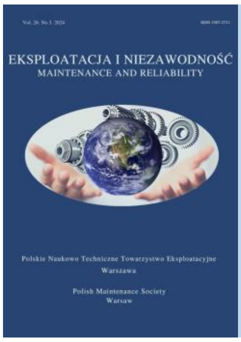

Eksploatacja i Niezawodnosc – Maintenance and Reliability

Volume 26 (2024), Issue 3

journal homepage: http://www.ein.org.pl

Article citation info:
Wei Y, Ling X, Liu S, Bayesian calculation of optimal burn-in policy for heterogeneous items under two-dimensional warranty, Eksploatacja i Niezawodnosc – Maintenance and Reliability 2024: 26(3) http://doi.org/10.17531/ein/186824

Bayesian calculation of optimal burn-in policy for heterogeneous items under two-dimensional warranty

Indexed by: Web of Science Group

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Highlights

- A new burn-in model of heterogeneous items with two-dimensional warranty is considered.
- We screen the items according to the failure information of the items during burn-in.
- A Bayesian method has been proposed to address the uncertainty of parameters in the model.

Abstract

Various burn-in procedures have been greatly used to screen weak items and reduce warranty costs. This paper proposes a new burn-in model for heterogeneous items with non-renewing two-dimensional warranty. All failures within burn-in and warranty are assumed to be repaired through the minimal repair. Then we screen the items according to the failure information of the items during burn-in. We establish a cost-based model to optimize the mean total cost of each item put into the market. We demonstrate that the optimal burn-in time or usage rate should reach its upper bound under some conditions. In practice, the reliability and mean total cost of an item may be random due to the uncertainty of parameters in the model. Therefore, we also propose a Bayesian method to calculate the mean total cost and optimal burn-in policy of an item, which fully considers the uncertainty of parameters in the model. An example is also given to demonstrate the proposed burn-in model and Bayesian method.

Keywords
burn-in, Bayesian analysis, two-dimensional warranty, heterogeneous population, minimal repair, reliability.

1. Introduction

In reality, the early failure (i.e., infant mortality) of items has greatly increased the cost of warranty services provided by manufacturers. Burn-in is a usual method for manufacturers to screen and reduce early failure. The burn-in procedure refers to letting the item operate in the burn-in environment for a period before putting it into the market. Jensen and Petersen (1982) and Kuo and Kuo (1983) introduced the common methods and the early literature on burn-in modeling.

In the market, the items are usually sold with a warranty service, which may bring higher costs to manufacturers. The burn-in procedure can reduce the warranty cost of items through reducing the early failure of items, it has attracted extensive attention (Sheu and Chien 2005; Ye et al 2012). The warranty policies can be classified into two types: one-dimensional (1D) warranty policy with only one variable constrained and two-dimensional (2D) warranty policy with two variables constrained. The burn-in models of items with 1D warranty have been greatly studied (see, Mi, 1997; Yun et al., 2002; Sheu and Chien, 2005; Wu et al., 2007; Ye et al., 2012; Shafiee et al., 2013; Chen et al., 2021; Hu et al., 2021). The burn-in models of items with 2D warranty have also

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received attention recently. The 2D warranty policy is widely used for some complex or heavy items, e.g., vehicles, printers and aircraft. For example, new vehicles on the market usually have a 2D warranty with 3-years or 36,000-miles, whichever comes first (Wang and Xie, 2018). Recently, there has been much research on the hot topic of 2D warranty policy (Wang et al., 2018; Dong et al., 2018; Dai et al., 2021; Mitra, 2021; Zhang et al., 2022; Gupta and Bhattacharya, 2022). In practice, items with 2D warranty also experience early failure. Attardi et al. (2005), Rai and Singh (2006), and Majeske (2007) analyzed vehicles warranty data and found that early failures accounted for approximately 5.6% of total failures, these vehicles typically have a warranty of no less than three years or 60,000 kilometers. Saleh et al. (2019) analyzed helicopter failures in the U.S. civilian fleet and found that there was a maintenance error infant mortality rate, that is, a large number of failure accidents occurred within a short flight hour after maintenance. Of these accidents, 31% occurred within the first 10 flight hours. Li et al. (2019) provided an example of a civilian helicopter gearbox, which includes a warranty of no less than two years or 600 hours of operating time. Therefore, we should also consider the corresponding burn-in model to reduce 2D warranty cost of items. Ye et al. (2013) first developed and analyzed two new burn-in models for repairable items with normal and defect failure modes under 2D warranty. Recently, Li et al. (2019) established a burn-in model for repairable items considering preventive maintenance, 2D warranty and extended warranty. Wei et al. (2022) considered a burn-in model for heterogeneous items with 2D combination warranty.

