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ON ONE OSCILLATORY CRITERION FOR THE SECOND ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. The Riccati equation method is used to establish an oscillatory criterion for second order linear ordinary differential equations. An oscillatory condition is obtained for the generalized Hill's equation. By means of examples the obtained result is compared with some known oscillatory criteria.

Keywords: Riccati equation, normal and extremal solutions, integral and interval oscillatory criteria, the generalized Hill's equation.

Mathematics Subject Classification: 34C10.

1. INTRODUCTION

Oscillatory analysis of second order linear ordinary differential equations is one of the important problems in the qualitative theory of differential equations and it is the subject of numerous papers (see [16] and cited works therein [1, 2, 4, 5, 7–15, 17]).

Let $q(t)$ be a continuous real function on $[t_0; +\infty)$. Consider the equation

$$\phi''(t) + q(t)\phi(t) = 0. \tag{1.1}$$

Throughout the following we assume that the solutions of the considered equations are real-valued.

Definition 1.1. Equation (1.1) is said to be oscillatory if each of its solutions has arbitrary large zeroes.

The study of oscillatory behavior of second order linear ordinary differential equations has developed in two directions: the goal of the first one is to derive oscillatory property of the equation from the properties of its coefficients on the whole half line (integral oscillatory criteria: see for example Leighton's theorem in [16], Wintner's theorem in [11], Hartman's theorem in [10, Theorem 52], and the works of I.V. Kamenev [8], J. Yan [17], W.-L. Liu and H.-J. Li [14], J. Deng [2], A. Elbert [4], H.Kh. Abdullah [1], G.A. Grigorian [5]); the second one – which is radical – studies the oscillatory behavior of equations on the finite interval (interval oscillatory criteria: see Wong's theorem in [10], G.A. Grigorian [5], Sturm's theorem in [7], Q. Kong [9], J.G. Sun, C.H. Ou and J.S.W. Wong [15], M.K. Kwong, J.S.W. Wong [12]). Then the equation is oscillatory if it is oscillatory on the countable set of intervals. The feature of this direction is that out of countable intervals there is no condition (except conditions like local integrability or continuity) posed on the coefficients of the equation. Probably this fact explains the phenomenon of the existence of oscillations; Eq. (1.1) with extremal effect: $\int_{t_0}^{+\infty} q(\tau)d\tau = -\infty$ (cf. [9]) (It is easy to construct an example of such an effect by using the Sturm comparison theorem.) In many cases the integral oscillatory criteria us allow to establish oscillatory behavior of linear equations easily. Recently M.K. Kwong [11] obtained new integral criteria, describing the broad classes of oscillatory equations in terms of $q(t)$. We note his following result. Let

$$Q(t) \equiv \int\limits_{t_0}^{t} \tau^2 q(\tau)d\tau \quad \text{and} \quad Q_+(t) \equiv \max\{Q(t), 0\}, \quad t \geq t_0.$$

Theorem 1.2 ([11, Theorem 11])**.** *Let the following conditions be satisfied:*

1) *for some $k > 0$, $\alpha > 2$ and for sufficiently large T the inequality*

$$\int\limits_{t_0}^{T} \frac{Q_+^2(t)}{t^2} dt \geq kT^\alpha$$

holds,

2) *there exist $\delta > \varepsilon > 0$ and an infinite number of intervals $[s_n; s_n + \delta]$ such that the measure $\{t \in [s_n; s_n + \delta] : Q(t) \geq 0\} \geq \varepsilon$.*

Then Eq. (1.1) *is oscillatory.*

In this paper we prove an oscillatory criterion for Eq. (1.1). The proof is based on the Riccati equation method. As a consequence, from this criterion is derived an oscillatory condition for the generalized Hill's equation. For the examples the obtained result is compared with some known oscillatory criteria.

2. RICCATI EQUATION

Consider Riccati equation

$$x'(t) + x^2(t) + q(t) = 0, \quad t \geq t_0. \tag{2.1}$$

Definition 2.1. A solution of Eq. (2.1) is said to be t_1-regular if it exists on the interval $[t_1; +\infty)$, $(t_1 \geq t_0)$.

