Malinowski Jacek, 0000-0002-4413-1868
Systems Research Institute, Polish Academy of Sciences, Warszawa, Poland,
jacek.malinowski{at}ibspan.wa.pl

Optimizing repair/replacement policy if cumulative distribution function of time-to-repair is rational function

Keywords

repair/replacement policy, time-to-repair, cumulative distribution function, rational function, optimization, distribution fitting

Abstract

A problem of finding the optimal repair/replacement policy of a technical object is under investigation. Depending on the distribution of the time-to-repair and certain cost parameters, a decision is taken whether to repair a failed object or replace it immediately after it fails. If a repair is chosen and it is not completed within a certain period, it is interrupted and the object is replaced by a new one. The optimization task consists in formulating the conditions for choosing between immediate replacement and repair, and deriving the analytical equation for the maximum duration of a repair. The objective function is the expected cost of restoring the object to the operating condition. It is assumed that the cumulative distribution function (CDF) of the time-to-repair is a rational function (a quotient of polynomials). The properties of such CDFs are analyzed and the solutions of several optimization tasks with different CDFs that are rational functions are presented. The issue of fitting the time-to-repair distribution to empirical data is also addressed.

1. Introduction

The probability distributions of time-to-failure (TTF) and time-to-repair (TTR) have key significance in constructing maintenance models used in reliability theory. In large part, they are used to describe the behavior of two-state systems which can be either functioning or failed, but can also be applied in modeling multi-state systems with discrete or continuous state-space, as distributions of sojourn times in individual states for semi-Markov processes (Grabski, 2015) or distributions of crossing times for continuous deterioration processes (Nakagawa, 2011). It is practically impossible to summarize the literature on maintenance policies in a paper of limited length, because the list of publications would possibly contain hundreds if not thousands of items. For this reason, the usual literature review is limited to several works providing a general overview of the considered topic. Comprehensive surveys of various maintenance models can be found in (Nakagawa, 2005, 2008) or in the recent paper (Zhao et al., 2022). A newly published book (Jardine & Tsang, 2021) is partly industry-oriented and gives a broad insight into the issues of physical asset management (PAM) and covers subject areas of replacement, spare parts provisioning, inspection scheduling, resource management, and application of some emerging technologies (AI, IoT, Industry 4.0, digital twin, Predictive Maintenance 4.0, Blockchain) in PAM. It puts a lot of focus on optimization techniques. As regards optimization of maintenance procedures, a detailed survey of relevant literature can be found in (De Jonge & Scarf, 2019). More specialized issues are elaborated in the following works: maintenance of systems with delayed failures (Cha & Filkenstein,

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2019), minimal, imperfect and perfect repairs (Rebaiaia & Ait-kadi, 2021; Tadj et al., 2011), shock and damage models (Zhao & Nakagawa, 2018), or repair vs replacement – a problem similar to that considered in the current paper (Safaei et al., 2019). All the latter papers contain substantial bibliographies on the respective topics.

Unlike most studies on maintenance policies, the present paper focuses on the repair part of a failure and repair process. It addresses a less frequently analyzed problem of deciding whether to repair or replace a failed object, and, in case of a prolonged repair, when to interrupt it and replace the object with a new one. In addition, the author departs from the usual assumption that TTR has Weibull or lognormal distribution. Instead, distributions whose cumulative distribution functions (CDFs) are rational functions are considered. This means that the CDF of such a distribution is given by $p(x)/q(x)$, where p and q are polynomials satisfying the following conditions:

a) $p(0)/q(0) = 0$,

b) $p(x)/q(x)$ increases in x for $x > 0$,

c) $\lim_{x\to\infty} p(x)/q(x) = 1$.

For brevity, the above defined distributions will be called P/Q distributions.

As shown in Sections 3 and 4, if TTR has a P/Q distribution, then the corresponding repair rate function first increases from zero to some maximum value and then slowly decreases to zero. Such shape of this function is proper to systems whose remaining repair time is likely to increase along with the duration of a repair, if the repair has not been completed within a certain timeframe. Moreover, the CDF of a P/Q distribution is a rational function, which can be convenient for numerical computations. These are two main reasons for considering P/Q distributed repair times. Thus, P/Q distributions make a useful alternative to log-normal or Weibull ones, commonly used as repair time distributions. For examples of application of the latter two see Jokiel-Rokita (Jokiel-Rokita & Magiera, 2011). To the best of the author's knowledge, P/Q distributions have not been used for maintenance modeling in previous literature.

