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NEW RANKING METHOD FOR FUZZY NUMBERS BY THEIR EXPANSION CENTER

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Abstract

Based on the area between the curve of the membership function of a fuzzy number and the horizontal real axis, a characteristic as a new numerical index, called the expansion center, for fuzzy numbers is proposed. An intuitive and reasonable ranking method for fuzzy numbers based on this characteristic is also established. The new ranking method is applicable for decision making and data analysis in fuzz environments. An important criterion of the goodness for ranking fuzzy numbers, the geometric intuitivity, is also introduced. It guarantees coinciding with the natural ordering of the real numbers.

1 Introduction

Fuzzy number [5, 9, 18] is one of the most important mathematical concepts concerning fuzziness. Ranking fuzzy numbers is a necessary link of analyzing fuzzy information in optimization, data mining, decision making, and other related areas. Since the concepts of fuzzy sets and fuzzy numbers were introduced in the sixties of the last century, many significant contributions have been made in ranking and ordering fuzzy numbers [1-4, 6-8, 10-13, 16, 17, 19]. They have respect intuition based on some geometric characteristics (e.g., area, distance, or centroid), and can be used for various purposes. Mostly, these methods can only rank fuzzy numbers but allow different fuzzy numbers to have the same rank, or order some special types of fuzzy numbers, such as the triangular (or, more generally, trapezoidal) fuzzy numbers. Recently, a method for totally ordering all fuzzy numbers is proposed [15]. By using the total ordering defined on the set of all fuzzy numbers, any equivalence relation on this set can generate a ranking for fuzzy numbers.

In this paper, a new concept of expansion center is proposed. Based on this concept, an alternative ranking method for fuzzy numbers is established. It is intuitive and reasonable. For some common types of fuzzy numbers, the calculation for ranking is not complex and, therefore, it is convenient to be used in fuzzy data mining and fuzzy decision making.

After the introduction, this paper is arranged as follows. In Section 2, the concept of fuzzy number and some common types of fuzzy numbers are recalled. The new concept of expansion center of fuzzy numbers is presented in Section 3. Its existence and uniqueness is proved in this section. The formula for calculating the expansion center is also presented for those common types of fuzzy numbers. For other fuzzy numbers, a soft computing method may be adopted in the calculation. Then Section 4 shows the relevant ranking method. By using the expansion center, an equivalence relation on the set of all fuzzy numbers is defined in Section 5 and a total ordering on the quotient space is also shown there. Some conclusions are summarized in Section 6.

2 Fuzzy Numbers

Let R $= (-\infty, \infty)$. A fuzzy subset of R, denoted by $\widetilde{e}$, is called a fuzzy number if its membership function [9] $m_e : \mathrm{R} \to [0, 1]$ satisfies the following conditions.

(**FN1**) Set $\{x|m_e(x) \geq \alpha\}$, the α-cut of $\widetilde{e}$ (denoted by e_α), is a closed interval, $[e^l_\alpha, e^r_\alpha]$, for every $\alpha \in (0, 1]$.

(**FN2**) Set $\{x|m_e(x) > 0\}$, the support set of $\widetilde{e}$ (denoted by supp(e)), is bounded.

Condition (FN1) implies that $\widetilde{e}$ is a convex fuzzy subset of R, in the meaning that

$$m_e(t) \geq \min([m_e(t_1), m_e(t_2)]$$

for any real numbers t, t_1, t_2 satisfying $t_1 < t < t_2$, and is equivalent to the following three conditions.

(**FN1.1**) There exists at least one real number a_0 such that $m_e(a_0) = 1$.

(**FN1.2**) Function m_e is nondecreasing on$(-\infty, a_0]$ and nonincreasing on $[a_0, \infty)$.

(**FN1.3**) Function m_e is upper semi-continuous, or say,m_e is right-continuous (i.e., $\lim_{x\to x_0+} m_e(x) = m_e(x_0)$) when $x_0 < a_0$ and is left-continuous (i.e.,$\lim_{x\to x_0-} m_e(x) = m_e(x_0)$) when $x_0 > a_0$.

The set of all fuzzy numbers is denoted by $\mathcal{N}_F$. Now, let us recall some special but common types of fuzzy numbers in $\mathcal{N}_F$.

