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FACIAL GRACEFUL COLORING OF PLANE GRAPHS

Július Czap

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Abstract. Let G be a plane graph. Two edges of G are facially adjacent if they are consecutive on the boundary walk of a face of G. A facial edge coloring of G is an edge coloring such that any two facially adjacent edges receive different colors. A facial graceful k-coloring of G is a proper vertex coloring $c : V(G) \to \{1, 2, \dots, k\}$ such that the induced edge coloring $c' : E(G) \to \{1, 2, \dots, k-1\}$ defined by $c'(uv) = |c(u) - c(v)|$ is a facial edge coloring. The minimum integer k for which G has a facial graceful k-coloring is denoted by $\chi_{fg}(G)$. In this paper we prove that $\chi_{fg}(G) \leq 14$ for every plane graph G and $\chi_{fg}(H) \leq 9$ for every outerplane graph H. Moreover, we give exact bounds for cacti and trees.

Keywords: facial edge coloring, plane graph.

Mathematics Subject Classification: 05C10, 05C15.

1. INTRODUCTION

In 1967, Alexander Rosa published a paper [17] which has served as a starting point and motivation for many interesting problems (see e.g. [9]). One of these problems is the following. A proper vertex coloring $c : V(G) \to \{1, 2, \dots, k\}$ of a graph G is called a graceful k-coloring if the edge coloring c' defined by $c'(uv) = |c(u) - c(v)|$ for every edge uv of G is also proper. The minimum integer k for which G has a graceful k-coloring is called the graceful chromatic number of G, denoted by $\chi_g(G)$. The task is to determine $\chi_g(G)$ for a given graph G. This concept was introduced by Bi *et al.* [3, 4] in 2017 and has attracted quite some attention since then. The parameter $\chi_g(G)$ is well defined because if we color the vertices of an n-vertex graph with colors $2^0, 2^1, 2^2, \dots, 2^{n-1}$ we obtain a graceful coloring. Observe that, if G is a graph with maximum degree Δ, then $\chi_g(G) \geq \Delta + 1$. A lower bound for $\chi_g(G)$ in terms of the minimum degree of G was obtained by English and Zhang [7]. If G is a graph with minimum degree $\delta \geq 2$, then $\chi_g(G) \geq \lceil \frac{5}{3}\delta \rceil$. In [3], it is shown that if G is a graph of order n such that $\delta(G) > \frac{n}{2}$, then $\chi_g(G) > n$.

Graceful colorings of trees have been intensively studied. It is known that if G is a tree with maximum degree Δ, then $\chi_g(G) \leq \lceil \frac{5}{3}\Delta \rceil$ and this bound is the best possible, see [3, 7]. Laavanya and Yamini [13] determined the exact values of $\chi_g(G)$ for trees with maximum degree 4. Bi *et al.* [3] determined the graceful chromatic number of all caterpillars. English and Zhang [7] derived the exact values of the graceful chromatic number for a class of rooted trees.

Suparta *et al.* [20] investigated the graceful chromatic number of the Cartesian products $C_m \times P_n$ for $m \geq 3$, $n \geq 2$, and $C_m \times C_n$ for $m, n \geq 3$, where C_m and P_m denote the cycle and the path with m vertices, respectively. Kristiana *et al.* [12] investigated the graceful chromatic number of some generalized Petersen graphs. Graceful colorings of special graphs can be found in conference papers [1, 2, 10, 14, 16].

If a graceful coloring satisfies the additional property that every induced edge color is odd, then it is called an odd-graceful coloring. Such a coloring was studied by Suparta *et al.* [19]. They showed that if a graph admits an odd-graceful coloring, then it is bipartite. They derived upper bounds for the odd-graceful chromatic number of caterpillars, ladders, and prisms.