2. Repair/replacement policy and optimal time of repair interruption

Let us assume that a failed technical system undergoes necessary repair according to the following rule: a repair takes a random amount of time that cannot exceed certain limit value L; if a repair is not completed within a period of length L, then it is interrupted and the system is replaced with a new one. The repair time, unlimited by L, will be denoted by T. Let $F(x)$, $f(x)$ and $r(x)$ denote the CDF, probability density function (PDF), and repair rate function of T, i.e.

$$F(x) = \Pr(T \le x) = \int_0^x f(s)ds \tag{1}$$

$$r(x) = f(x)/[1 - F(x)]. \tag{2}$$

We also assume that the repair cost is proportional to the duration of a repair, where c is the cost rate per unit time, and the replacement cost is R. Such a repair policy is applicable when the repair time cannot be determined or estimated in advance and a repair has to be performed in order to know whether its duration exceeds L or not. Clearly, an optimization problem arises of finding L that minimizes the expected post-failure maintenance cost. The above assumptions yield that this cost, denoted here by K, is a random variable given by the following formula:

$$K = \begin{cases} c \cdot T, & T \le L \\ c \cdot L + R, & T > L. \end{cases} \tag{3}$$

We can represent K as the sum of two random variables K_1 and K_2:

$$K_1 = c \cdot \min(T, L), \quad K_2 = R \cdot \chi_{\{T>L\}} \tag{4}$$

where c is a characteristic function on the sample space. We thus have:

$$E(K) = E(K_1) + E(K_2) =$$

$$= c \cdot E[\min(T, L)] + R \cdot \Pr(T > L). \tag{5}$$

In order to compute $E[\min(T,L)]$, we will use the following lemma, helpful in computing moments of probability distributions.

Lemma 1

The k-th moment of a nonnegative random variable X can be computed from the following formula:

$$E[\min(X^k, L^k)] = \int_0^L kx^{k-1} \Pr(X > x)\, dx. \tag{6}$$

Proof:
Integration by parts yields:

$$\int_0^L kx^{k-1}\Pr(X > x)\,dx = L^k \Pr(X > L) +$$

$$+ \int_0^L x^k dF(x), \qquad (7)$$

where $F(x)$ is the distribution function of X. In view of the following equality

$$\min(X^k, L^k) = \begin{cases} X^k, & X \le L \\ L^k, & X > L \end{cases} \qquad (8)$$

the right-hand side of (7) is the expected value of $\min(X^k, L^k)$, which ends the proof.
From (6) it follows immediately that

$$E(X^k) = \int_0^\infty kx^{k-1}\Pr(X > x)\,dx. \qquad (9)$$

Using (5) and Lemma 1 we obtain:

$$E(K) = c \cdot \int_0^L \Pr(T > t)\,dt +$$

$$+ R \cdot \Pr(T > L). \qquad (10)$$

The right-hand side of (10) can be regarded as a function of L. It will be denoted by $j(L)$ and referred to as the cost function. Differentiation of $j(L)$ yields:

$$\frac{d\varphi(L)}{dL} = c \cdot \Pr(T > L) - R \cdot f(L). \qquad (11)$$

By equating the right-hand side of (11) to zero, we obtain the values of L for which the considered maintenance policy can be optimal in the sense of minimum expected cost. However, we have to keep in mind that a function may not have a minimum at a point where its derivative is equal to zero, or it may have a local minimum there. Thus, apart from finding the zeroes of $dj(L)/dL$, it is necessary to examine the behavior of $j(L)$ on the whole $[0,\infty)$ interval. It may happen that, apart from having a local minimum inside that interval, $j(L)$ reaches its global minimum at $L = 0$ or $L = \infty$. In the first case the optimal policy is to replace the system on failure without performing a repair, while in the second – to continue a repair indefinitely and never to replace. The next lemma shows that under certain conditions on the repair rate function, the optimal repair policy is either to replace a system immediately after a failure or to continue a repair until completion.

Lemma 2

1. Let $r(x) < c/R$ for $x \ge 0$. Then $j(L)$ increases in L, and the optimal policy is to replace the system on failure (do not repair).
2. Let $r(x) > c/R$ for $x \ge 0$. Then $j(L)$ decreases in L, and the optimal policy is to continue a repair until completion (do not replace).
3. Let $r(x)$ be unimodal (there exists $x_{\max}$ such that $r(x)$ increases for $x < x_{\max}$ and decreases for $x > x_{\max}$), and such that $r(0) < c/R$, $r(x_0) = c/R$ for some $x_0 > 0$, and $r(x) > c/R$ for $x > x_0$. Then, if $c \cdot E(T) < R$, the optimal policy is to continue a repair until completion (do not replace), and if $c \cdot E(T) \ge R$, the optimal policy is to replace on failure.

Proof:

1. If $r(x) < c/R$ for $x \ge 0$, then (11) implies that $j'(L)$ is positive for $L > 0$, i.e. $j(L)$ increases in L in the interval $[0,\infty)$, hence $j(L)$ has minimum at $L = 0$. This means that no repair is to be carried out and the system must be replaced immediately after a failure.
2. The proof is analogous to that of part 1.
3. Under the given assumptions, $j'(L)$ is positive for $L \in [0,x_0)$, equal to zero for $L = x_0$, and negative for $L > x_0$, hence $j(L)$ increases for $L \in [0,x_0)$, has maximum at $L = x_0$, and decreases for $L > x_0$. Also, $j(L)$ converges to $c \cdot E(T)$ in view of (10). Since $j(0) = R$, $j(L)$ has minimum at $L = 0$ if $c \cdot E(T) \ge R$ (do not repair), or at infinity if $c \cdot E(T) < R$ (do not replace).