All aforementioned literature only assumes that the model parameters are known or estimated by frequency statistics. This does not take into account the uncertainty of the parameters in the model. However, in reality, the parameters in the model are usually uncertain, so it is necessary to consider the uncertainty of these parameters. The uncertainty of related parameters is usually characterized through Bayesian statistics. Perlstein et al. (2001) proposed a Bayesian method to calculate the cost-based optimal burn-in policy for heterogeneous items, where the item comes from a heterogeneous population modeled by a mixed Exponential distribution. Kwon and Keats (2002) established a Bayesian burn-in model for limited failure population, where normal item does not fail within their technical life and defective item fails shortly after it starts running. Yuan and Kuo (2010) proposed a cost-based Bayesian burn-in model, where the item follows the Weibull-Exponential distribution. Ulusoy et al. (2011) considered a Bayesian burn-in model of heterogeneous items to jointly optimize item reliability and expected total cost, where the item follows the mixed Weibull distribution. Yuan et al. (2016) considered a new Bayesian burn-in model for degradation-based items, where the two-phase degradation patterns of an item are described by a bi-exponential model.

However, the above literature only considered the Bayesian methodology of the burn-in model for 1D warranty items. There are many differences between the Bayesian method of the burn-in model for 2D warranty items and that for 1D warranty items, because the uncertainty of parameters in the distribution of customer random usage rate and the effect of the usage rate on item aging should be considered. Therefore, we developed a Bayesian method to solve the optimal burn-in policy of items with 2D warranty. Furthermore, items heterogeneity should not be ignored in reality, otherwise, it will lead to misleading in engineering (Finkelstein and Cha, 2013). Therefore, this paper considers that the items population consists of a mixture of two different subpopulations. In addition, many studies directly scrap or replace items that fail within burn-in procedure, regardless of their repairability. However, for some complex and expensive items, the entire item is not discarded by the failure in the burn-in procedure but can be repaired (Cha, 2000). This paper considers that all failures of item within burn-in and warranty periods are repaired by minimal repair. The burn-in model with minimal repair for item failures during burn-in is also considered in Finkelstein and Cha (2013), Ye et al. (2013), Cha and Badía (2016) and Li et al. (2019). The minimal repair will not change the attribute (strong or weak) and failure rate of the item, that is, the failed strong (weak) item is still a strong (weak) item after repair and its failure rate will not change. Then, we screen the items according to the failure information during burn-in procedure. If the number of failures for an item during burn-in exceeds the burn-in screening threshold, it will be scrapped and replaced with a new one; otherwise, it will be put into the market.

We summarize the contributions of this paper as follows. To the best of our knowledge, this paper is the first to study Bayesian burn-in method for items with 2D warranty, which is different from existing Bayesian burn-in method for items with 1D warranty. This method fully considers the uncertainty of item parameters and updates the parameters of the model by combining prior knowledge with observed data. And then we obtain a more accurate and reliable optimal burn-in policy. In addition, we also consider the minimal repair during the burn-in procedure based on the previous burn-in model under 2D warranty. This is more applicable to the burn-in of some complete machines or modular subsystems, where components that fail during the burn-in period are repaired or replaced instead of directly discarding the entire machines or system. This can also better save costs and reserve greater profit margins for manufacturers.

The rest of this paper is organized below. Section 2 presents the burn-in model considered in this paper. Section 3 establishes a burn-in model and analyzes its optimal burn-in policy. Section 4 develops a Bayesian method for mixed Weibull distribution. Section 5 gives a numerical example to illustrate the proposed model and method. Section 6 concludes this paper.

2. Model Formulation

For convenience, Table 1 lists the main notations to be used.

Table 1. Notations.