Each rectangular fuzzy number can be regarded as a closed interval. We use $\mathcal{N}_I$, which is a proper subset of $\mathcal{N}_F$, to denote the set of all rectangular fuzzy numbers. The membership function of a rectangular number $\widetilde{e}$ has a form as

$$m_e(x) = \begin{cases} 1 & \text{if } x \in [a_l, a_r] \\ 0 & \text{otherwise} \end{cases}, \qquad (1)$$

and can be simply denoted by vector $[a_l \ \ a_r]$, where real-valued parameters a_l and a_r satisfy $a_l \leq a_r$. Its core is closed interval $[a_l, a_r]$.

Another common special type of fuzzy numbers is the triangular fuzzy numbers whose membership function has a form

$$m_e(x) = \begin{cases} \frac{x-a_l}{a_0-a_l} & \text{if } x \in [a_l, a_0) \\ 1 & \text{if } x = a_0 \\ \frac{x-a_r}{a_0-a_r} & \text{if } x \in (a_0, a_r] \\ 0 & \text{otherwise} \end{cases}, \qquad (2)$$

where real-valued parameters a_l, a_0, and a_r satisfy $a_l \leq a_0 \leq a_r$. The set of all triangular fuzzy numbers is denoted by $\mathcal{N}_T$, which is also a proper subset of $\mathcal{N}_F$. Such a fuzzy number can be simply denoted by a vector $[a_l \ a_0 \ a_r]$. Its core is singleton $\{a_0\}$.

As a generalization of both rectangular fuzzy numbers and triangular fuzzy numbers, trapezoidal fuzzy numbers are also an important common type of fuzzy numbers, whose membership function has a form

$$m_e(x) = \begin{cases} \frac{x-a_l}{a_b-a_l} & \text{if } x \in [a_l, a_b) \\ 1 & \text{if } x \in [a_b, a_d] \\ \frac{x-a_r}{a_d-a_r} & \text{if } x \in (a_d, a_r] \\ 0 & \text{otherwise} \end{cases}, \qquad (3)$$

where a_l, a_b, a_d and a_r are real-valued parameters with $a_l \leq a_b \leq a_d \leq a_r$. The set of all trapezoidal fuzzy numbers is denoted by $\mathcal{N}_p$, which is a proper subset of $\mathcal{N}_F$ as well. Such a trapezoidal fuzzy number can be simply denoted by a vector $[a_l \ a_b \ a_d \ a_r]$. Its core is closed interval $[a_b, a_d]$.

Definition 1. Fuzzy number $\widetilde{e}$ is said to be symmetric iff there exists real number t_0 such that $m_e(t_0 - t) = m_e(t_0 + t)$ for any $t \in$ R. Real number t_0 is called the symmetry center of fuzzy number $\widetilde{e}$.

Any rectangular fuzzy number with membership function (1) is symmetric. Any triangular fuzzy number with membership function (2) is symmetric iff $a_0 - a_l = a_r - a_0$. Any trapezoidal fuzzy number with membership function (3) is symmetric iff $a_b - a_l = a_r - a_d$.

3 A General Discussion On Rankin And Total Ordering For Fuzzy Numbers

Let E be a nonempty set of fuzzy numbers. A relation on E , denoted by R, is a subset of E $\times E$, the product set of E and E itself, i.e., $R \subseteq E \times$E . For any $\widetilde{a}, \widetilde{b} \in E$, we say that $\widetilde{a}$ is related to $\widetilde{b}$ iff $(\widetilde{a}, \widetilde{b})$, denoted by $\tilde{a}R\tilde{b}$ simply.

Relation R is said to be reflexive iff $\tilde{a}R\tilde{a}$ for every $\tilde{a} \in E$. Relation R is said to be symmetric iff $\tilde{a}R\tilde{b}$ implies $\tilde{b}R\tilde{a}$ for any $\tilde{a}, \tilde{b} \in E$. Relation R is said to be antisymmetric iff $\tilde{a}R\tilde{b}$ and $\tilde{b}R\tilde{a}$ imply $\tilde{a} = \tilde{b}$ for any $\tilde{a}, \tilde{b} \in E$. Relation R is said to be transitive iff $\tilde{a}R\tilde{b}$ and $\tilde{b}R\tilde{c}$ imply $\tilde{a}R\tilde{c}$ for any $\tilde{a}, \tilde{b},\ \tilde{c} \in E$. Relation R is said to be totally comparable iff either $\tilde{a}R\tilde{b}$ or $\tilde{b}R\tilde{a}$ holds for any $\tilde{a}, \tilde{b} \in E$.