In this paper we consider facial graceful coloring of plane graphs. A graph is planar, if it can be drawn in the plane so that its edges intersect only at their endvertices. A plane graph is a particular drawing of a planar graph in the Euclidean plane $\mathbb{R}^2$ such that no edges intersect except at their endvertices. Let G be a plane graph with vertex set $V(G)$ and edge set $E(G)$. The connected components of $\mathbb{R}^2 \setminus G$ form the set $F(G)$ of faces of G. Each plane graph has exactly one unbounded face, called the outer face. Outerplane graphs are plane graphs such that every vertex is incident with the outer face. The rotation at a vertex v of G is the clockwise order of its incident edges. Two edges e_1, e_2 of G are facially adjacent if they have a common endvertex, say v, and they are consecutive in the rotation at v. A facial edge coloring of a plane graph G is an edge coloring such that no two facially adjacent edges are assigned the same color. Facial edge coloring was studied already in the last century (see e.g. [18]) but serious progress in this area has been achieved only in the last years, see [5, 6]. Observe that the classical proper edge coloring and the facial edge coloring coincide in the class of subcubic plane graphs.

A facial graceful k-coloring of a plane graph G is a proper vertex coloring $c : V(G) \to \{1, 2, \dots, k\}$ such that the induced edge coloring $c' : E(G) \to \{1, 2, \dots, k-1\}$ defined by $c'(uv) = |c(u) - c(v)|$ is a facial edge coloring. The minimum integer k for which G has a facial graceful k-coloring is called the facial graceful chromatic number of G, denoted by $\chi_{fg}(G)$. By the above mentioned observation we have $\chi_{fg}(G) = \chi_g(G)$ for every plane graph with maximum degree at most 3.

In this paper we prove that $\chi_{fg}(G) \leq 14$ for every plane graph G and $\chi_{fg}(H) \leq 9$ for every outerplane graph H. Moreover, we obtain tight upper bounds for the facial graceful chromatic number in the classes of cacti and trees.

2. GENERAL UPPER BOUND

A vertex coloring of a plane graph is an ℓ-facial coloring if any two distinct vertices on a facial walk of length ℓ have distinct colors. Notice that 1-facial coloring is the usual proper coloring. Observe that any facial graceful coloring is a 2-facial coloring, but not every 2-facial coloring is a facial graceful one. The tree T depicted in Figure 1 has a 2-facial coloring with colors $1, 2, 3, 4$, but no such coloring is a facial graceful one. In any 2-facial coloring of T, the vertices a, b, c, d receive different colors and no such coloring (with colors $1, 2, 3, 4$) can be extended to a facial graceful coloring of the whole tree T.

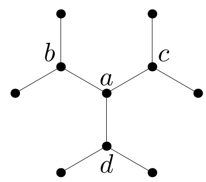

Fig. 1. The tree T

Lemma 2.1. *Let c be a vertex coloring of a plane graph G. Then c is a facial graceful coloring of G if and only if*

(i) *c is a 2-facial coloring, and*
(ii) *$c(y) \neq \frac{c(x)+c(z)}{2}$ for any two facially adjacent edges xy and yz.*

Proof. Let xy and yz be two facially adjacent edges of G. Let c be a vertex coloring of G.

If c is a facial graceful coloring, then $c(x) \neq c(y)$ and $c(y) \neq c(z)$, since c is a proper coloring. Moreover, $c(x) \neq c(z)$, otherwise $|c(x) - c(y)| = |c(y) - c(z)|$. So c is a 2-facial coloring. Next, $c(y) \neq \frac{c(x)+c(z)}{2}$, since otherwise $|c(x)-c(y)| = |c(y)-c(z)|$.

Now assume that c satisfies the conditions (i) and (ii). By (i), c is a proper coloring. Suppose to the contrary that $|c(x) - c(y)| = |c(y) - c(z)|$.

First assume that $c(y) > \max\{c(x), c(z)\}$. In this case

$$c(y) - c(x) = |c(x) - c(y)| = |c(y) - c(z)| = c(y) - c(z),$$

which implies $c(x) = c(z)$, a contradiction.

If $c(y) < \min\{c(x), c(z)\}$, then

$$c(x) - c(y) = |c(x) - c(y)| = |c(y) - c(z)| = c(z) - c(y),$$

which implies $c(x) = c(z)$, a contradiction.

