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MODIFIED SEMI-ANALYTICAL APPROACH FOR DUFFING EQUATION

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Abstract: This research endeavour-investigates the enhanced adaptation of the Laplace-based variational iteration method (VIM) tailored specifically for tackling the Duffing Equation. This is accomplished by incorporating the Lagrange multiplier as a strategic tool to effectively address the inherent natural frequency within the Duffing Equation. Using a meticulous comparative analysis, here are juxtapose the analytical outcomes generated by the modified VIM approach with the numerical solution obtained through the application of the renowned Runge-Kutta Fehlberg method (RKF45), implemented by using the powerful mathematical software, MAPLE. Furthermore, by exploring the profound influence of diverse initial conditions on the resulting solution, a diverse array of distinct graphical representations is presented.

Keywords: nonlinear dynamics, modified variational iterative method, Simulink model, mechanical vibration.

1. INTRODUCTION

In the 1990s, many mathematicians work to solve the dribbled flow in fractional derivatives and nonlinear differential equations [1-3] and they introduced a method that is known as the variational iterative method (VIM) and now in recent times VIM has been applied immensely as an important mathematical tool for solving nonlinear differential equations in various field of sciences (for instance, visit [4-10]). For linear problems, the Lagrange multiplier is identified and used to obtain the exact solution with single iteration.

This method is very desired in the directory of techniques for nonlinear models and high summons record articles that are concerned with the "Variational iterative method" which has been into account. Also, this method is more reliable and effective than other existing strategies, for example, the Adomian decomposition and perturbation method etc. The other benefits of VIM are that we can reduce the measurements of computation and still maintain the precision of the numerical solution. However, the strength of this technique is the ability to handle a large group of analytical applications as well as numerical applications in real-life problems. The Integral Transform-based VIM (ITVIM) is a mathematical technique that combines the variational iteration method (VIM) with integral transforms, such as the Laplace transform, to solve differential equations. This hybrid approach leverages the advantages of both methods, allowing for the efficient solution of a wide range of differential equations, including ordinary differential equations (ODEs) and partial differential equations (PDEs). The Laplace transform-based VIM offers a robust approach for solving a wide array of differential equations, providing researchers and engineers with a valuable tool for understanding and analysing dynamic systems in different fields.

The main concern of this method is about nonlinear oscillators like Fangzhu Oscillators [11], lowfrequency property of fractal vibration model [12], microelectro mechanical system oscillators [13, 14] time-fractional problems [15, 16] etc. The VIM was modified with the help of Laplace transformation by Rehman et al. [17]. In this work the nonlinear differential equation is solved by using the method, known as the Modified VIM (MVIM) and periodic solution is obtained. The MVIM retains an approximate solution for each time level. The tactics MVIM identify incontestable use and the Lagrange multiplier is so easy to handle than that of Variational principle [18-22]. We assume a nonlinear oscillator that is expressed by equation:

$$y''(t) + f(y) = 0, \tag{1}$$

with following initial conditions

$$y(0) = A, \qquad y'(0) = 0.$$

We can write Eq. (1) as

$$y'' + \omega^2(y) + r(y) = 0, \tag{2}$$

where ω is unknown frequency,

$$r(y) = f(y) - \omega^2 y, \tag{3}$$

and inVIM, for the Eq. (2) the convolution is given by [23] and known as the correction functional which is

$$v_{n+1}(t) = v_n(t) + \int_0^t \eta(t,\xi)[v''_n(\xi) + \omega^2 v_n(\xi) + \tilde{r}(v_n)]d\xi, n = 0,1,2,3,\ldots \tag{4}$$

In the above expression the general Lagrange multiplier is η, and by using VIM [24-27] with respect to v_n, it can be alternatively calculated by stabilizing the conditions of Eq. (4). The symbol n represents the nth approximation of the solution and $\tilde{r}$ denotes the restricted variation. By assembling stabilised $v_n(t)$, the convolution will give the value of η . Now we obtain a relationship between amplitude and angular frequency and we incorporate