3. P/Q distributions

In this section we will introduce a family of distributions whose properties make them appropriate for modeling TTR of many types of systems, both technical and non-technical. They will be called P/Q distributions, where P/Q denotes a rational function (a quotient of polynomials), i.e. a distribution of this type has the following CDF:

$$F(x) = \frac{w(x)}{w(x) + v(x) + \beta} \qquad (12)$$

where $w(x)$ and $v(x)$ are polynomials with positive coefficients and zero constant terms, and

$0 \le \deg(v) < \deg(w)$. Clearly, if $\deg(v) = 0$ then $v \equiv 0$. Thus, the equivalent of the survival function (the survival function as such characterizes the distribution of TTF) is given by

$$S(x) = 1 - F(x) = \frac{v(x)+\beta}{w(x)+v(x)+\beta} \tag{13}$$

and the PDF by

$$f(x) = \frac{w'(x)w(x)+w'(x)v(x)+w'(x)\beta}{[w(x)+v(x)+\beta]^2} +$$

$$-\frac{w(x)w'(x)+w(x)v'(x)}{[w(x)+v(x)+\beta]^2} =$$

$$= \frac{w'(x)[v(x)+\beta]-w(x)v'(x)}{[w(x)+v(x)+\beta]^2}. \tag{14}$$

Let us note that the numerator in the last expression must be nonnegative so that $f(x)$ can be a PDF. Thus, it must hold that

$$w'(x)[v(x) + \beta] - w(x)v'(x) \ge 0. \tag{15}$$

For this reason $w(x)$ and $v(x)$ cannot be chosen arbitrarily, e.g. if $w(x) = x^3 + x$, $v(x) = 4x^2$ and $b = 1/3$ then $f(x) < 0$ for $x = 1/2$. From (13) and (14) we readily obtain the repair rate function:

$$r(x) = \frac{f(x)}{1-F(x)} =$$

$$= \frac{1}{w(x)+v(x)+\beta}\left[w'(x) - \frac{w(x)v'(x)}{v(x)+\beta}\right]. \tag{16}$$

Remark: It is not necessary that powers of x in $w(x)$ and $v(x)$ be integer numbers. This ensures greater flexibility in fitting $F(x)$ given by (12) to sample data. Some examples given in Section 4 show the behavior of repair rate and cost functions for non-integer powers of x.

3.1. Distributions whose CDFs are quotients of poly-logarithmic functions

Distributions referred to in the title of this section are a modification of P/Q distributions, obtained by replacing x with $\ln(x + 1)$, i.e. the CDF od such a distribution is a quotient of poly-logarithmic functions defined below:

$$F(x) = \frac{w(\ln(x+1))}{w(\ln(x+1))+v(\ln(x+1))+\beta}. \tag{17}$$

The respective PDF and repair rate function are given by the following formulas:

$$f(x) = \frac{w'(y)w(y)+w'(y)v(y)+w'(y)\beta}{(x+1)[w(y)+v(y)+\beta]^2} +$$

$$-\frac{w(y)w'(y)+w(y)v'(y)}{(x+1)[w(y)+v(y)+\beta]^2} =$$

$$= \frac{w'(y)[v(y)+\beta]-w(y)v'(y)}{(x+1)[w(y)+v(y)+\beta]^2} \tag{18}$$

and

$$r(x) = \frac{f(x)}{1-F(x)} =$$

$$= \frac{1}{(x+1)[w(y)+v(y)+\beta]}\left[w'(y) - \frac{w(y)v'(y)}{v(y)+\beta}\right] \tag{19}$$

where $y = \ln(x + 1)$. For example, if $w(y) = y^k$ and $v(y) = 0$, then

$$S(x) = \beta/[\ln(x+1)]^k \tag{20}$$

and

$$r(x) = \frac{k[\ln(x+1)]^{k-1}}{(x+1)\{[\ln(x+1)]^k+\beta\}}. \tag{21}$$

4. Repair rate and cost functions for selected P/Q distributions

Now we shall present the graphs of a number of repair rate and cost functions to illustrate how they behave depending on the CDF of the repair time and cost parameters. The first two figures illustrate the case $w(x) = x^2$, $v(x) = 0$. Figure 1 presents the repair rate function $r(x)$ with the horizontal lines indicating the values of c/R for $c = 8, 10, 12$ and $R = 100$. The points at which these lines cross the increasing/decreasing $r(x)$ are the local maxima/minima of the respective cost functions (see Lemma 2) whose graphs are shown in Figure 2. In all three cases the local minima are also the global ones (for $c = 12$ the value of j at its local minimum is only slightly less than R), hence in each case the optimal policy is to interrupt a repair at the time equal to the respective local minimum.

Figures 3 and 4 present the graphs of the repair rate and cost functions for $w(x) = x^{1.75}$ (non-integer power of x) and $v(x) = 0$. Let us note that the graph in Figure 3 is less steep than that in Figure 1 and the horizontal line indicating c/R for $c = 12$ ($c/R = 0.12$) lies above the graph of $r(x)$.