Definition 2.2. The t_1-regular solution $x(t)$ of Eq. (2.1) is said to be t_1-normal if there exists $\delta > 0$ such, that any solution $x_1(t)$ of Eq. (2.1) with $x_1(t_1) \in (x(t_1)-\delta; x(t_1)+\delta)$ is t_1-regular. Otherwise, the solution $x(t)$ is said to be t_1-extremal.

Let $\mathbb{R}$ stand for the set of real numbers. Denote by $reg(t_1)$ the set of such $x_{(0)} \in \mathbb{R}$, for which the solution $x(t)$ of Eq. (2.1) with $x(t_1) = x_{(0)}$ is t_1-regular.

Lemma 2.3. *If Eq.* (2.1) *has a t_1-regular solution, then it has sole t_1-extremal solution $x_*(t)$, and $reg(t_1) = [x_*(t_1); +\infty)$.*

See the proof in [6].

Let $x(t)$ be a t_1-regular solution of Eq. (2.1). Consider the integral

$$\nu_x(t) \equiv \int\limits_t^{+\infty} \exp\biggl\{-2\int\limits_t^{\tau} x(s)ds\biggr\}d\tau, \quad t \geq t_1.$$

Theorem 2.4 ([6, Theorem 2.A])**.** *The integral $\nu_x(t)$ is convergent for each $t \geq t_1$ if and only if $x(t)$ is t_1-normal.*

3. OSCILLATORY CRITERION

Denote by Ω the set of positive and continuously differentiable on $[t_0; +\infty)$ functions. For any $f \in \Omega$ denote

$$I_{q,f} \equiv \int\limits_{t_0}^{+\infty} \exp\biggl\{\int\limits_{t_0}^{t} \frac{d\tau}{f(\tau)} \int\limits_{t_0}^{\tau} \biggl[2f(s)q(s) - \frac{1}{2}\frac{f'(s)^2}{f(s)}\biggr]ds\biggr\}dt.$$

Denote

$$A^{\pm}_{q,\lambda} \equiv \biggl\{t \geq t_0 : \pm\biggl(\lambda + \int\limits_{t_0}^{t} q(\tau)d\tau\biggr) \geq 0\biggr\}, \quad \lambda \in \mathbb{R}.$$

Theorem 3.1. *For some $f \in \Omega$ let the following conditions be satisfied:*

1) $I_{q,f} = +\infty$,
2) *there exists an infinitly large sequence $\{\theta_n\}_{n=1}^{+\infty}$ such that*

$$S \equiv \sup_{n\geq 1}\biggl\{\frac{1}{f(\theta_n)}\int\limits_{t_0}^{\theta_n}\biggl[4f(\tau)q(\tau) - \frac{f'(\tau)^2}{f(\tau)}\biggr] - 4\int\limits_{t_0}^{\theta_n} q(\tau)d\tau\biggr\} < +\infty,$$

and let for some $\lambda \in \mathbb{R}$

3)

$$\int\limits_{A^{+}_{q,\lambda}} d\tau = +\infty,$$

4)
$$\int\limits_{A^-_{q,\lambda}} \Big(\lambda + \int\limits_{t_0}^{\tau} q(s)ds\Big)^2 d\tau = +\infty.$$

Then Eq. (1.1) *is oscillatory.*

Proof. Suppose Eq. (1.1) is not oscillatory. Then Eq. (2.1) has a t_1-regular solution for some $t_1 \geq t_0$ (see [7, p. 332]). In Eq. (2.1) make a change

$$x(t) = y(t) - \lambda - \int\limits_{t_0}^{t} q(\tau)d\tau, \quad t \geq t_0. \tag{3.1}$$

We will arrive at the equation

$$y'(t) + y^2(t) - 2\Bigg(\lambda + \int\limits_{t_0}^{t} q(\tau)d\tau\Bigg)y(t) + \Bigg(\lambda + \int\limits_{t_0}^{t} q(\tau)d\tau\Bigg)^2 = 0, \quad t \geq t_0. \tag{3.2}$$

By virtue of Lemma 2.3, Eq. (2.1) has the t_1-extremal solution $x_*(t)$. Let $y_*(t)$ be the solution of Eq. (3.2) with $y_*(t_1) = x_*(t_1) - \lambda - \int\limits_{t_0}^{t_1} q(\tau)d\tau$. By virtue of (3.1), $y_*(t)$ is t_1-regular. Let us show that

$$y_*(t) \to -\infty \quad \text{for} \quad t \to +\infty. \tag{3.3}$$

By virtue of (3.2), we have

$$y_*(t) = y_*(t_1) - \int\limits_{t_1}^{t} \Bigg[y_*(\tau) - \lambda - \int\limits_{t_0}^{\tau} q(s)ds\Bigg]^2 d\tau, \quad t \geq t_1. \tag{3.4}$$