Symbol	Meaning
π	the proportion of strong items in the heterogeneous population.
X, X_1, X_2	the first failure time of items in heterogeneous population, strong subpopulation, and weak subpopulation.
$F(\cdot), F_1(\cdot), F_2(\cdot)$	the distribution functions of X, X_1 and X_2.
R	the random usage rate of the customer population.
$G(r)$	the distribution function of R.
r_0	nominal usage rate.
ξ_1, ξ_2	the acceleration coefficients of strong and weak items.
$F_i(t\|r)$	the distribution functions of X_i under the usage rate r.
$f_i(t\|r)$	the density functions of X_i under the usage rate r.
$F(t\|r)$	the distribution functions of X under the usage rate r.
t_b, r_b, n_b	the burn-in usage rate, the burn-in time, and the burn-in screening threshold.
$\lambda_i(t\|r_b)$	the failure rate of X_i during burn-in period.
$N_i(t\|r_b)$	the number of failures of X_i under the usage rate r_b.
W, U	the limits of age and usage for the 2D warranty policy.
T_r	the actual length of warranty time given $R = r$.
$\pi(t_b, r_b, n_b)$	the proportion of strong items in the population after burn-in screening.
$F_i^w(t\|r, t_b, r_b)$	the distribution function of X_i given $R = r$ within warranty region, $i = 1,2$.
$\lambda_i^w(t\|r, t_b, r_b)$	the failure rate of X_i given $R = r$ within warranty region, $i = 1,2$.
$N_i^w(t\|r, t_b, r_b)$	the number of failures of X_i given $R = r$ within warranty, $i = 1,2$.
$N^w(T_r\|r, t_b, r_b, n_b)$	the number of minimal repairs within warranty region given $R = r$.
$\lambda^w(t\|r, t_b, r_b, n_b)$	the intensity function of $N^w(T_r\|r, t_b, r_b)$.
$N^w(t_b, r_b, n_b)$	the number of minimal repairs within warranty region.
T_b	the operating time until the first item passes burn-in.
S_n	the occurrence time of the nth repair of an item during burn-in.
$H_n(t)$	the distribution function of S_n.
$Y_1, \dots, Y_n$	the independent and identically distributed random variables with the distribution function $H_{n_b+1}(t\|S_{n_b+1} \le t_b), t \le t_b$.
$M - 1$	the random number of replacements until the first item passes burn-in.
$c_1(r_b)$	the burn-in operation cost per unit time for each item.
ξ_0	the shape parameter of $c_1(r_b)$.
c_0, c_2, c_3, c_4	the setup cost, the replacement cost per item during burn-in, the cost of each minimal repair during burn-in, and the minimal repair cost per failure within warranty region.

$E[C(t_b, r_b, n_b)]$	the mean total cost per item.
t_b^*, r_b^*, n_b^*	the optimal burn-in time, usage rate and screening threshold.

2.1. Failures modelling

This paper considers that all items are from a heterogeneous population consisting of strong and weak subpopulations, and that all failures of items are repaired by minimal repair. Suppose that the proportion of strong items in the heterogeneous population is π. We use X, X_1 and X_2 to represent the first failure time of items in heterogeneous population, strong subpopulation and weak subpopulation, respectively. Denote the distribution functions of X, X_1 and X_2 by $F(\cdot)$, $F_1(\cdot)$ and $F_2(\cdot)$, respectively. In addition, we consider that the usage rate of different customers may be different, but the usage rate of each customer is constant. Let R with distribution function $G(r)$ be the random usage rate of the customer population, where $r \in \mathcal{R}$. Manufacturers can obtain this distribution through some surveys or fitting actual data. Common usage rate distributions include Uniform distribution, Gamma distribution, Lognormal distribution, Weibull distribution, etc.

The burn-in procedure is always carried out in an accelerated environment (Block and Savits 1997). The Accelerated Failure Time model is often used to describe the effect of usage rate and accelerated environment on item aging (Blischke et al., 2011; Zaharia, 2019; Lone and Panahi, 2022). Therefore, we use the Accelerated Failure Time model to model the effect of usage rate on item operation. Let the item be designed for a nominal usage rate r_0 and the virtual age of X_i at time t under usage rate r be given by $(r/r_0)^{\xi_i}t, i = 1,2$. The notations ξ_1 and ξ_2 are the acceleration coefficients of strong and weak items respectively and they describe the effect of accelerated environment on item aging. Then the distribution functions of X_i and X under the usage rate r are

$$F_i(t|r) = F_i((r/r_0)^{\xi_i}t|r_0),$$

and

$$F(t|r) = \pi F_1(t|r) + (1-\pi)F_2(t|r).$$

respectively, $i = 1,2$. Then the density function of X_i under the usage rate r is

$$f_i(t|r) = (r/r_0)^{\xi_i} f_i((r/r_0)^{\xi_i}t|r_0), i = 1,2.$$

2.2. Failures during burn-in

The burn-in should be carried out for a fixed length of time in a specific environment. Here we use the usage rate r_b, the burn-in time t_b and the burn-in screening threshold n_b to characterize the burn-in environment, length of time and screening condition. We use $\mathcal{T}_\mathrm{b}$, $\mathcal{R}_\mathrm{b}$ and $\mathcal{N}_\mathrm{b}$ to represent the value ranges of t_b, r_b and n_b, respectively.