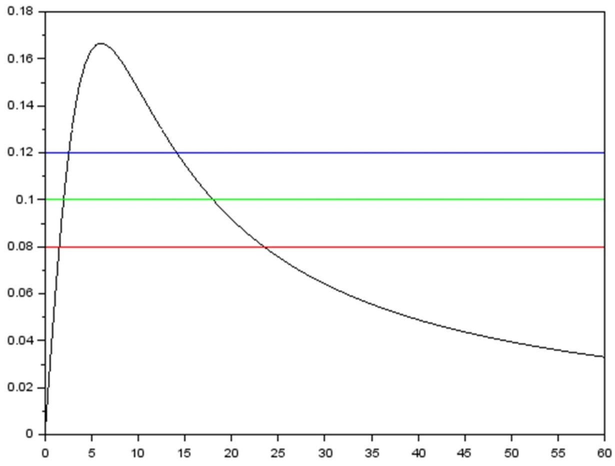

Figure 1. Repair rate function for $w(x) = x^2$, $v(x) = 0$, $b = 36$.

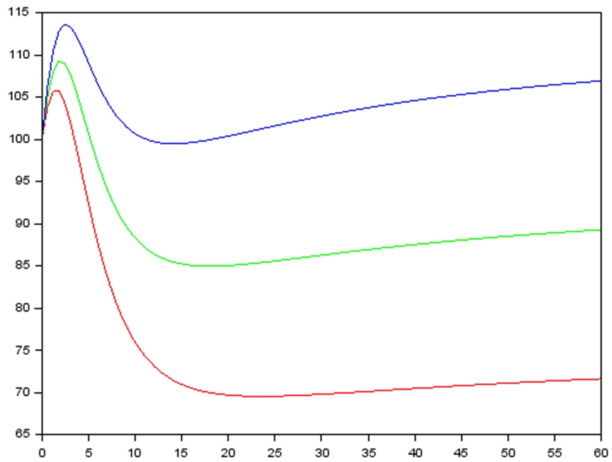

Figure 2. Cost functions for $w(x) = x^2$, $v(x) = 0$, $b = 36$, $c = 8$ (red), $c = 10$ (green), $c = 12$ (blue), $R = 100$.

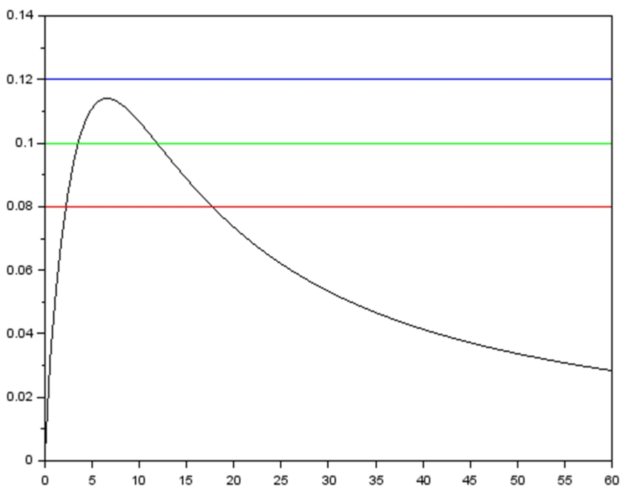

Figure 3. Repair rate function for $w(x) = x^{1.75}$, $v(x) = 0$, $b = 36$.

In Figure 4 we can see how the shape of the cost function $j(L)$ changes as the parameter c increases. For $c = 8$ the local minimum of $j(L)$ near $L = 18$ is also its global minimum, which means that if a repair is not completed until then, the system must be replaced. For $c = 10$ the function $j(L)$ has one local minimum near $L = 12$, but, since the value of $j(L)$ at that point exceeds its global minimum at $L = 0$, the optimal policy in this case is *do not repair and replace on failure*. For $c = 12$ it holds that $r(x) < c/R = 0.12$, where $x \geq 0$, thus, according to Lemma 2, the optimal policy is the same as in the previous case (immediate replacement).

Figures 5 and 6 show the behavior of the repair rate and cost functions for $w(x) = x^2$, $v(x) = x$ (nonzero $v(x)$) and $b = 36$. Let us note that the graph in Figure 5 *lies under* the graph in Figure 1. As a result, the cost functions in Figure 6 decrease more slowly and then increase faster than the cost functions in Figure 2.

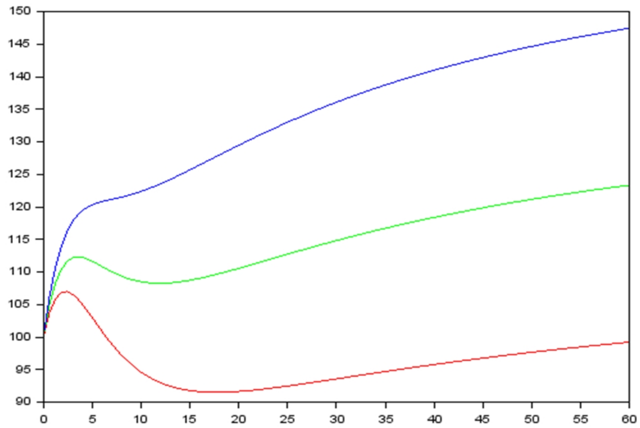

Figure 4. Cost functions for $w(x) = x^{1.75}$, $v(x) = 0$, $b = 36$, $c = 8$ (red), $c = 10$ (green), $c = 12$ (blue), $R = 100$.