Suppose the relation (3.3) is false. Then it follows from (3.4) that $y_*(t)$ decreases and has a finite limit on $+\infty$:

$$y_*(+\infty) \equiv \lim_{t\to+\infty} y_*(t) \quad (y_*(t) \downarrow y_*(+\infty) \neq -\infty). \tag{3.5}$$

Two cases are possible:

$$\text{a)}\, y_*(+\infty) < 0, \quad \text{b)}\, y_*(+\infty) \geq 0.$$

Let case a) hold. Then it follows from (3.5) that $y_*(t) \leq -\varepsilon$, $t \geq t_2$, for some $\varepsilon > 0$ and $t_2 \geq t_1$. From here, from condition 3) and (3.4) it follows, that $y_*(t) \to -\infty$ for $t \to +\infty$, which contradicts (3.5). Let case b) hold. Then from (3.5) it follows that $y_*(t) \geq 0$, $t \geq t_1$. From here, from the condition 4) and from (3.4) it follows that

$y_*(t) \to -\infty$ for $t \to +\infty$, which again contradicts (3.5). So the relation (3.3) holds. By condition 2) and relation (3.5) choose $n = n_0$ so large that

$$y_*(\theta_{n_0}) - \lambda + S/4 < 0, \tag{3.6}$$

and put $t_2 \equiv \theta_{n_0}$. Show that the solution $x_0(t)$ of Eq. (3.1) with

$$x_0(t_2) = \frac{1}{f(t_2)} \int\limits_{t_0}^{t_2} \left[\frac{f'(\tau)^2}{4f(\tau)} - f(\tau)q(\tau)\right] d\tau \tag{3.7}$$

is t_2-normal. From (3.1) it follows

$$x_*(t_2) = y_*(t_2) - \lambda - \int\limits_{t_0}^{t_2} q(\tau)d\tau.$$

From this, (3.6) and (3.7) it follows that $x_*(t_2) < x_0(t_2)$. By virtue of Lemma 2.3, it follows from here that $x_0(t)$ is t_2-normal. By virtue of (2.1), we have

$$f(t)x_0'(t) + f(t)x_0^2(t) + f(t)q(t) = 0, \quad t \geq t_2.$$

Integrating this equality from t_0 to t we obtain

$$f(t)x_0(t) + \int\limits_{t_2}^{t} [f(\tau)x_0^2(\tau) - f'(\tau)x_0(\tau)]d\tau = f(t_2)x_0(t_2) - \int\limits_{t_2}^{t} f(\tau)q(\tau)d\tau, \quad t \geq t_2.$$

Completing the square in the left hand side of this equality and dividing both sides of the obtained by $f(t)$ we will come to the equality

$$x_0(t) + \frac{1}{f(t)} \int\limits_{t_2}^{t} f(\tau)\left[x_0(\tau) - \frac{f'(\tau)}{2f(\tau)}\right]^2 d\tau = \frac{c}{f(t)} + \frac{1}{f(t)} \int\limits_{t_0}^{t} \left[\frac{f'(\tau)^2}{4f(\tau)} - f(\tau)q(\tau)\right] d\tau, \ t \geq t_2, \tag{3.8}$$

where

$$c \equiv f(t_2)x_0(t_2) - \int\limits_{t_0}^{t_2} \left[\frac{f'(\tau)^2}{4f(\tau)} - f(\tau)q(\tau)\right] d\tau.$$

By virtue of (3.7), $c = 0$. Therefore, from (3.8) we get

$$-2x_0(t) \geq \frac{1}{f(t)} \int\limits_{t_0}^{t} \left[2f(\tau)q(\tau) - \frac{f'(\tau)^2}{2f(\tau)}\right] d\tau, \quad t \geq t_2.$$