Laplace transform in the well-known VIM. Then we will apply this method on the equations arising from Duffing equation. Applying MVIM to the Duffing equation offers valuable insights into the behaviour of nonlinear systems, including understanding complex dynamics, resonance phenomena, bifurcation analysis; and sensitivity to system parameters. These implications are not only important for theoretical understanding but also have practical applications in engineering and related fields. The Duffing equation is a nonlinear second-order differential equation frequently used to model the behaviour of various physical systems, including mechanical, electrical, and biological systems. It was named after the German engineer and physicist Georg Duffing, who extensively studied this equation in the early 20th century. In mechanical engineering, it describes the behaviour of certain mechanical systems, such as a nonlinear oscillator with cubic stiffness. In electrical engineering, it represents the response of certain electronic circuits to external inputs. Moreover, it has been applied in biology to model the dynamics of biological systems, such as the motion of the human heart under certain conditions. Studying the Duffing equation and its solutions provides valuable insights into the behavior of nonlinear systems and help to use them in various scientific and engineering disciplines.

The generalized Duffing Equation is defined by Petrova [28] and given as:

$$s''(t) + \delta s'(t) + \alpha s(t) + \beta s^3(t) = \varphi(t), \tag{5}$$

where α, β and δ are positive constants and $\varphi(t)$ is a well-known function.

The above equation is a nonlinear second-order differential equation. If $\beta = 0$, then this equation will reduce to linear equation. The Duffing equation shows the chaotic nonlinear behaviour of dynamical system. If $\beta > 0$ the equation denotes a rigid springs and if $\beta < 0$, the equation denotes the smooth springs and, in this case the phase portrait will be closed [29]. The many terms from the qualitative theory of ODE can be easily illustrated by using the Eq.(5) i.e. limit cycle [30] and chaotic behaviour. Tao et al. [31] introduced a promising method for solving fractional differential equations by combining the homotopy perturbation method with the Aboodh transform. This innovative hybrid technique offers a straightforward way to obtain an approximate solution, which converges rapidly to the exact solution, requiring minimal computational effort.

Anjum and He [32] proposed a simplified method utilising the Laplace transform to calculate the multiplier, thereby making this approach accessible to researchers dealing with diverse nonlinear problems. Anjum et al. [33] explored the Elzaki transform, a modified version of the Laplace transform known for its effectiveness in dealing with nonlinear oscillators. The study demonstrated that a single iteration using this transform results in a highly accurate solution, highlighting its practical utility for researchers. In this paper we will solve by taking a special case of Duffing equation and applying the MVIM on this equation.

2. APPLICATIONS OF MVIM TO DUFFING EQUATION

Problem 1: Firstly, we take $\delta = 0$, $\alpha = 0$ and $\varphi(t) = 0$ in Eq. (5) of Duffing equation,

$$s'' + s^3 = 0, \quad when \quad s(0) = A, s'(0) = 0. \tag{6}$$

This equation represents the Duffing oscillator. In order to solve the Duffing equation, with the help of Laplace transform inVIM, we will consider the nonlinear oscillator form

$$s'' + \omega^2 s^3 + g(s) = 0\,, \tag{7}$$

where $\quad g(s) = (1-\omega^2)s^3$.

Consider the correctional formula for VIM

$$\mathcal{L}[v_{n+1}(t)] = \mathcal{L}[v_n(t)] - \mathcal{L}\int_0^t \frac{1}{\omega}\sin\omega(t-\xi)\left(v_n''(\xi) + \omega^2 v^3{}_n(\xi) + g(v_n)\right)d\xi, \tag{8}$$

and now by using convolution for the Laplace transformation, we get

$$\mathcal{L}[s_{n+1}(t)] = \mathcal{L}[s_n(t)] - \frac{1}{\omega}\mathcal{L}\,[\sin\omega t]\mathcal{L}[s''{}_n + \omega^2 s^3{}_n + g(s_n)]. \tag{9}$$