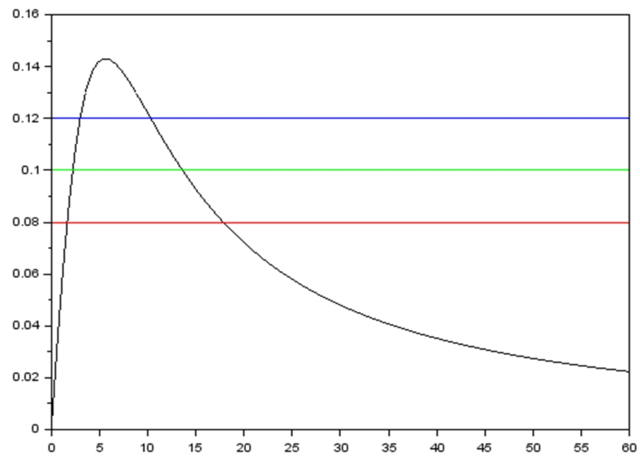

Figure 5. Repair rate function for $w(x) = x^2$, $v(x) = x$, $b = 36$.

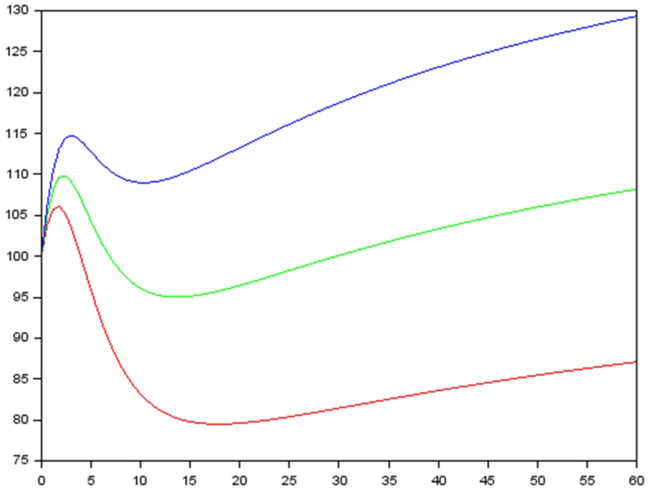

Figure 6. Cost functions for $w(x) = x^2$, $v(x) = x$, $b = 12$, $c = 6$ (red), $c = 8$ (green), $c = 12$ (blue), $R = 60$.

Finally we present the graphs of repair rate and cost functions if the time-to-repair CDF is a quotient of poly-logarithmic functions. Let $w(y) = y^2$ and $v(y) = 0$. It can be seen that with the same scale and cost parameters, the graph of the repair rate function in Figure 7 is much more flat than the graph in Figure 1, which results in almost linearly increasing cost functions. As indicated by Figure 8, and confirmed by Lemma 2, the optimal policy is to replace the system on failure.

Let now $w(y) = y^5$, i.e. the power of $\ln(x + 1)$ is changed to 5. The scale and cost parameters being the same, the repair rate function in Figure 9 decreases faster than that in Figure 1, which results in a faster increase of the cost functions (Figure 10 compared to Figure 2) after they reach their minimum values.

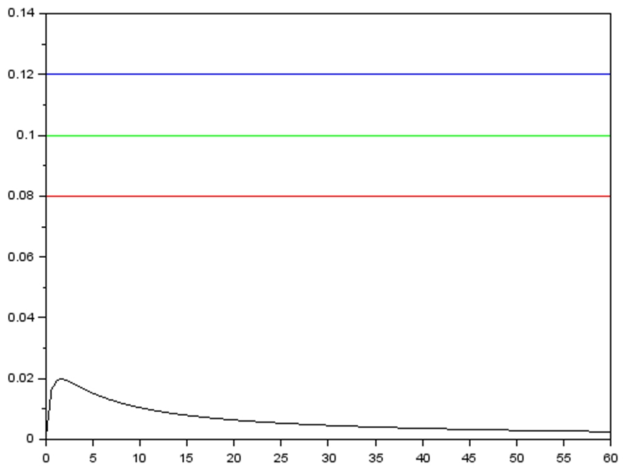

Figure 7. Repair rate function for $w(x) = [\ln(x + 1)]^2$, $v(x) = 0$, $b = 36$.