Finally, without loss of generality, assume that $c(x) < c(y)$ and $c(z) > c(y)$. In this case $c(y) - c(x) = |c(x) - c(y)| = |c(y) - c(z)| = c(z) - c(y)$, which implies $c(y) = \frac{c(x)+c(z)}{2}$, a contradiction. □

A subset $\{a_1, a_2, \ldots\}$ of the set $\{1, 2, \ldots, n\}$ is called 3-AP-free if it does not contain any three elements a_p, a_q, a_r such that $a_p - a_q = a_q - a_r$, i.e. it does not contain any three consecutive members of an arithmetic progression. By $r(n)$ we denote the cardinality of the largest such subset. The study of the function $r(n)$ was initiated by Erdős and Turán [8] in 1936, and since the study of $r(n)$ has attracted a lot of attention. Nevertheless, the exact value of $r(n)$ is known only for a few n. Table 1 contains the values of $r(n)$ and also an example of the largest 3-AP-free set for $n \leq 14$.

Table 1

n	$r(n)$	one of the largest 3-AP-free sets
1	1	$\{1\}$
$2, 3$	2	$\{1, 2\}$
4	3	$\{1, 2, 4\}$
$5, 6, 7, 8$	4	$\{1, 2, 4, 5\}$
$9, 10$	5	$\{1, 2, 4, 8, 9\}$
$11, 12$	6	$\{1, 2, 4, 5, 10, 11\}$
13	7	$\{1, 2, 4, 5, 10, 11, 13\}$
14	8	$\{1, 2, 4, 5, 10, 11, 13, 14\}$

In general, the largest 3-AP-free sets are not necessarily unique. For example, there are six such sets, namely $\{1, 2, 4, 5, 10, 11, 13\}$, $\{1, 2, 4, 5, 10, 12, 13\}$, $\{1, 2, 4, 8, 10, 11, 13\}$, $\{1, 2, 4, 9, 10, 12, 13\}$, $\{1, 3, 4, 6, 10, 12, 13\}$, and $\{1,3,4,9,10,12,13\}$, for $n = 13$. On the other hand, the largest 3-AP-free set is unique for $n = 14$.

Lemma 2.2. *Let c be a 2-facial coloring of a plane graph G. If the set of colors used by c is 3-AP-free, then c is a facial graceful coloring.*

Proof. Suppose that c is a 2-facial coloring, the set of colors used by c is 3-AP-free, but c is not a facial graceful coloring. Then, by Lemma 2.1, $c(y) = \frac{c(x)+c(z)}{2}$ for some two facially adjacent edges xy and yz. This implies that $c(x) - c(y) = c(y) - c(z)$, so the colors $c(x)$, $c(y)$, $c(z)$ form an arithmetic progression, a contradiction. □

Theorem 2.3. *If G is a plane graph, then $\chi_{fg}(G) \leq 14$.*

Proof. Kráľ, Madaras, and Škrekovski [11] proved that every plane graph admits a 2-facial coloring with at most 8 colors. If we use the colors from the 3-AP-free set $\{1, 2, 4, 5, 10, 11, 13, 14\}$ in a 2-facial coloring of G, then, by Lemma 2.2, we obtain a facial graceful coloring. □

If a plane graph G contains the configuration depicted in Figure 2, then $\chi_{fg}(G) \geq 7$. Consequently, there is a plane graph G such that $\chi_{fg}(G) = 7$.

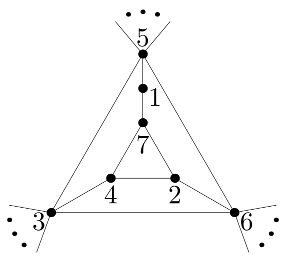

Fig. 2. A configuration with no facial graceful 6-coloring

Problem 2.4. Is there a plane graph G such that $\chi_{fg}(G) \geq 8$?

Theorem 2.5. *If G is an outerplane graph, then $\chi_{fg}(G) \le 9$.*

Proof. Montassier and Raspaud [15] proved that every outerplane graph has a 2-facial coloring using at most 5 colors. If we use the colors from the 3-AP-free set $\{1, 2, 4, 8, 9\}$ in a 2-facial coloring of G, then we obtain a facial graceful coloring. □

By Theorem 3.4, there are outerplane graphs with $\chi_{fg}(G) = 6$.

Problem 2.6. Is there an outerplane graph G such that $\chi_{fg}(G) \ge 7$?

3. FACIAL GRACEFUL COLORING OF CACTI

An edge of a plane graph not incident with the outer face is called inner edge. A cactus is a connected outerplane graph with no inner edges (i.e., a connected outerplane graph in which any two cycles have at most one vertex in common). A vertex v of a graph G is a cut-vertex if $G - v$ has more components than G.