Then

$$\nu_{x_0}(t_2) \geq M \int\limits_{t_2}^{+\infty} \exp\left\{\int\limits_{t_0}^{t} \frac{d\tau}{f(\tau)} \int\limits_{t_0}^{\tau} \left[2f(s)q(s) - \frac{f'(s)^2}{2f(s)}\right] ds\right\} dt, \tag{3.9}$$

where

$$M \equiv \exp\Big\{-\int\limits_{t_0}^{t_2}\frac{d\tau}{f(\tau)}\int\limits_{t_0}^{\tau}\Big[2f(s)q(s)-\frac{f'(s)^2}{2f(s)}\Big]ds\Big\}.$$

Since $x_0(t)$ is t_2-normal by virtue of Theorem 2.4 the left hand side of inequality (3.9) is finite, whereas from condition 1) it follows that its right hand side is equal to $+\infty$. The obtained contradiction proves the theorem. □

Remark 3.2. For $f(t) \equiv 1$, condition 2) of Theorem 3.1 always holds.

Example 3.3. Consider equation

$$\phi''(t)+\Big[\sum_{k=1}^{n} a_k\frac{\cos(\lambda_k t^{\alpha_k})}{t^{\beta_k}}\Big]\phi(t)=0, \quad t\geq t_0>0, \tag{3.10}$$

where a_k, λ_k, α_k, β_k, $k=\overline{1,n}$ are some constants, $a_1\neq 0$, $\lambda_k\neq 0$, $\alpha_k>0$, $k=\overline{1,n}$, $\alpha_1\leq 1$, $\alpha_1+\beta_1\leq 3/2$, $\alpha_1+\beta_1<\alpha_k+\beta_k$, $k=\overline{2,n}$. We have

$$\begin{aligned}&\int\limits_{t_0}^{t}\Big[\sum_{k=1}^{n}a_k\frac{\cos(\lambda_k\tau^{\alpha_k})}{\tau^{\beta_k}}\Big]d\tau\\ &=\sum_{k=1}^{n}\Big[\frac{a_k}{\lambda_k}\frac{\sin(\lambda_k t^{\alpha_k})}{t^{\alpha_k+\beta_k-1}}+\frac{(1-\alpha_k-\beta_k)a_k}{\lambda_k}\int\limits_{t}^{+\infty}\frac{\sin(\lambda_k\tau^{\alpha_k})}{\tau^{\alpha_k+\beta_k-2}}d\tau\Big]+c_0(t_0), \quad t\geq t_0,\end{aligned} \tag{3.11}$$

where

$$c_0(t_0)\equiv\sum_{k=1}^{n}\Big[\frac{a_k}{\lambda_k}\frac{\sin(\lambda_k t_0^{\alpha_k})}{t_0^{\alpha_k+\beta_k-1}}d\tau+\frac{(1-\alpha_k-\beta_k)a_k}{\lambda_k}\int\limits_{t_0}^{+\infty}\frac{\sin(\lambda_k\tau^{\alpha_k})}{\tau^{\alpha_k+\beta_k-2}d\tau}\Big].$$

It is not difficult to see that

$$c_0(t_0)=\frac{a_1}{\lambda_1 t_0^{\alpha_1+\beta_1-1}}\big[\sin(\lambda_1 t_0^{\alpha_1})+o(1)\big] \quad \text{for} \quad t_0\to+\infty.$$

Therefore, without loss of generality we will assume that $c_0(t_0)=0$, $t_0>0$. Then from (3.11) we get

$$\int\limits_{t_0}^{t}\Big[\sum_{k=1}^{n}a_k\frac{\cos(\lambda\tau^{\alpha_k})}{\tau^{\beta_k}}\Big]d\tau=\frac{a_1}{\lambda_1 t_0^{\alpha_1+\beta_1-1}}\Big[\sin(\lambda_1 t_0^{\alpha_1})+o(1)\Big] \quad \text{for} \quad t\to+\infty.$$