Definition 2. Relation R is called a ranking on E iff it is reflexive, transitive, and totally comparative. A ranking R on E is called a total ordering iff it is also antisymmetric.

From Definition 2, we know that two different fuzzy numbers may share the same rank, but they must have different order. This is just the difference of ranking and total ordering.

Definition 3. Relation R is called an equivalence relation on E iff it is reflexive, symmetric, and transitive.

Generally, based on one or more selected characteristics of fuzzy numbers in a given set of fuzzy numbers, with a special type often, an equivalence relation and an opposite (antisymmetric and not reflexive) but transitive relation on this set of fuzzy numbers can be defined. Using these two relations, a ranking can be defined. It is easy to define a total ordering on its quotient space with respect to the equivalence relation, but not on the set of all fuzzy numbers themselves. An example is given in Section VII.

A ranking or a total ordering, as a relation, should be defined on the whole set of fuzzy numbers with the selected type, such as on $\mathcal{N}_T$ or $\mathcal{N}_P$, even on $\mathcal{N}_F$. In literature, some works define a ranking for only a few (two, three, or four) fuzzy numbers appearing in the present example or real problem by using a reference index that depends on some characteristic belonging only these given fuzzy numbers, such as the infimum or the supremum of their support sets, or even the average of the infimum and the supremum. Unfortunately, such a method is not stable in the following meaning: the calculated ranks for given fuzzy numbers will be disrupted by adding a new fuzzy number. Hence, in our current work, the rankings or total orderings are defined on the whole set of some selected type of fuzzy numbers, or even on the set of all fuzzy numbers. Thus, the defined ranking or total ordering is stable.

There are infinitely many different ranking methods that can be defined on the set of all fuzzy numbers, or on an infinite set of fuzzy numbers with selected type, such as triangular or trapezoidal fuzzy numbers. Two ranking methods defined on the same domain being different means that, in the domain, there are at least two different fuzzy numbers that have different results of ranking. Hence, only comparing a newly introduced ranking method with some existing ranking methods by using different ranking results for some selected fuzzy numbers does not make much sense. In our opinion, it is critical to establish some criteria of goodness for judging the ranking methods. In Section VI of this work, we introduce and discuss the geometric intuitivity of ranking and total ordering methods for fuzzy numbers.

4 The Expansion Center Of Fuzzy Numbers

Let $\widetilde{e}$ be a fuzzy number and m_e be its membership function.

Lemma 1. For any real number x, definite integrals $\int_{-\infty}^{x} m_e(t)\,dt$ and $\int_{x}^{\infty} m_e(t)\,dt$, as well as $\int_{-\infty}^{\infty} m_e(t)\,dt$ exist and are finite.

Proof. This is a direct result of the fact that membership function m_e is piecewise monotone. The finiteness comes from the fact that the support set of any fuzzy number is bounded.

Lemma 2. $\int_{-\infty}^{\infty} m_e(t)\,dt = 0$ if and only if $\widetilde{e}$ is a crisp real number.

Proof. If $\widetilde{e}$ is a crisp real number, then its membership function is zero everywhere except one point on the real line. Hence, the value of definite integral $\int_{-\infty}^{\infty} m_e(t)\,dt$ must be zero. Conversely, a proof by contradiction is adopted to show that if $\int_{-\infty}^{\infty} m_e(t)\,dt = 0$, then the value of membership function m_e is zero everywhere except one point on the real line. In fact, assume that there are two points t_1 and t_2 with $t_1 < t_2$ such that $m_e(t_1) > 0$ and $m_e(t_2) > 0$. Let $c = \min[m_e(t_1)\,,\ m_e(t_3)]$. Then $s > 0$. By the convexity of fuzzy numbers, we have