By substituting the value of $g(s_n)$, we have

$$\mathcal{L}[s_{n+1}(t)] = \mathcal{L}[s_n(t)] - \frac{1}{\omega}\mathcal{L}\,[\sin\omega t]\mathcal{L}[s_n'' + \omega^2 s^3{}_n + (1-\omega^2)s_n{}^3, \tag{10}$$

$$\mathcal{L}[s_{n+1}(t)] = \mathcal{L}[s_n(t)] - \frac{1}{\omega}\mathcal{L}\,[\sin\omega t]\mathcal{L}[s_n'' + s_n{}^3]. \tag{11}$$

Use trail function $s_0(t) = A\cos wt$, then the Eq. (11) becomes, when $n = 0$,

$$\mathcal{L}[s_1(t)] = \mathcal{L}[s_0(t)] - \frac{1}{\omega}\mathcal{L}\,[\sin\omega t]\mathcal{L}[s_0'' + s_0{}^3]. \tag{12}$$

By substituting the value of $s_0(t) = A\cos\omega t$, and $s_0''(t) = -A\omega^2\cos\omega t$, we get,

$$\mathcal{L}[s_1(t)] = \mathcal{L}[A\cos\omega t] - \frac{1}{\omega}\mathcal{L}\,[\sin\omega t]\mathcal{L}[-A\omega^2\cos\omega t + A^3\cos\omega t^3], \tag{13}$$

$$\mathcal{L}[s_1(t)] = \mathcal{L}[A\cos\omega t] - \frac{1}{\omega}\mathcal{L}\,[\sin\omega t]\mathcal{L}[-A\omega^2\cos\omega t + \frac{A^3}{4}(\cos 3\omega t + 3\cos\omega t)]. \tag{14}$$

By solving the above, we have following equation

$$\mathcal{L}[s_1(t)] = \mathcal{L}[A\cos\omega t] + \left[A\omega - \frac{3A^3}{4\omega}\right]\mathcal{L}\,[\sin\omega t]\mathcal{L}[\cos\omega t] - \frac{A^3}{4\omega}\mathcal{L}\,[\sin\omega t]\mathcal{L}[\cos 3\omega t]. \tag{15}$$

Now apply the inverse Laplace transform, the first order approximate solution is of the form

$$s_1(t) = [A\cos\omega t] + \left[A\omega - \frac{3A^3}{4\omega}\right]\left[\frac{1}{2}\mathrm{t}\sin\omega t\right] - \frac{A^3}{4\omega}\left[\frac{1}{8\omega}(\cos\omega t - \cos 3\omega t)\right], \tag{16}$$

$$s_1(t) = A\cos\omega t + \left[A\omega - \frac{3A^3}{4\omega}\right]\left[\frac{1}{2}\mathrm{t}\sin\omega t\right] - \frac{A^3}{4\omega}\left[\frac{1}{8\omega}(\cos\omega t - \cos 3\omega t)\right]. \tag{17}$$

In the above, the second term is a secular term because it increases in amplitude with time, so avoiding the secular term in approximate solution requires that

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$$A\omega - \frac{3A^3}{4\omega} = 0,$$

$$\omega^2 = \frac{3}{4}A^2,$$

$$\omega = \sqrt{\tfrac{3}{4}}A. \tag{18}$$

The above expression of angular frequency is same as is obtained by the first-order Hamiltonian approach, second-order Hamiltonian approach and third-order Hamiltonian approach in Eq.(2) by Yildirim et al. [34]. So periodic solution in this case becomes same as that of first-order Hamiltonian approach, second-order Hamiltonian approach and third-order Hamiltonian approach while approximate solution is

$$s_1(t) = A\cos - \tfrac{A^3}{4\omega}\left[\tfrac{1}{8\omega}(\cos\omega t - \cos 3\omega t)\right]. \tag{19}$$

Problem 2: We will take $\varphi(t) = f_0 \cos \omega t$ in Eq. (5). This leads to a new case of Duffing equation,

$$s''(t) + \delta s'(t) + \alpha s(t) + \beta s^3(t) = f_0 \cos\omega t\,, \quad when \quad s(0) = A,\ s'(0) = 0\,. \tag{20}$$