Hence it is clear that for $\lambda=0$ that conditions 3) and 4) of Theorem 3.1 are fulfilled. From (3.11) we derive

$$\int\limits_{t_0}^{t}d\tau\int\limits_{t_0}^{\tau}\Big[\sum_{k=1}^{n}a_k\frac{\cos(\lambda s^{\alpha_k})}{s^{\beta_k}}\Big]ds=-\frac{a_1}{\lambda_1^2 t^{\alpha_1+\beta_1-2}}\Big[\cos(\lambda_1 t^{\alpha_1})+c_1(t)\Big]+c_2(t),$$

where

$$c_1(t) \equiv \frac{\lambda_1^2}{a_1} \sum_{k=1}^{n} \frac{a_k}{\lambda_k^2} \frac{\cos(\lambda_k t^{\alpha_k})}{t^{\alpha_k+\beta_k-\alpha_1-\beta_1}} = o(1),$$

and

$$c_2(t) \equiv \sum_{k=1}^{n} \frac{(1-\alpha_k-\beta_k)a_k}{\lambda_k} \int_{t_0}^{t} d\tau \int_{\tau}^{+\infty} \frac{\sin(\lambda_k s^{\alpha_k})}{s^{\alpha_k+\beta_k-2}} ds = O(1)$$

for $t \to +\infty$. Then assuming $f(t) \equiv 1$ and taking into account Remark 3.2 we conclude that for Eq. (3.10) conditions 1) and 2) of Theorem 3.1 are fulfilled. Therefore, Eq. (3.10) is oscillatory.

Let $H_1(t)$ and $H_2(t)$ be real continuous and periodic functions on $[t_0; +\infty)$ with periods T_1 and T_2, correspondingly, and let T_1/T_2 be irrational. Denote $H(t) \equiv H_1(t) + H_2(t)$, $t \geq t_0$. Consider generalized Hill's equation

$$\phi''(t) + H(t)\phi(t) = 0, \quad t \geq t_0. \tag{3.12}$$

Corollary 3.4. *If*

$$\frac{1}{T_1} \int_{t_0}^{t_0+T_1} H_1(\tau)d\tau + \frac{1}{T_2} \int_{t_0}^{t_0+T_2} H_2(\tau)d\tau \geq 0,$$

then Eq. (3.12) *is oscillatory.*

Proof. We prove only for the case

$$\int_{t_0}^{t_0+T_k} H_k(\tau)d\tau = 0, \quad k = 1, 2. \tag{3.13}$$

The proof in the general case can be derived from the realized proof by using the Sturm comparison criterion (see [7, p. 334]). Denote

$$h_k(t) \equiv \int_{t_0}^{t} H_k(\tau)d\tau, \quad t \geq t_0, \quad k = 1, 2.$$

It is easy to derive from (3.13) that $h_k(t)$ is a periodic function of period T_k $(k = 1, 2)$. Denote

$$\overline{h}_k \equiv \frac{1}{T_k} \int_{t_0}^{t_0+T_k} h_k(\tau)d\tau, \quad k = 1, 2.$$

Then

$$h_k(t) = \overline{h}_k + h_k^0(t), \quad t \geq t_0, \quad k = 1, 2, \tag{3.14}$$

where

$$\int\limits_{t_0}^{t_0+T_k} h_k^0(\tau)d\tau = 0, \quad k = 1, 2. \tag{3.15}$$

By virtue of the mean value theorem, the equality $\overline{h}_k = h_k(\xi_k)$ holds for some $\xi_k \in [t_0; t_0 + T_k]$ $(k = 1, 2)$. Then since

$$h_k(t) = h_k(\xi_k) + \int\limits_{\xi_k}^{t} H_k(\tau)d\tau, \quad t \geq t_0,\ k = 1, 2,$$

we deduce from (3.14) and (3.15) that

$$h_k^0(t) = \int\limits_{\xi_k+nT_k}^{t} H_k(\tau)d\tau, \quad t \geq t_0, \quad k = 1, 2, \tag{3.16}$$

for a fixed $n \in \{1, 2, \ldots\}$. Evidently,

$$\min_{t\in[t_0;t_0+T_k]} h_k(t) < \overline{h}_k < \max_{t\in[t_0;t_0+T_k]} h_k(t), \quad k = 1, 2.$$

In view of this, we assume ξ_2 such that

$$\varepsilon(t) \equiv \overline{h}_2 - h_2(t) \geq 0, \quad t \in [\xi_2 - \delta; \xi_2], \tag{3.17}$$

for some $\delta > 0$. Denote

$$M \equiv \min\{M_1, M_2\},$$

where

$$M_1 \equiv \max_{t\in[t_0;t_0+T_1]} h_1^0(t), \quad M_2 \equiv \Big|\min_{t\in[t_0;t_0+T_1]} h_1^0(t)\Big|.$$