$$c_e = \begin{cases} a_l + \sqrt{\frac{1}{2}(a_b - a_l)(a_d + a_r - a_l - a_b)} & \text{if } a_l + a_d + a_r \le 3\, a_b \\ a_r - \sqrt{\frac{1}{2}(a_r - a_d)(a_d + a_r - a_l - a_b)} & \text{if } a_l + a_b + a_r \ge 3\, a_d \\ \frac{(a_l + a_b + a_d + a_r)}{4} & \text{otherwise} \end{cases} \quad (4)$$

$$c_e = \begin{cases} a_l + \sqrt{\frac{1}{2}(a_0 - a_l)(a_r - a_l)} & \text{if } a_l + a_r \le 2\, a_0 \\ a_r - \sqrt{\frac{1}{2}(a_r - a_0)(a_r - a_l)} & \text{otherwise} \end{cases} \quad (5)$$

$m_e(t) \ge s$ for every $t(t_1, t_2)$. Thus, $\int_{-\infty}^{\infty} m_e(t)\,dt \ge \int_{t_1}^{t_2} m_e(t)\,dt \ge s(t_2 - t_1) > 0$. This contradicts the condition $\int_{-\infty}^{\infty} m_e(t)\,dt = 0$.

Lemma 3. If $\int_{-\infty}^{b} m_e(t)\,dt > 0$ and $\int_{d}^{\infty} m_e(t)\,dt > 0$ for some real numbers b and d with $b < d$, then $\int_{b}^{d} m_e(t)\,dt > 0$.

Proof. From $\int_{-\infty}^{b} m_e(t)\,dt > 0$, we may find point $t_1 < b$ such that $m_e(t_1) > 0$. Similarly, we may find point $t_2 > d$ such that $m_e(t_2) > 0$. Denote $\min[m_e(t_1), m_e(t_2)]$ by s. Then $s > 0$. Since the convexity of $\widetilde{e}$, we have $m_e(t) > s$ for any t between t_1 and t_2. Thus, $\int_{b}^{d} m_e(t)\,dt > s(d - b) > 0$.

Definition 4. Let $\widetilde{e}$ be a fuzzy number but not a crisp real number. A real number c_e is called the expansion center of $\widetilde{e}$ iff $\int_{-\infty}^{c_e} m_e(t)\,dt = \int_{c_e}^{\infty} m_e(t)\,dt$. For any real number, its expansion center is just itself.

The role of expansion center for a given fuzzy number in fuzzy mathematics is similar to the one of medium (50 percentile) in statistics. The former is one of the numerical indexes of fuzziness possessed by a fuzzy number, while the latter is used for describing the randomness of a random variable. For any crisp real number, its expansion center is defined as itself. For any symmetric fuzzy number, its expansion center coincides with the symmetry center.

Theorem 1. For any given fuzzy number, its expansion center exists and is unique.

Proof. We only need to prove the conclusion for fuzzy numbers that are not crisp real numbers. Denote a given non crisp fuzzy number by $\widetilde{e}$. From Lemma 1, we know that definite integrals $\int_{-\infty}^{x} m_e(t)\,dt$ and $\int_{x}^{\infty} m_e(t)\,dt$ exist and are finite for any real number x and can be regarded as functions of x. Let $\int_{-\infty}^{\infty} m_e(t)\,dt = r$ and $f(x) = \int_{-\infty}^{x} m_e(t)\,dt - \int_{x}^{\infty} m_e(t)\,dt$, where $x \in$ R. Function f is continuous. Since $\widetilde{e}$ is not a crisp real number and supp(e) is bounded, we have $0 < r < \infty$ and $\lim_{x \to -\infty} f(x) = -r$ as well as $\lim_{x \to \infty} f(x) = r$. By the well-known Intermediate Value Theorem of Continuous Function, we know that there exists a real number c_e such that $f(c_e) = 0$ and, therefore, $\int_{-\infty}^{c_e} m_e(t)\,dt = \int_{c_e}^{\infty} m_e(t)\,dt$, that is, c_e is an expansion center of $\widetilde{e}$. Furthermore, we have $\int_{-\infty}^{c_e} m_e(t)\,dt = \int_{c_e}^{\infty} m_e(t)\,dt = r/2$. Now, let's prove the uniqueness of the expansion center. Assume that, for given fuzzy number $\widetilde{e}$, there exists another expansion center c_e'. Without any loss of generality, we may assume that $c_e' < c_e$. From both $\int_{-\infty}^{c_e} m_e(t)\,dt = \int_{c_e}^{\infty} m_e(t)\,dt = r/2$ and $\int_{-\infty}^{c_e'} m_e(t)\,dt = \int_{c_e'}^{\infty} m_e(t)\,dt = r/2$, we obtain $\int_{c_e'}^{c_e} m_e(t)\,dt = \int_{c_e}^{c_e'} m_e(t)\,dt$. So, $\int_{c_e'}^{c_e} m_e(t)\,dt = 0$. By using Lemma 3, we get $c_e' = c_e$. The proof is now complete.