This equation represents the Duffing oscillator. In order to solve Duffing equation, by making Laplace transform in VIM, we will consider the nonlinear oscillator form

$$s'' + \omega^2 s + g(s) = 0, \tag{21}$$

where

$$g(s) = \delta s'(t) + \alpha s(t) + \beta s^3(t) - f_0 \cos\omega t - \omega^2 s.$$

Consider the correctional formula for VIM

$$\mathcal{L}[v_{n+1}(t)] = \mathcal{L}[v_n(t)] - \mathcal{L}\int_0^t \tfrac{1}{\omega}\sin\omega(t-\xi)\left(v_n''(\xi) + \omega^2 v_n(\xi) + g(v_n)\right)d\xi. \tag{22}$$

Now by using convolution for the Laplace transformation, we get

$$\mathcal{L}[s_{n+1}(t)] = \mathcal{L}[s_n(t)] - \tfrac{1}{\omega}\mathcal{L}\,[\sin\omega t]\mathcal{L}[s_n''(t) + \omega^2 s_n + g(s_n)]. \tag{23}$$

By substituting the value of $g(s_n)$, we have

$$\mathcal{L}[s_{n+1}(t)] = \mathcal{L}[s_n(t)] - \tfrac{1}{\omega}\mathcal{L}\,[\sin\omega t]\mathcal{L}[s_n''(t) + \omega^2 s_n + \delta s'(t) + \alpha s(t) + \beta s^3(t) - f_0\cos\omega t \tag{24}$$

$$\mathcal{L}[s_{n+1}(t)] = \mathcal{L}[s_n(t)] - \tfrac{1}{\omega}\mathcal{L}\,[\sin\omega t]\mathcal{L}[s_n''(t) + \delta s_n'(t) + \alpha s_n(t) + \beta s_n^{\,3}(t) - f_0\cos\omega t]. \tag{25}$$

Using the trail function $s_0(t) = A\cos\omega t$, and when $n = 0$, the above equation becomes

$$\mathcal{L}[s_1(t)] = \mathcal{L}[s_0(t)] - \tfrac{1}{\omega}\mathcal{L}\,[\sin\omega t]\mathcal{L}[s_0''(t) + \beta s_0^{\,3}(t) + \delta s_0'(t) + \alpha s_0(t) - f_0\cos\omega t]. \tag{26}$$

By substituting the value of $s_0(t) = A\cos\omega t$, $s_0'(t) = -A\omega\sin\omega t$, and $s_0''(t) = -A\omega^2\cos\omega t$, we get,

$$\mathcal{L}[s_1(t)] = \mathcal{L}[A\cos\omega t] - \tfrac{1}{\omega}\mathcal{L}\,[\sin\omega t]\mathcal{L}[-A\omega^2\cos\omega t + \alpha A\cos\omega t - \delta A\omega\sin\omega t + \beta A^3\cos\omega t^3 - f_0\cos\omega t], \tag{27}$$

$$\mathcal{L}[s_1(t)] = \mathcal{L}[A\cos\omega t] - \tfrac{1}{\omega}\mathcal{L}\,[\sin\omega t]\mathcal{L}[-A\omega^2\cos\omega t + \alpha A\cos\omega t - \delta A\omega\sin\omega t + \tfrac{\beta A^3}{4}cos\,3\omega t + \tfrac{3\beta A^3}{4}cos\,\omega t - f_0\cos\omega t], \tag{28}$$

$$\mathcal{L}[s_1(t)] = \mathcal{L}[A\cos\omega t] - \tfrac{1}{\omega}\mathcal{L}\,[\sin\omega t]\mathcal{L}[(-A\omega^2 + \alpha A + \tfrac{3\beta A^3}{4} - f_0)\,cos\,\omega t - \delta A\omega\sin\omega t + \tfrac{\beta A^3}{4}cos\,3\omega t], \tag{29}$$

and by solving these, obtain following equation

$$\mathcal{L}[s_1(t)] = \mathcal{L}[A\cos\omega t] - \tfrac{1}{\omega}\left(-A\omega^2 + \alpha A + \tfrac{3\beta A^3}{4} - f_0\right)\mathcal{L}\,[\sin\omega t]\mathcal{L}[\cos\omega t] - \delta A\mathcal{L}\,[\sin\omega t]\mathcal{L}[\sin\omega t] - \tfrac{\beta A^3}{4\omega}\mathcal{L}\,[\sin\omega t]\mathcal{L}[\cos 3\omega t]. \tag{30}$$