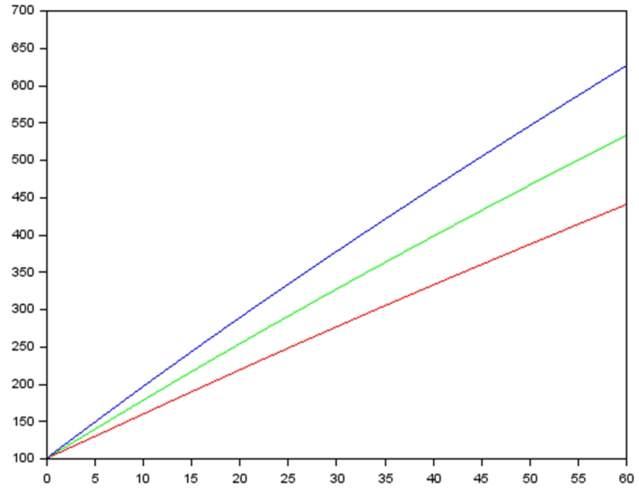

Figure 8. Cost functions for $w(x) = [\ln(x + 1)]^2$, $v(x) = 0$, $b = 36$, $c = 8$ (red), $c = 10$ (green), $c = 12$ (blue), $R = 100$.

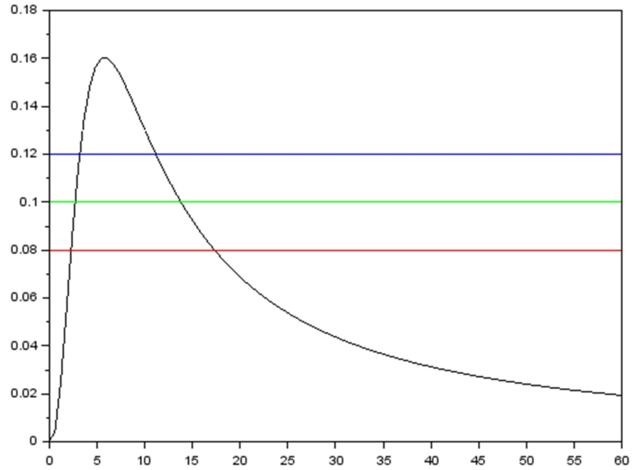

Figure 9. Repair rate function for $w(x) = [\ln(x + 1)]^5$, $v(x) = 0$, $b = 36$.

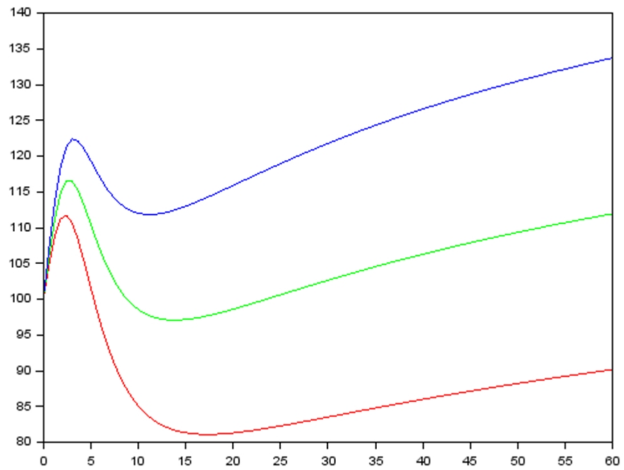

Figure 10. Cost functions for $w(x) = [\ln(x + 1)]^5$, $v(x) = 0$, $b = 36$, $c = 8$ (red), $c = 10$ (green), $c = 12$ (blue), $R = 100$.

5. Moments of P/Q distributions

The following lemma specifies conditions under which a P/Q distributed random variable has finite moments, needed e.g. for the purpose of estimating its parameters, which issue is addressed in Section 7.

Lemma 3
The random variable X whose CDF is given by (12) has finite k-th moment if and only if $\deg(w) > 1$ and $\deg(w) - \deg(v) > k$.

Proof: Let $\deg(v) > 0$. According to (1) and (9), we have:

$$E(X^k) = k\int_0^\infty x^{k-1}\frac{v(x)+\beta}{w(x)+v(x)+\beta} >$$

$$> k\int_0^\infty x^{k-1}\frac{v(x)}{w(x)+v(x)+\beta}dx =$$

$$= k\int_0^\infty \left(\frac{w(x)}{v(x)x^{k-1}} + \frac{1}{x^{k-1}} + \frac{\beta}{v(x)x^{k-1}}\right)^{-1} dx >$$

$$> k\int_1^\infty \left(\frac{w(x)}{v(x)x^{k-1}} + \frac{1}{x^{k-1}} + \frac{\beta}{v(x)x^{k-1}}\right)^{-1} dx. \quad (22)$$

Let $\deg(w) - \deg(v) = k$, i.e.

$$w(x) = \sum_{j=1}^{n}\alpha_j x^j,$$

$$v(x) = \sum_{j=1}^{n-k}\beta_j x^j. \quad (23)$$

The following sequence of inequalities holds for $x \geq 1$:

$$\frac{w(x)}{v(x)x^{k-1}} = \frac{\alpha_1 x+\cdots+\alpha_{n-1}x^{n-1}+\alpha_n x^n}{\beta_1 x^k+\cdots+\beta_{n-k}x^{n-1}} <$$

$$< \frac{\alpha_1 x+\cdots+\alpha_{n-1}x^{n-1}+\alpha_n x^n}{\beta_{n-k}x^{n-1}} =$$

$$= \frac{\alpha_1 x^{2-n}+\cdots+\alpha_{n-2}x^{-1}+\alpha_{n-1}+\alpha_n x}{\beta_{n-k}} <$$

$$< \frac{\alpha_1+\cdots+\alpha_{n-2}+\alpha_{n-1}+\alpha_n x}{\beta_{n-k}}. \quad (24)$$