Theorem 3.1 ([3])**.** *Let C_n be the cycle on $n \ge 3$ vertices. Then $\chi_{fg}(C_5) = 5$ and $\chi_{fg}(C_n) = 4$ for $n \ne 5$.*

Corollary 3.2. *In every facial graceful 5-coloring of C_5 all five colors $1, 2, 3, 4, 5$ appear on the vertices of C_5. If $n \ne 5$, then in every facial graceful 4-coloring of C_n the colors 1 and 4 appear on some vertices.*

Lemma 3.3. *Let C_n be the cycle on $n \ge 3$ vertices and let $i \in \{1, 2, 3, 4, 5, 6\}$. Then there is a facial graceful coloring $c : V(C_n) \to \{1, 2, 3, 4, 5, 6\}$ such that at least one vertex receives color i.*

Proof. Let $C_n = v_1v_2 \dots v_nv_1$ and let c be a facial graceful coloring of C_n. Observe that the coloring c' defined by $c'(v_i) = c(v_i) + 1$ is also a facial graceful one. Hence, Corollary 3.2 guarantees the existence of a required coloring. □

Theorem 3.4. *If G is a cactus, then $\chi_{fg}(G) \le 6$. Moreover, this bound is tight.*

Proof. Suppose there is a counterexample to Theorem 3.4. Let G be a counterexample with the minimum number of vertices.

First assume that the minimum degree of G is 1. Let u be a vertex of degree 1 in G and let v be its neighbor. Let vx and vy be the edges facially adjacent to vu ($x = y$ if the vertex v has degree two). The graph $G - u$ is a cactus with fewer vertices than T, so it admits a facial graceful coloring c with colors $1, 2, \dots, 6$. Let i be a color from the set $\{1, 2, 3\} \setminus \{|c(v) - c(x)|, |c(v) - c(y)|\}$. Let $k \in \{1, 2, \dots, 6\}$ be a color which satisfies $|c(v) - k| = i$. The facial graceful coloring c of $G - u$ can be extended to a facial graceful coloring of G by coloring u with color k, a contradiction.

Now assume that G has no vertices of degree 1. By Theorem 3.1, G is not a cycle. Consequently, G contains a cycle $C = v_1v_2 \dots v_nv_1$ which is incident with exactly one cut-vertex, say u, in G. Without loss of generality, assume that $u = v_1$. Let r be the edge facially adjacent to v_1v_2 different from v_1v_n, v_2v_3 and let s be the edge facially adjacent to v_1v_n different from $v_1v_2, v_{n-1}v_n$, see Figure 3 for illustration.

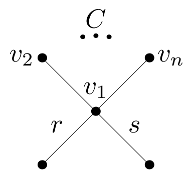

Fig. 3. A configuration in G

Let H be the graph obtained from G by removing the vertices and edges incident with C except for v_1. The graph H has fewer vertices than G so it has a facial graceful coloring c with colors $1, 2, \ldots, 6$. In the following we extend this coloring to a facial graceful coloring of G. We distinguish two cases according to the degree of v_1 in G.

First assume that v_1 has degree at least four in G. In this case the edges r, s have different colors, since they are facially adjacent in H. By Lemma 3.3, the cycle C has a facial graceful coloring $c' : V(C_n) \to \{1, 2, 3, 4, 5, 6\}$ such that $c'(v_1) = c(v_1)$. If neither v_1v_2 and r nor v_1v_n and s have the same color, then the colorings c of H and c' of C_n define a facial graceful coloring of G. Otherwise, we recolor the cycle C in the following way: $c''(v_1) = c'(v_1)$, $c''(v_i) = c'(v_{n+2-i})$ for $i \geq 2$. Now, neither v_1v_2 and r nor v_1v_n and s have the same color. Therefore, c and c'' give a facial graceful coloring of G.

Finally, assume that v_1 has degree 3 in G. Let e be the edge incident with v_1 in H. First we modify the facial graceful coloring of H. At least one of the colors $1, 2, 3$ does not appear on the edges facially adjacent to e in H. We choose one such color and recolor e with the chosen color, and then we recolor v_1 in order to obtain a facial graceful coloring of H. Clearly, this is always possible. Now, it is sufficient to show that this new coloring of H can be extended to a facial graceful coloring of G. We color the vertices $v_1, v_2, \ldots, v_n$ (in this order) in the following way. Assume that $n = 3k + i$, where $i \in \{0, 1, 2\}$. In each of the following cases, first we use the pattern P_{3+i} and then $(k - 1)$-times the pattern P_3.