Applying the inverse Laplace transform, the formulation for the first order approximate solution is

$$s_1(t) = A\cos\omega t - \tfrac{1}{\omega}\left(-A\omega^2 + \alpha A + \tfrac{3\beta A^3}{4} - f_0\right)\left(\tfrac{1}{2}t\sin\omega t\right) - \tfrac{\delta A}{2\omega}(\sin\omega t - t\omega\cos 3\omega t) - \tfrac{\beta A^3}{4\omega}\left(\tfrac{1}{8\omega}(\cos\omega t - \cos 3\omega t)\right), \tag{31}$$

$$s_1(t) = A\cos\omega t - \tfrac{1}{\omega}\left(-A\omega^2 + \alpha A + \tfrac{3\beta A^3}{4} - f_0\right)\left(\tfrac{1}{2}t\sin\omega t\right) - \tfrac{\delta A}{2\omega}\sin\omega t + \tfrac{\delta A}{2\omega}t\omega\cos 3\omega t - \tfrac{\beta A^3}{4\omega}\left(\tfrac{1}{8\omega}(\cos\omega t - \cos 3\omega t)\right). \tag{32}$$

In this, the second term is a secular term because it increases in amplitude with time, and so avoiding the secular term in approximate solution requires that

$$-\tfrac{1}{\omega}\left(-A\omega^2 + \alpha A + \tfrac{3\beta A^3}{4} - f_0\right) = 0,$$

$$A\omega^2 - \alpha A - \tfrac{3\beta A^3}{4} + f_0 = 0,$$

$$A\omega^2 = \alpha A + \tfrac{3\beta A^3}{4} - f_0,$$

$$\omega^2 = \alpha + \tfrac{3\beta A^2}{4} - \tfrac{f_0}{A},$$

$$\omega = \sqrt{\alpha + \tfrac{3\beta A^2}{4} - \tfrac{f_0}{A}}. \tag{33}$$

The above expression shows the angular frequency of the system using the periodic solution approach while approximate solution is

$$s_1(t) = A\cos\omega t - \tfrac{\delta A}{2\omega}\sin\omega t - \tfrac{\beta A^3}{4\omega}\left[\tfrac{1}{8\omega}(\cos\omega t - \cos 3\omega t)\right]. \tag{34}$$

3. RESULT AND DISCUSSION OF PROBLEM 1

In this problem, we take a Duffing Equation with a special case to obtain analytical solution by modified VIM and now we will compare our results with the numerical method that is Runge-Kutta Fehlberg method (RKF45). The Figs. 1, 2 and 3 show the comparison between analytical solutions obtained by MVIM and numerical solutions by RKF45 whichconforms the validity of MVIM:

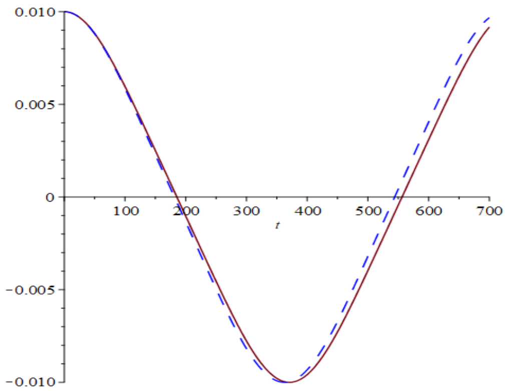

Fig. 1. Comparison between VIM with Laplace and RKF45 for A = 0.01. RKF45, Runge-KuttaFehlberg method: VIM, Variational iteration method.

In this session, we have characterised the error analysis of the analytical solution by MVIM and numerical solution by RKF45. In Tab. 1, error term E1 conforms the validity of the solution by MVIM.