It follows from (3.15) that $M > 0$. Then from (3.17) we have that for enough small value of $\delta > 0$ the following inequalities hold:

$$0 \leq \varepsilon(t) \leq \frac{M}{8}, \quad t \in [\xi_2 - \delta; \xi_2] \tag{3.18}$$

(because $\varepsilon(\xi_2) = \overline{h}_2 - h_2(\xi_2) = 0$). Since T_1/T_2 irrational, the set

$$\{t_0 + mT_2(modT_1) : m = 1, 2, \ldots\}$$

is everywhere dense in $[t_0; t_0 + T_1]$. In view of this, we choose the natural numbers n_0 and m_0 such that

$$0 < \xi_2 + m_0T_2 - \xi_1 - n_0T_1 < \delta \tag{3.19}$$

and put $t_1 = \xi_1 + n_0T_1$. Denote

$$g_k(t) \equiv \int\limits_{t_1}^{t} H_k(\tau)d\tau, \quad t \geq t_1,\ k = 1, 2.$$

From (3.16) we see that

$$g_1(t) = h_1^0(t), \quad t \geq t_1. \tag{3.20}$$

It is evident that

$$g_2(t) = \overline{g}_2 + h_2^0(t), \quad t \geq t_1, \tag{3.21}$$

where

$$\overline{g}_2 = h_2(\xi_2 + m_0 T_2) - h_2(\xi_1 + n_0 T_1).$$

From (3.18) and (3.19) it follows that

$$0 \leq \overline{g}_2 \leq \frac{M}{8}. \tag{3.22}$$

Consider the functions

$$F_k(t) \equiv \int_{t_1}^{t} d\tau \int_{t_1}^{\tau} H_k(s) ds, \quad t \geq t_1,\, k = 1, 2.$$

We have

$$F_k(t) = \int_{t_1}^{t} g_k(\tau) d\tau, \quad t \geq t_1,\, k = 1, 2.$$

Then from (3.15) and (3.20) it follows that $F_1(t)$ is a periodic function, and from (3.15) and (3.21) that $F_2(t) = \overline{g}_2 t + F_2^0(t)$, $t \geq t_1$, where $F_2^0(t)$ is a periodic function. From here and from (3.22) it follows that for $f(t) \equiv 1$ conditions 1) and 2) of Theorem 3.1 are fulfilled. Let $\eta_+(\eta_-)$ be a maximum (minimum) point of the function $h_1^0(t)$ on $[t_1; t_1 + T_1]$, and let $h_2(\eta_0) = 0$ for some $\eta_0 \in [t_1; t_1 + T_2]$. Choose $\Delta > 0$ so small that $\Delta \leq \delta$,

$$h_1(t) \geq \frac{h_1^0(\eta_+)}{2}, \quad |t - \eta_+| \leq \Delta, \tag{3.23}$$

$$h_1(t) \geq \frac{h_1^0(\eta_-)}{2}, \quad |t - \eta_-| \leq \Delta, \tag{3.24}$$

$$|h_2^0(t)| \leq \frac{M}{8}, \qquad |t - \eta_0| \leq \Delta. \tag{3.25}$$

Since the set $\{t_1 + mT_2 (mod T_1) : m = 1, 2, \ldots\}$ is everywhere dense in $[t_1; t_1 + T_1]$, we can choose the sequences of natural numbers $\{n_k^\pm\}_{k=1}^{+\infty}$ and $\{m_k^\pm\}_{k=1}^{+\infty}$ such that

$$|n_k^\pm T_1 + \eta_\pm - m_k^\pm T_2 - \eta_0| < \Delta, \quad k = 1, 2, \ldots.$$

Then (3.23) and (3.25) imply that

$$h(t) \equiv h_1(t) + h_2(t) \geq 0, \quad t \in [n_k^+ + \eta_+ - \Delta; n_k^+ + \eta_+ + \Delta],$$

and from (3.24) and (3.25) we get

$$h(t) \leq \frac{h_1(\eta_-)}{4}, \quad t \in [n_k^- + \eta_- - \Delta; n_k^- + \eta_- + \Delta],\, k = 1, 2, \ldots.$$

Therefore, for $\lambda = 0$ conditions 3) and 4) of Theorem (3.1) are fulfilled. So we showed that for Eq. (3.12) all conditions of Theorem (3.1) are fulfilled. Therefore, Eq. (3.12) is oscillatory. The proof is complete. □

