \setminus \Sigma_n$. Now, since $\overline{E}$ and $\overline{E}^*$ have the same degree of undecidability, it follows that *E* is the intersection of a $\Pi_n \setminus \Sigma_n$-set with a Σ_1-set, which is in Π_n - a contradiction.

Assume $E \in \Pi_n$. Because $\overline{E} \in \Sigma_n$, any projection of $\overline{E}$ is in Σ_n. So, $E = \Phi_D(T, E) \in \Sigma_n$. Hence, $E \in \Sigma_n \cap \Pi_n = \Delta_n$. □

Note. If, for instance, *D* is empty then its nonmonotonic closure *E* coincides with *Cn*(*T*), which for some recursive *T* is in $\Sigma_1 \setminus \Pi_1$ (r.e. but non-recursive, that is).

3.2 Asymptotic computation of *E*

Let $E \in \Delta_2$, that is, let $E = \lim_{n \to \infty} f(n)$ for some recursive relation *f*. By the continuousness of the operator Φ_D, $\Phi_D(T, E) = \lim_{n \to \infty} \Phi_D(T, f(n))$. Therefore, $\Phi_D(T, f(n))$ is an asymptotic computation of *E* as well.

Example 3.2.1: Autoepistemic Logic

Autoepistemic logic allows for a modal operator □ (which is ***not*** related to the operator ◊ used in the definition of defaults in this paper) instead of quantifiers. Its nonmonotonic rules of inference are:

$$\frac{true \mid \Diamond\psi}{\neg\Box\neg\psi}$$

where ψ is a first-order sentence. The operator Φ is also closed under consequences of modal logic ***S5***, in particular, closed under the monotonic rule

$$\frac{\varphi}{\Box\varphi}.$$

If follows that for □-free recursive sets *T*, the nonmonotonic closure of *T* relative the above is in Δ_2 (in Δ_1 if $Cn(T)$ is recursive). More specifically, it is recursive in $Cn(T) \bowtie \overline{Cn(T)}$. Therefore, any asymptotic computation f(n) of $Cn(T) \bowtie \overline{Cn(T)}$ yields, by the continuousness of the operator Φ, an asymptotic computation $\Phi_D(T, f(n))$ of the closure.

However, if *T* contains sentences with occurrences of □ then the above closure may or may not be in Δ_2. Of course, if it is not in Δ_2 then, by the Theorem 3.1.4, if it is arithmetical then it is in $\Delta_{n+1} \setminus (\Sigma_n \cup \Pi_n)$ for some $n \geq 2$. □

References

1. Enderton H.B., *Elements of Recursion Theory*, Handbook of Mathematical Logic, (Barwise, J., ed.), pp. 527-566, North Holland, 1977.
2. Rogers Jr.H., *Theory of Recursive Functions and Effective Computablity*, MIT Press, 1992.
3. Suchenek M.A., *Evaluation of Queries under Closed-World Assumption*, Journal of Automated Reasoning, Volume 18, pp 357-398, 1997.
4. Suchenek M.A., *Evaluation of Queries under Closed-World Assumption II*, Journal of Automated Reasoning, Volume 25, pp 247-289, 2000.
5. Suchenek M. A., A review of: *G. Antoniou, "Nonmonotonic Reasoning"*, The Bulletin of Symbolic Logic, Volume 6, Issue 4, pp 484 and on, December 2000.