Tab. 1. Error analysis for problem

Time steps	RKF45	MVIM	E_1
1.	0.00999999500495420	0.009999996250	0.000000000622523
2.	0.00999998001985614	0.009999985000	0.00000000249007
3.	0.00999995504478561	0.009999966250	0.00000000560261
4.	0.00999992007985009	0.009999940000	0.00000000996007
5.	0.00999987512518472	0.009999906250	0.0000000155624
6.	0.00999982018095232	0.009999865000	0.0000000224095
7.	0.00999975524734341	0.009999816251	0.0000000305018
8.	0.00999968032457617	0.009999760001	0.0000000398382
9.	0.00999959541289647	0.009999696252	0.0000000504196
10.	0.00999950051257784	0.009999625002	0.0000000622447

MVIM, modified Variational iteration method; RKF45, Runge-Kutta Fehlberg method.

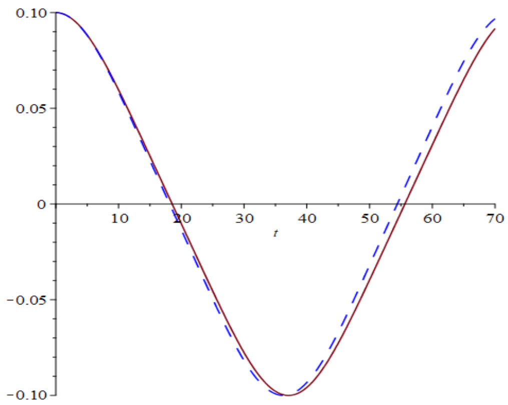

Fig. 2. Comparison between VIM with Laplace and RKF45 for A = 0.1. RKF45, Runge-Kutta Fehlberg method; VIM, Variational iteration method.

In this session, we have characterised the error analysis of the analytical solution by MVIM and numerical solution by RKF45. In Tab. 2, error term Error E2 again conforms the validity of the solution by MVIM.

Tab. 2. Error analysis for problem

Time steps	RKF45	MVIM	E_2
1.	0.0999950010279834	0.09999625002	0.000000624496
2.	0.0999800056260519	0.09998500037	0.00000249737
3.	0.0999550182421241	0.09996625190	0.00000561683
4.	0.0999200462264396	0.09994000600	0.00000997989
5.	0.0998750998315593	0.09990626465	0.0000155824
6.	0.0998201922123647	0.09986503037	0.0000224191
7.	0.0997553394260590	0.09981630627	0.0000304834
8.	0.0996805604321661	0.09976009598	0.0000397678
9.	0.0995958770925311	0.09969640374	0.0000502633
10.	0.0995013141713202	0.09962523432	0.0000619601

MVIM, modified Variational iteration method; RKF45, Runge-Kutta Fehlberg method.

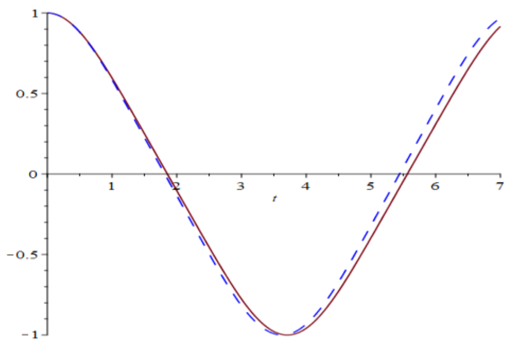

Fig. 3. Comparison between, VIM with Laplace and RKF45 for A = 1. RKF45, Runge-Kutta Fehlberg method; VIM, Variational iterative method.

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In this session, we have characterised the error analysis of the analytical solution by MVIM and numerical solution by RKF45. In Tab. 3, error term E3 again conforms the validity of the solution by MVIM.

Tab. 3. Error analysis for problem

Time steps	RKF45	MVIM	E_3
1.	0.995012463963846	0.9962523432	0.00061994
2.	0.980197633634482	0.9850374625	0.002419914
3.	0.955985874143270	0.9664394171	0.005226771
4.	0.923053269838254	0.9405976051	0.008772168
5.	0.882266426229066	0.9077057190	0.012719646
6.	0.834616758953844	0.8680102938	0.016696767
7.	0.781152682325995	0.8218088592	0.020328088
8.	0.722917072594592	0.7694477095	0.023265318
9.	0.660895283270746	0.7113193077	0.025212012
10.	0.595976638587752	0.6478593447	0.025941353

MVIM, modified Variational iteration method; RKF45, Runge-Kutta Fehlberg method.