Case 1. The color of e is 1.
Case 1.1. If the color of v_1 is 1, then $P_3 = 1, 4, 3$, $P_4 = 1, 4, 2, 3$, and $P_5 = 1, 4, 5, 2, 3$.
Case 1.2. If the color of v_1 is 2, then $P_3 = 2, 5, 4$, $P_4 = 2, 5, 3, 4$, and $P_5 = 2, 5, 6, 3, 4$.
Case 1.3. If the color of v_1 is 3, then $P_3 = 3, 6, 5$, $P_4 = 3, 6, 1, 5$, and $P_5 = 3, 6, 1, 2, 5$.
Case 1.4. If the color of v_1 is 4, then $P_3 = 4, 1, 2$, $P_4 = 4, 1, 3, 2$, and $P_5 = 4, 1, 3, 6, 2$.
Case 1.5. If the color of v_1 is 5, then $P_3 = 5, 2, 3$, $P_4 = 5, 2, 4, 3$, and $P_5 = 5, 2, 1, 4, 3$.
Case 1.6. If the color of v_1 is 6, then $P_3 = 6, 3, 4$, $P_4 = 6, 3, 1, 4$, and $P_5 = 6, 3, 1, 5, 4$.

Case 2. The color of e is 2.
Case 2.1. If the color of v_1 is 1, then $P_3 = 1, 4, 2$, $P_4 = 1, 4, 6, 2$, and $P_5 = 1, 4, 3, 6, 2$.
Case 2.2. If the color of v_1 is 2, then $P_3 = 2, 5, 3$, $P_4 = 2, 5, 1, 3$, and $P_5 = 2, 5, 6, 1, 3$.
Case 2.3. If the color of v_1 is 3, then $P_3 = 3, 6, 4$, $P_4 = 3, 6, 1, 4$, and $P_5 = 3, 6, 1, 2, 4$.
Case 2.4. If the color of v_1 is 4, then $P_3 = 4, 1, 3$, $P_4 = 4, 1, 6, 3$, and $P_5 = 4, 1, 2, 6, 3$.
Case 2.5. If the color of v_1 is 5, then $P_3 = 5, 2, 4$, $P_4 = 5, 2, 6, 4$, and $P_5 = 5, 2, 1, 6, 4$.
Case 2.6. If the color of v_1 is 6, then $P_3 = 6, 3, 5$, $P_4 = 6, 3, 2, 5$, and $P_5 = 6, 3, 4, 2, 5$.

Case 3. The color of e is 3.
Case 3.1. If the color of v_1 is 1, then $P_3 = 1, 5, 2$, $P_4 = 1, 5, 4, 2$, and $P_5 = 1, 5, 4, 6, 2$.
Case 3.2. If the color of v_1 is 2, then $P_3 = 2, 4, 1$, $P_4 = 2, 4, 3, 1$, and $P_5 = 2, 4, 3, 6, 1$.
Case 3.3. If the color of v_1 is 3, then $P_3 = 3, 4, 1$, $P_4 = 3, 4, 2, 1$, and $P_5 = 3, 4, 2, 5, 1$.
Case 3.4. If the color of v_1 is 4, then $P_3 = 4, 2, 5$, $P_4 = 4, 2, 3, 5$, and $P_5 = 4, 2, 3, 1, 5$.
Case 3.5. If the color of v_1 is 5, then $P_3 = 5, 3, 6$, $P_4 = 5, 3, 4, 6$, and $P_5 = 5, 3, 4, 1, 6$.
Case 3.6. If the color of v_1 is 6, then $P_3 = 6, 2, 5$, $P_4 = 6, 2, 3, 5$, and $P_5 = 6, 2, 3, 1, 5$.

In each of these cases we obtain a facial graceful coloring of G.