It also holds that

$$\frac{1}{x^{k-1}} + \frac{\beta}{v(x)x^{k-1}} =$$

$$= \frac{1}{x^{k-1}} + \frac{\beta}{\beta_1 x^k+\cdots+\beta_{n-k}x^{n-1}} <$$

$$< 1 + \frac{\beta}{\beta_1+\cdots+\beta_{n-k}}. \quad (25)$$

In view of (24) and (25), if $x \geq 1$ then

$$\left(\frac{w(x)}{v(x)x^{k-1}} + \frac{1}{x^{k-1}} + \frac{\beta}{v(x)x^{k-1}}\right)^{-1} > \frac{1}{cx+d} \quad (26)$$

where

$$c = \frac{\alpha_n}{\beta_{n-k}}$$

$$d = \frac{\alpha_1+\cdots+\alpha_{n-1}}{\beta_{n-k}} + 1 + \frac{\beta}{\beta_1+\cdots+\beta_{n-k}}. \quad (27)$$

From (22) and (26) we obtain

$$E(X^k) > k\int_1^\infty \frac{1}{cx+d}dx =$$

$$= \frac{k}{c}\ln(cx+d)\Big|_1^\infty = \infty \quad (28)$$

which means that the k-th moment is infinite.
Let now $\deg(w) - \deg(v) = k + 1$, i.e.

$$w(x) = \sum_{j=1}^{n}\alpha_j x^j,$$

$$v(x) = \sum_{j=1}^{n-k-1}\beta_j x^j. \quad (29)$$

We then have:

$$E(X^k) = k\int_0^\infty x^{k-1}\frac{v(x)+\beta}{w(x)+v(x)+\beta}dx =$$

$$= k\int_0^1 x^{k-1}\frac{v(x)+\beta}{w(x)+v(x)+\beta}dx +$$

$$+k\int_1^\infty x^{k-1}\frac{v(x)+\beta}{w(x)+v(x)+\beta}dx <$$

$$< k\int_0^1 \frac{\beta_1+\cdots+\beta_{n-k-1}+\beta}{\beta}dx +$$

$$+k\int_1^\infty x^{k-1}\frac{v(x)+\beta}{w(x)}dx \leq$$

$$\leq k\frac{\beta_1+\cdots+\beta_{n-k-1}+\beta}{\beta} +$$

$$+k\int_1^\infty \frac{\beta_1 x^k+\cdots+\beta_{n-k-1}x^{n-2}+\beta}{\alpha_n x^n}dx. \quad (30)$$

If $x \geq 1$ then

$$\beta_1 x^k + \dots + \beta_{n-k-1} x^{n-2} \leq$$

$$\leq (\beta_1 + \dots + \beta_{n-k-1}) x^{n-2}. \tag{31}$$

Let us note that $n - 2 \geq k$, because $n = \deg(w) = \deg(v) + k + 1$ and $\deg(v) \geq 1$. From (30) and (31) we obtain:

$$E(X^k) < k\left(\frac{c}{\beta} + 1\right) +$$

$$+ k \int_1^\infty \left(\frac{c}{\alpha_n x^2} + \frac{\beta}{\alpha_n x^n}\right) dx \tag{32}$$

where $c = b_1 + \dots + b_{n-k-1}$. The integral in (32) is finite, as follows from the integration rules for power functions. Thus, the k-th moment of X is also finite.

Remark 1: The proof in the case $\deg(v) = 0$ is similar, but simpler.
Remark 2: Lemma 2 holds true for non-integer powers of x in $w(x)$ and $v(x)$. E.g. $E(X) < \infty$ for $w(x) = x^{2.5}$ and $v(x) = x^{1.4}$. The proof is analogous to that for integer values.

6. Pareto distribution as P/Q one

According to Arnold (Arnold, 2015), a nonnegative random variable X with the following survival function:

$$\Pr(X > x) = \left(\frac{\beta}{x+\beta}\right)^\alpha \tag{33}$$

has a Pareto distribution with the scale parameter b and the shape parameter a, where both parameters are positive. This type of distribution is frequently used in financial mathematics and for modeling natural and social phenomena. Let us note that if $a = 1$ then a Pareto distributed random variable has CDF of the P/Q type. Indeed, in view of (13), if $a = 1$ then $w(x) = x$ and $v(x) = 0$.