4. RESULT AND DISCUSSION OF PROLEM 2

In this problem, we consider a Duffing Equation with a special case to obtain analytical solution by MVIM and the results are compared with the numerical method that is RKF45.

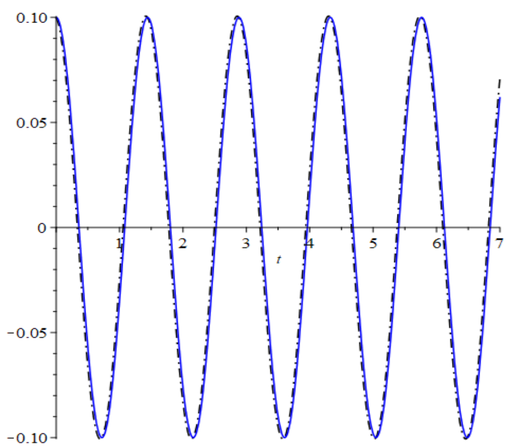

Fig. 4. Comparison between, VIM with Laplace and RKF45 for A = 0.1. RKF45, Runge-Kutta Fehlberg method; VIM, Variational iterative method.

The Figs. 4, 5 and 6 show the comparison between analytical solutions obtained by MVIM and numerical solution by RKF45 which conforms the validity of MVIM. MVIMs, including the Laplace-based variant, offer a flexible and powerful approach to solving a wide range of differential equations. However, users should be aware of its limitations and the specific characteristics of the problems they intend to solve. Additionally, it's important to stay updated with the latest research in this field, as ongoing developments might address some of the current limitations [35-37].

Note that we have characterised the error analysis of the analytical solution by MVIM and numerical solution by RKF45 in this session. In Tab. 4, error term E4 conforms the validity of the solution by MVIM.

Tab. 4. Error analysis for problem

Time steps	RKF45	MVIM	E4
1.	0.090647789	0.090647651	0.000000068965
2.	0.064339347	0.064339933	0.00000002931
3.	0.025999877	0.025997625	0.000000112601
4.	-0.01720434	-0.017207461	0.00000156036
5.	-0.05719678	-0.057193943	0.00000141875
6.	-0.086484702	-0.08648247	0.00000111599
7.	-0.099596785	-0.099594713	0.00000103591
8.	-0.094045679	-0.094078066	0.0000161936
9.	-0.070989376	-0.070964401	0.0000124876
10.	-0.034789867	-0.034577059	0.000106404

MVIM, modified Variational iteration method; RKF45, Runge-Kutta Fehlberg method.

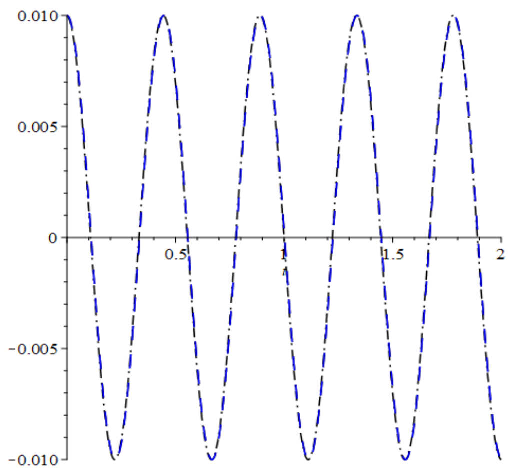

Fig. 5. Comparison between, VIM with Laplace and RKF45 for A = 0.01. RKF45, Runge-Kutta Fehlberg method; VIM, Variational iterative method.

In this part, we have characterised the error analysis of the analytical solution by MVIM and numerical solution by RKF45. In Tab. 5, error term Error E5 again conforms the validity of the solution by MVIM.