Now we show that there are infinitely many cacti with no facial graceful 5-coloring. Let G be a cactus which contains a cycle $C_5 = v_1v_2v_3v_4v_5v_1$ on five vertices, and let all of these vertices have degree 3 in G. Suppose that G admits a facial graceful 5-coloring. Any two vertices of C_5 are on a facial walk of length at most 2, therefore no two of them have the same color. Consequently, all five colors $1, 2, 3, 4, 5$ appear on the vertices of C_5. Let v_i be the vertex of color 3. Then all the edges incident with v_i have color either 1 or 2 ($3-1 = 5-3 = 2$ and $3-2 = 4-3 = 1$). The vertex v_i has degree 3 and hence at least two of the incident edges have the same color, a contradiction. □

4. FACIAL GRACEFUL COLORING OF TREES

Lemma 4.1. *Every tree admits a 2-facial coloring with at most 4 colors.*

Proof. Suppose there is a counterexample to Lemma 4.1. Let T be a counterexample with the minimum number of vertices. Clearly, T has at least five vertices. Let u be a leaf of T and let v be its neighbor. Let vx and vy be the edges facially adjacent to vu ($x = y$ if the vertex v has degree two). The tree $T - u$ has fewer vertices than T, so it admits a 2-facial coloring c with at most 4 colors. If we color the vertex u with a color distinct from $c(v), c(x), c(y)$, then we obtain a 2-facial coloring of T using at most 4 colors, a contradiction. □

Theorem 4.2. *If T is a tree, then $\chi_{fg}(T) \leq 5$. Moreover, the bound is tight.*

Proof. By Lemma 4.1, T admits a 2-facial coloring with at most 4 colors. If we use the colors $1, 2, 4, 5$ in a 2-facial coloring of T, then, by Lemma 2.2, we obtain a facial graceful coloring of T.

Now we show that the bound is tight. Consider a tree T such that it contains a vertex v of degree 3. Let v_1, v_2, v_3 be its neighbors. If v_1, v_2, v_3 have degree 3, then $\chi_{fg}(T) = 5$. Suppose that T admits a facial graceful 4-coloring. Observe that one of the vertices v, v_1, v_2, v_3 receives color 3. Then the incident edges have color 1 or 2. Consequently, at least two of them have the same color, a contradiction. □

4.1. FACIAL GRACEFUL 3-COLORING

Theorem 3.1 implies that if $\chi_{fg}(G) = 3$ and G is connected, then G is a tree. In the following we characterize all trees which admit a facial graceful 3-coloring.

Any tree in this paper is embedded in the plane. The particular embedding is very important. The tree depicted in Figure 4 with the embedding on the left has a facial graceful 3-coloring, and with the embedding on the right, its facial graceful chromatic number is at least 4 (see Theorem 4.5).

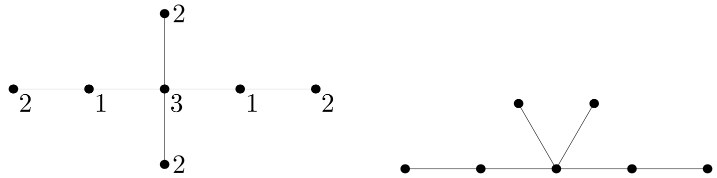

Fig. 4. Two different embeddings of the same tree

Lemma 4.3. *Let T be a tree on at least three vertices. Then T admits a facial edge coloring with two colors if and only if T has no vertex of degree $2k+1$ for $k \geq 1$.*

Proof. First assume that every inner vertex (i.e. vertices of degree at least two) has an even degree. If T is a star, then we color the edges incident with the central vertex with colors a and b alternately. If T is not a star, then pick any vertex of T to be the root. We color the edges of T starting from the root to the leaves. In each step it is sufficient to find a suitable edge coloring of a star with one precolored edge.

Now assume that T has a vertex v of degree $2k+1$ for some $k \geq 1$. In this case the star with central vertex v has no facial edge coloring with two colors, so the same holds for T. □

Lemma 4.4. *Let T be a tree which admits a facial graceful 3-coloring c. Then*

(i) *every vertex of degree at least two receives color either 1 or 3 under c, and*
(ii) *only the leaves have odd degree in T.*

Proof. If a vertex v receives color 2 in a facial graceful 3-coloring of T, then it is a leaf, since every edge incident with v has color 1 ($3-2=2-1=1$).