Example 1. Let $\widetilde{e}$ be a trapezoidal fuzzy number with membership function (3) shown in Section 2. Its expansion center c_e can be expressed as (4)

As a special case, when $\widetilde{e}$ be a triangular fuzzy number with membership function (2), its expansion center is (5)

As another special case, when $\widetilde{e}$ be a rectangular fuzzy number with membership function (1), i.e., when $a_b - a_l = a_r - a_d = 0$, expression (4) is reduced to $c_e = \frac{1}{2}(a_l + a_r)$. This coincides with the intuition.

For a given fuzzy number, in general cases, a numerical method may be used to calculate the values of the involved definite integrals and a soft computing technique, such as the genetic algorithm, can be adopted to optimize the location of its expansion center.

5 Ranking Fuzzy Number By Their Expansion Center

By using the expansion center of fuzzy numbers, we may propose a ranking method as follows.

Definition 5. Let $\widetilde{e}$ and $\widetilde{f}$ be two fuzzy numbers. We say that fuzzy number $\widetilde{e}$ precedes fuzzy number $\widetilde{f}$, denoted by $\widetilde{e} \preccurlyeq \widetilde{f}$, iff their expansion centers satisfy $c_e \leq c_f$.

Relation $\preccurlyeq$ is well defined on the set of all fuzzy numbers, $\mathcal{N}_F$. It does not satisfy the antisymmetry, i.e., $\widetilde{e}$ and $\widetilde{f}$ may be different even both $\widetilde{e} \preccurlyeq \widetilde{f}$ and $\widetilde{f} \preccurlyeq \widetilde{e}$ hold. So, it is not a total ordering on $\mathcal{N}_F$. However, this relation can be used to rank fuzzy numbers.

Example 2. Let fuzzy numbers $\widetilde{e}$ and $\widetilde{f}$ have membership functions

$$m_e(x) = \begin{cases} 1 & \text{if } x \in [0,\ 1] \\ \frac{5-x}{4} & \text{if } x \in (1,\ 5] \\ 0 & \text{otherwise} \end{cases}$$

and

$$m_f(x) = \begin{cases} x & \text{if } x \in [0,\ 1] \\ 2-x & \text{if } x \in (1,\ 1.5] \\ 0.5 & \text{if } x \in (1.5,\ 3.5] \\ 0 & \text{otherwise} \end{cases}$$

shown in Figures 1 and 2 respectively. $\widetilde{e}$ is a trapezoidal fuzzy number but $\widetilde{f}$ is not. Their expansion centers are c_e 1.5359 and $c_f = 1.5625$ respectively. So, $\widetilde{e} \preccurlyeq \widetilde{f}$.

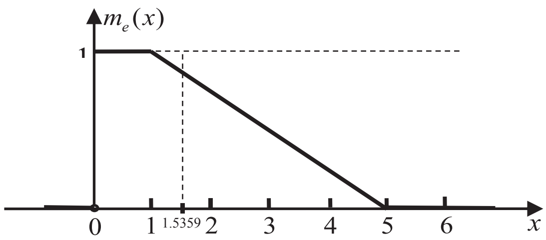

Figure 1. The membership function of fuzzy number $\widetilde{e}$ and its expansion center $c_e \approx 1.536$ in Example 2

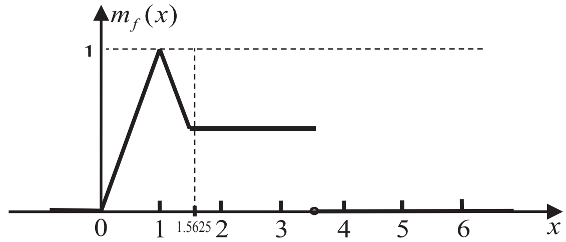

Figure 2. The membership function of fuzzy number $\widetilde{f}$ and its expansion center $c_f = 1.5625$ in Example 2

6 Geometric Intuitivity Of Ranking And Total Ordering

The geometric intuitivity can be defined by the geometric location of the non-zero parts of membership functions corresponding to the given fuzzy numbers.