Remark 3.5. The condition of Corollary 3.4 is simpler than the condition of Coppel's theorem [10, Theorem 56] (in the sense that for the calculation of $\lim_{t\to+\infty} \frac{1}{t}\int_{a+t}^{t} H(\tau)d\tau$ for sure we use the quantities $\frac{1}{T_k}\int_{t_0}^{t_0+T_k} H_k(\tau)d\tau$, $k = 1, 2$, and we need not prove the uniform convergence of $\lim_{t\to+\infty} \frac{1}{t}\int_{a+t}^{t} H(\tau)d\tau$).

Let $H_0(t)$ be a continuous real function on $[t_0; +\infty)$ such that the integral $\int_{t_0}^{+\infty} H_0(\tau)d\tau$ is convergent.

Remark 3.6. Slightly changing the proof of Corollary 3.4 it can be shown that the equation

$$\phi''(t) + [H(t) + H_0(t)]\phi(t) = 0, \quad t \ge t_0,$$

is oscillatory if $H(t)$ satisfies the condition of Corollary 3.4.

Example 3.7. The equation

$$\phi''(t)+[a+a_1\cos(\lambda_1 t+\omega_1)+a_2\cos(\lambda_2 t+\omega_2)+\sum_{k=1}^{n} b_k t^{\alpha_k}\cos(\mu_k t^{\beta_k})]\phi(t) = 0, \ \ t \ge t_0 > 0, \tag{3.26}$$

where $a, a_j, \lambda_j, \omega_j (j = 1, 2), b_k, \alpha_k, \mu_k, \beta_k (k = \overline{1, n})$ are some constants, $a_j\lambda_j \ne 0$, $j = 1, 2$, $\mu_k \ne 0$, $k = \overline{1, n}$, $\lambda_1/\lambda_2 \ne n/m$, $n, m \in \mathbb{Z}$, for $a \ge 0$, and $\alpha_k - \beta_k + 1 < 0$, $k = \overline{1, n}$ is oscillatory.

Example 3.8. Consider equation

$$\phi''(t) + \left[\frac{\mu}{t^2} + \gamma\frac{\cos(\sqrt{t})}{t}\right]\phi(t) = 0, \quad t \ge t_0 > 0, \quad \mu \ge \frac{1}{4}, \gamma \in \mathbb{R}. \tag{3.27}$$

Take $f(t) = t$, $t \ge t_0$. We have

$$2\int_{t_0}^{t} \cos\sqrt{s}ds = 2\sqrt{t}\sin\sqrt{t} - 2\sqrt{t_0}\sin\sqrt{t_0} + \int_{t_0}^{+\infty} \frac{\sin\sqrt{s}}{\sqrt{s}}ds, \quad t \ge t_0. \tag{3.28}$$

Without loss of generality we choose $t_0 > 0$ such that

$$2\sqrt{t_0}\sin\sqrt{t_0} = \int_{t_0}^{+\infty} \frac{\sin\sqrt{s}}{\sqrt{s}}ds$$

This is possible for

$$2\sqrt{t_0}\sin\sqrt{t_0} - \int_{t_0}^{+\infty} \frac{\sin\sqrt{s}}{\sqrt{s}}ds = 2\sqrt{t_0}(\sin\sqrt{t_0} + o(1)) \quad \text{for} \quad t_0 \to +\infty.$$

Then using (3.28) it is easy to show that conditions 1) and 2) of Theorem 3.1 for Eq. (3.27) are fulfilled. It is easy to show that

$$\int_{t_0}^{t}\left[\frac{\mu}{\tau^2}-\gamma\frac{\cos\sqrt{\tau}}{\tau}\right]d\tau$$

$$=\frac{\mu}{t_0}+\gamma\int_{t_0}^{+\infty}\frac{\sin\sqrt{\tau}}{\tau\sqrt{\tau}}d\tau-\frac{2\gamma}{\sqrt{t_0}}\sin\sqrt{t_0}+\frac{2\gamma}{\sqrt{t}}\sin\sqrt{t}-\frac{\mu}{t}-\gamma\int_{t}^{+\infty}\frac{\sin\sqrt{\tau}}{\tau\sqrt{\tau}}d\tau,\quad t\geq t_0,$$