The vertex coloring c of T uses the colors $1,2,3$, therefore, the induced facial edge coloring uses only the colors $1,2$. Consequently, the degree of every inner vertex of T must be even, see Lemma 4.3. □

Theorem 4.5. *Let T be a tree on at least three vertices and let T^- be the tree obtained from T by removing all leaves. Then $\chi_{fg}(T)=3$ if and only if*

(i) *T admits a facial edge coloring c with two colors, and*
(ii) *T^- is monochromatic under c.*

Proof. First assume that $\chi_{fg}(T)=3$. By Lemma 4.4, only the leaves have odd degree in T. Lemma 4.3 implies that T admits a facial edge coloring with two colors. Observe that this facial edge 2-coloring is unique up to permutation of the colors. This facial edge 2-coloring of T is also given by a facial graceful 3-coloring of T. By Lemma 4.4, the vertices of T^- are colored with 1 and 3, therefore all of its edges have the same color.

Now assume that (i) and (ii) hold. Let c be a facial edge coloring of T with colors x and y. Assume that the edges of T^- have color x. We define a facial graceful 3-coloring of T in the following way. First we color the vertices of degree at least two, thereafter the leaves. T^- is a tree so it has a proper vertex coloring with two colors. We use colors 1 and 3. This proper vertex coloring of T^- induces a partial vertex coloring of T. It remains to color the leaves of T. Let v be a vertex of T of degree $k \geq 2$ and let $e_1, e_2, \ldots, e_k$ be the edges incident with v, listed in their clockwise order around v. By Lemma 4.3, the degree of v in T is even. Without loss of generality, assume that the edges $e_1, e_3, \ldots, e_{k-1}$ have color x and the edges $e_2, e_4, \ldots, e_k$ have color y. Let v_i be the second endvertex of e_i. The vertices $v_2, v_4, \ldots, v_k$ are leaves, since the edges $e_2, e_4, \ldots, e_k$ have color y. We color these leaves with color 2. If the color of v is 1 (3), then we color the vertices $v_1, v_3, \ldots, v_{k-1}$ with color 3 (1). Note that if a vertex v_i, $i \in \{1, 3, \ldots, v_{k-1}\}$, belongs to T^- then it is already colored with 3 (1). □

4.2. FACIAL GRACEFUL 4-COLORING OF TREES

Lemma 4.6. *If T is a tree with no vertex of degree $2k+1$, $k \geq 1$, then it has a 2-facial coloring with at most 3 colors.*

Proof. If T is a star, then we color the central vertex with color a and the adjacent vertices alternately with b and c. If T is not a star, then pick any vertex of T to be the root. We color the vertices of T starting from the root to the leaves. In each step it is sufficient to find a suitable coloring of a star. □

Theorem 4.7. *If T is a tree with no vertex of degree $2k+1$, $k \geq 1$, then $\chi_{fg}(T) \leq 4$. Moreover, the bound is tight.*

Proof. By Lemma 4.6, T admits a 2-facial coloring with at most 3 colors. If we use the colors $1, 2, 4$ in a 2-facial coloring of T, then we obtain a facial graceful coloring. □

Observe that by Theorem 4.5 and Theorem 4.7 we can determine the exact value of $\chi_{fg}(T)$ for any tree T with no vertex of degree $2k+1$, $k \geq 1$. By Theorem 4.7, if T is a tree with $\chi_{fg}(T) = 5$, then T necessarily contains a vertex of degree $2k+1$ for some $k \geq 1$. In the following we prove that there are trees which contain only vertices of odd degree, and they admit a facial graceful 4-coloring.

Lemma 4.8. *Let T be a tree and let T^- be the tree obtained from T by removing all leaves. If no two edges of T^- are facially adjacent in T, then $\chi_{fg}(T) \leq 4$.*

Proof. First we find a proper vertex coloring of T^- with colors 1 and 4. This coloring gives a partial vertex coloring of T. After that we color the leaves with colors 2 and 3 in order to obtain a facial graceful coloring of T. □

We finish the paper with the following challenging problem.

Problem 4.9. Characterize all trees with facial graceful chromatic number equal to four.

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Július Czap
julius.czap@tuke.sk
https://orcid.org/0000-0002-9933-5884

Technical University of Košice
Faculty of Economics
Department of Applied Mathematics and Business Informatics
Němcovej 32, 040 01 Košice, Slovakia

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