7. Fitting P/Q distribution to sample data

Selected P/Q distributions can be fitted to sample data by means of a method developed by the author and called *method of curtailed moments*. The standard method of moments consists in estimating the unknown parameters $q_1,\dots,q_k$ of a random variable X by solving the set of equations

$$h_j(\hat{\theta}_1, \dots, \hat{\theta}_k) = \frac{1}{n}\sum_{i=1}^n x_i^j,\ j = 1, \dots, k \tag{34}$$

where q_j with caret is the estimator of the unknown parameter q_j, the functions h_j, $j = 1,\dots,n$ are such that $E(X^j)=h_j(q_1,\dots,q_k)$, and $x_1,\dots,x_n$ is a random sample from X. The right hand-side of (34) is the k-th raw sample moment that approximates $E(X^j)$. Note that the first k moments of X have to be finite so that the standard method of moments can be applied (see Lemma 3 in Section 5).
The method of curtailed moments uses the approximation of $E[\min(X^j,L^j)]$ instead of $E(X^j)$, where L is an arbitrarily chosen large number. It can be applied when X has infinite moments, which rules out the use of the regular method. The set of equations (34) is replaced with the following one:

$$h_j(\hat{\theta}_1, \dots, \hat{\theta}_k, L) = \frac{1}{n}\sum_{i=1}^n \min(x_i^j, L^j), \tag{35}$$

for $j = 1, \dots, k$.

As an example, let us consider the P/Q distribution with the following CDF

$$F(x) = \frac{x^k}{x^k+\beta} \tag{36}$$

where $k \geq 1$ and $b > 0$. In view of (6) we have:

$$E[\min(X^k, T^k)] = \int_0^L kx^{k-1} \frac{\beta}{x^k+\beta} dx =$$

$$= \beta[\ln(L^k + \beta) - \ln(\beta)] = \beta \ln\left(\frac{L^k+\beta}{\beta}\right). \tag{37}$$

The distribution given by (36) has one parameter b which can be estimated by b with caret found from the following equation:

$$h(\hat{\beta}, L) = \frac{1}{n}\sum_{i=1}^n \min(x_i^k, L^k), \tag{38}$$

where

$$h(\beta, L) = \beta \ln\left(\frac{L^k+\beta}{\beta}\right). \tag{39}$$

The equation (38) cannot be solved analytically with respect to b caret, thus a numerical method (e.g. the Newton-Raphson one) has to be used to estimate b.

8. Modification of P/Q repair rate function

As implied by (16) and (19), the repair rate function of a P/Q distribution or one with poly-logarithmic CDF converges to 0 in infinity. This can be seen in the graphs in Section 4. In consequence, the longer the duration of a repair, the smaller the chance that it will be completed. For example, such a situation is possible if the repair time is extended due to increasingly occurring unpredicted complications, but is rather unlikely to occur if a repair is carried out following a standard procedure. Thus, in many cases, the appropriately shaped repair rate function should converge in infinity to a non-zero positive value. Quite fortunately, the repair rate function of a P/Q distribution can be easily modified so as to satisfy this condition and remain analytically tractable. These criteria are met by the following repair rate function:

$$r_1(x) = r(x) + \frac{\gamma x}{\delta + x}, \tag{40}$$

where $r(x)$ is given by (16) while g and d are appropriately selected positive constants. It is easy to check that $r_1(0) = 0$ and $r_1(x)$ converges to g as $x \to \infty$.

Let us now derive a formula for $S_1(x) = 1 - F_1(x)$, where $F_1(x)$ is the CDF corresponding to $r_1(x)$. Since

$$r_1(x) = \frac{F_1'(x)}{1 - F_1(x)} = -\frac{d \ln\left(1 - F_1(x)\right)}{dx}, \tag{41}$$

we have:

$$\begin{aligned} S_1(x) &= \exp\left[-\int_0^x r_1(u)du\right] = \\ &= \exp\left[-\int_0^x r(u)du\right] \exp\left[-\int_0^x \frac{\gamma u}{\delta + u} du\right] = \\ &= S(x) \exp\left[-\gamma \int_0^x \left(1 - \frac{\delta}{\delta + u}\right) du\right] = \\ &= S(x) \exp(-\gamma x) \exp\left[\gamma\delta \int_0^x \frac{1}{\delta + u} du\right] = \\ &= S(x) \exp(-\gamma x) \exp\left(\gamma\delta \ln \frac{\delta + x}{\delta}\right), \end{aligned} \tag{42}$$

where $S(x)$ is given by (13). As shown in Section 2, the formula for $S_1(x)$ is needed to compute the expected post-failure maintenance cost and find the optimal value of L.

9. Conclusion

In the presented chapter, the problem of optimizing a repair/replacement policy for a two-state repairable system has been addressed. Depending on the properties of the repair rate function and the values of the cost parameters, the following options are possible:

- replace the system immediately after a failure,
- interrupt a repair when its duration exceeds a certain time limit,
- continue a repair until its completion.

The choice of the optimal option is based on Lemma 2 or, if this lemma is inapplicable, on the analysis of the cost function. The chapter focuses on systems with P/Q distributed repair times, i.e. the CDF of such a system's TTR is a rational function, where the polynomials in the numerator and denominator satisfy the condition (15). Thus defined CDF can be modified so that the corresponding repair rate function has a non-zero limit at infinity (see Section 8). The justification for considering P/Q distributed repair times is given in the last paragraph of the Introduction. If there are indications that the repair time follows a P/Q distribution, one may need to estimate its parameters. One such estimation method, developed by the author, is demonstrated in Section 7. In general, estimation of P/Q distributions and their modifications seems to be an interesting and practice-oriented area of research and will be a topic of future work.

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