Tab. 5. Error analysis for problem

Time steps	RKF45	MVIM	E5
1.	0.001594365	0.001594392	0.0000000136765
2.	-0.009491547	-0.009491583	0.0000000179675
3.	-0.004621835	-0.004621054	0.000000390476
4.	0.008017644	0.008018028	0.000000192285
5.	0.007178921	0.00717783	0.000000545386
6.	-0.005726789	-0.005729173	0.00000119161
7.	-0.009006871	-0.00900474	0.00000106561
8.	0.002859876	0.002857755	0.0000010607
9.	0.009919876	0.009916016	0.00000192992
10.	0.000398376	0.000304249	0.00000470636

MVIM, modified Variational iteration method; RKF45, Runge-Kutta Fehlberg method.

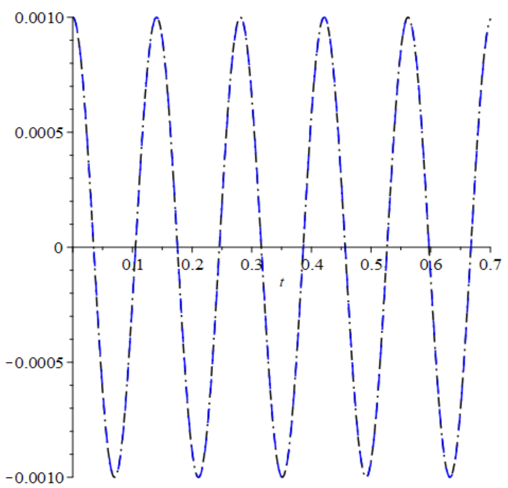

Fig. 6. Comparison between, VIM with Laplace and RKF45 for A = 0.001. RKF45, Runge-Kutta Fehlberg method; VIM, Variational iterative method

In this section, we have characterised the error analysis of the analytical solution by MVIM and numerical solution by RKF45. In Tab. 6, error term E6 again conforms the validity of the solution by MVIM.

Tab. 6. Error analysis for problem

Time steps	RKF45	MVIM	E6
1.	-0.000239068	-0.000239034	0.0000000167992
2.	-0.0008857	-0.000885725	0.0000000126032
3.	0.000662679	0.000662472	0.000000103627
4.	0.000569568	0.000569018	0.000000274801
5.	-0.00093679	-0.000934501	0.0000011442
6.	-0.000126891	-0.000122262	0.0000023141
7.	0.000996544	0.000992951	0.00000179618
8.	-0.000308648	-0.000352436	0.0000218943
9.	-0.000875433	-0.000824463	0.0000254851
10.	0.000905437	0.000746586	0.0000794254

MVIM, modified Variational iteration method; RKF45, Runge-Kutta Fehlberg method.

5. CONCLUSIONS

The Modified Laplace-Based variational Iteration Method VIM (MLVIM) has been applied to the Duffing equation and has yielded numerical results that are in good agreement with numerical solution.

Numerical simulations using the MLVIM have shown that the method is capable of capturing the nonlinear behavior exhibited by the Duffing equation. The approximate solutions obtained through the MLVIM approach closely match the expected trends and characteristics of the Duffing equation, including nonlinear oscillations, bifurcations, and chaos.

Comparisons between the MLVIM results and RKF45 have demonstrated a high degree of accuracy. The MLVIM has been successful in reproducing key features of the Duffing equation, such as the amplitude-frequency response, phase portraits, and frequency response curves.

Additionally, the MVIM has proven to be a versatile method for studying different variations of the Duffing equation, including those with additional nonlinear terms or external forcing. By appropriately modifying the iterative scheme and incorporating the necessary terms, the MLVIM has been able to handle these extensions and produce satisfactory numerical results.

It is worth noting that the accuracy of the MLVIM solutions depends on the number of iterations performed and the convergence behaviour of the method. In cases where the Duffing equation exhibits strong nonlinearity or complex dynamics, a higher number of iterations may be required to achieve desired accuracy.

We can conclde that the MLVIM has provided numerical results which are in good agreement with RKF45 for the Duffing equation. This method demonstrates its effectiveness in capturing the nonlinear behavior and complex dynamics exhibited by the Duffing equation, making it a valuable tool for analysing and understanding such systems.

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