Definition 6. For two fuzzy numbers $\tilde{e}$ and $\tilde{f}$, if the α-cuts $e_\alpha \leq f_\alpha$, i.e., $l_\alpha(e) \leq l_\alpha(f)$ and $r_\alpha(e) \leq r_\alpha(f)$ for every $\alpha \in (0,1]$, then $\tilde{e} \preccurlyeq \tilde{f}$.

Intuitively, if the curve of the membership function of $\tilde{e}$ is totally on the left (including the same) of the curve of the membership function of $\tilde{f}$, then the ranking method should issue $\tilde{e} \preccurlyeq \tilde{f}$. This charactonym guarantees that the ranking coincides with the natural ordering of the real numbers, i.e., regarding real numbers as special fuzzy numbers, their defined rank for fuzzy numbers is the same as their natural order.

In our case, the proposed ranking method obviously follows the above-mentioned definition. So, our ranking method adheres to the geometric intuitivity for rankings.

7 Equivalence Classes And A Total Ordering On The Quotient Space

By using the concept of expansion center, we may define a relation   on the set of all fuzzy numbers, $\mathcal{N}_F$, as follows.

Definition 5. We say that two fuzzy numbers $\widetilde{e}$ and $\widetilde{f}$ are equivalent, denoted as $\widetilde{e} \sim \widetilde{f}$, iff they have the same expansion center, i.e., $c_e = c_f$.

Example 3. Let fuzzy numbers $\widetilde{g}$ and $\widetilde{h}$ have membership functions

$$m_g(x) = \begin{cases} \frac{2x}{3} & \text{if } x \in [0,\ 1.5) \\ 1 & \text{if } x = 1.5 \\ \frac{6-2x}{3} & \text{if } x \in (1.5,\ 3] \\ 0 & \text{otherwise} \end{cases}$$

and

$$m_h(x) = \begin{cases} 1 & \text{if } x \in [0,\ 2) \\ \frac{4-x}{2} & \text{if } x \in (2,\ 4] \\ 0 & \text{otherwise} \end{cases} .$$

Figure 3 shows the membership functions of fuzzy numbers $\widetilde{g}$ and $\widetilde{h}$ with their expansion center. Their expansion centers c_g and c_h have the same value 1.5. So, $\widetilde{g}\ \widetilde{h}$, that is, these two fuzzy numbers have the same rank when the ranking method shown in Section 4 is used.

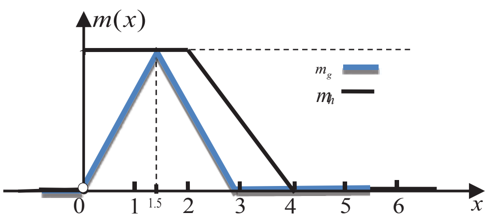

Figure 3. The membership functions of fuzzy numbers $\widetilde{g}$ and $\widetilde{h}$ with their expansion center $c_g = c_h$ = 1.5 in Example 3

Relation $\sim$ is reflective, symmetric, and transitive. Hence, it is an equivalence relation on $\mathcal{N}_F$. The collection of all equivalent classes with respect to equivalence relation $\sim$ forms the quotient space, denoted by $Q_\sim$. Regarding $\preccurlyeq$ as a relation on $Q_\sim, (Q_\sim, \preccurlyeq)$ is a total ordered set [14].

8 Conlusions

The introduced new concept of expansion center for fuzzy numbers is effective in ranking fuzzy numbers. This new way of ranking fuzzy numbers is intuitive and practicable. It provides an alternative choice in decision making and data mining within fuzzy environment.

9 Acknowledgement

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