$$\frac{\mu}{t}+\gamma\int_{t}^{+\infty}\frac{\sin\sqrt{\tau}}{\tau\sqrt{\tau}}d\tau=O\Big(\frac{1}{t}\Big)$$

for $t\to+\infty$ (therefore $\int_{t_0}^{+\infty}\big[\frac{\mu}{t}+\gamma\int_{t}^{+\infty}\frac{\sin\sqrt{\tau}}{\tau\sqrt{\tau}}d\tau\big]^2dt<+\infty$). Using these relations and taking

$$\lambda=\frac{2\gamma}{\sqrt{t_0}}\sin\sqrt{t_0}-\frac{\mu}{t_0}-\gamma\int_{t}^{+\infty}\frac{\sin\sqrt{\tau}}{\tau\sqrt{\tau}}d\tau$$

one can readily show that conditions 3) and 4) of Theorem 3.1 for Eq. (3.27) are fulfilled too. Therefore, Eq. (3.27) is oscillatory.

In the above examples the function $q(t)$ has at most power growth on $+\infty$. Now we give an example of oscillatory Eq. (1.1) with exponential growth of $q(t)$ on $+\infty$.

Example 3.9. Consider equation

$$\phi''(t)+e^{2t}\cos(e^t)\phi(t)=0,\quad t\geq t_0. \tag{3.29}$$

Without loss of generality we assume that $e^{t_0}=ctg(e^{t_0})$. Then

$$\int_{t_0}^{t}q(\tau)d\tau\equiv\int_{t_0}^{t}e^{2\tau}\cos(e^\tau)d\tau=e^t\sin(e^t)+\cos(e^t),\quad t\geq t_0. \tag{3.30}$$

It follows from here that for enough large integers k the following inequalities are fulfilled:

$$\int_{t_0}^{t}q(\tau)d\tau\geq 0 \text{ for } t\in\left[\ln\Big(2\pi k+\frac{\pi}{6}\Big);\ln\Big(2\pi k+\frac{5\pi}{6}\Big)\right],$$

$$\int_{t_0}^{t}q(\tau)d\tau\leq-\frac{1}{3}\Big(2\pi k+\frac{7\pi}{6}\Big)\ln\left[1+\frac{\frac{2\pi}{3}}{2\pi k+\frac{7\pi}{6}}\right] \text{ for } t\in\left[\ln\Big(2\pi k+\frac{13\pi}{6}\Big);\ln(2\pi k+3\pi)\right].$$

Therefore, when $\lambda=0$ for Eq. (3.29) conditions 3) and 4) of Theorem 3.1 are fulfilled. By integrating (3.30) from t_0 to t it is easy to verify that for $f(t)\equiv 1$ conditions

1) and 2) of Theorem 3.1 are fulfilled. So all conditions of Theorem 3.1 for Eq. (3.29) are satisfied. Therefore, Eq. (3.29) is oscillatory.

It is not difficult to verify that the oscillatory criteria of Ph. Hartman [7, Theorem 52] and I.V. Kamenev [8] are not applicable to equations (3.10), (3.26), (3.27) and (3.29). Theorem 1 of M.K. Kwong is not applicable to Eq. (3.10) for $\alpha_1 + \beta_1 \geq \frac{1}{2}$, as well as to Eq. (3.27). Although for $\beta_1 - \alpha_1 > 0$ the coefficient $q(t)$ in Eq. (3.10) is integrable, even in this case the criterion of J. Deng [2, Theorem 1] is not applicable to Eq. (3.10). The last one is not applicable to Eq. (3.27) too. The question of the applicability of the criteria of J.S.W. Wong [10, Theorem 1], Y.C. Sun, C.H. Ou and J.S.W. Wong [15, Corollaries 1-3], J. Yan [17, Theorem 1], W.-L. Liu and H.-J. Li [14, Theorems 1 and 2], A. Elbert [4, Theorem 2], Z. Zheng [18, Theorem 2], H.Kh. Abdullah [1, Theorem 2] to the equations (3.10) and (3.26) remains open. It also remains open the following question: which parameter function $g(t)$ should be selected (for each individual case), for proving the oscillation of equations (3.10), (3.26) and (3.29) by using the Hauptsatz's test [3, Theorem 1]?

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