Note that the failure rate of X_i during burn-in period is

$$\lambda_i(t|r_b) = \frac{f_i(t|r_b)}{1 - F_i(t|r_b)}$$
$$= (\frac{r_b}{r_0})^{\xi_i} \frac{f_i((\frac{r_b}{r_0})^{\xi_i}t|r_0)}{1 - F_i((\frac{r_b}{r_0})^{\xi_i}t|r_0)} \tag{2.1}$$
$$= (\frac{r_b}{r_0})^{\xi_i} \lambda_i((\frac{r_b}{r_0})^{\xi_i}t|r_0), i = 1,2$$

Ascher and Feingold (1984) demonstrated that the number of minimal repairs can be modelled by a nonhomogeneous Poisson process (NHPP) in which the intensity function is the failure rate function of item. Therefore, the number of failures of X_i under the usage rate r_b follows the NHPP (denoted by $N_i(t|r_b)$) with the intensity function $\lambda_i(t|r_b), i = 1,2$. For more literature on modeling the number of minimal repairs using NHPP, see Finkelstein and Cha (2013) and Cha and Finkelstein (2011).

We screen the items according to their failure information, if the number of failures of an item exceeds the screening threshold n_b within burn-in, it will be scrapped and replaced with a new one; otherwise, it will be put into the market. Therefore, the probability that the number of failures of X_i is not greater than the screening threshold n_b during the burn-in is

$$P\{N_i(t_b|r_b) \le n_b\} = \sum_{n=0}^{n_b} P\{N_i(t_b|r_b) = n\} \tag{2.2}$$

$$= \sum_{n=0}^{n_b} \frac{[\int_0^{t_b} \lambda_i(t|r_b)\, dt]^n}{n!} \exp\left\{-\int_0^{t_b} \lambda_i(t|r_b)\ \mathrm{d}t\right\}, i = 1,2.$$

Note that the minimal repair should not change the (strong or weak) attribute of the item. Then the probability of X passing the burn-in screening is

$$P\{N(t_b|r_b) \le n_b\} = \pi P\{N_1(t_b|r_b) \le n_b\} + (1-\pi) P\{N_2(t_b|r_b) \le n_b\}.$$

2.3. Failures during warranty period

In this paper, we consider that the 2D warranty policy is limited by age W and usage U, whichever comes first. Then the actual length of warranty time given $R = r$ is

$$T_r = \begin{cases} W, & r \le U/W, \\ U/r, & r > U/W. \end{cases}$$

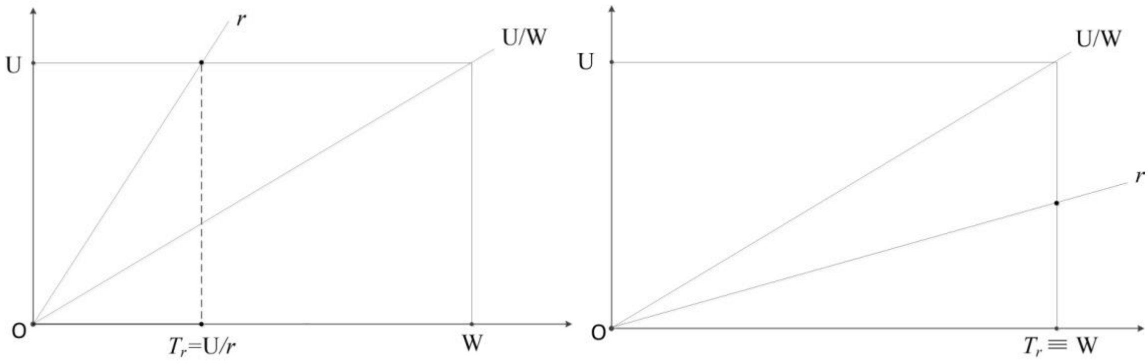
Figure 1 gives the actual length of warranty time T_r given $R = r$.

Note that the proportion of strong items in the population after burn-in screening is

$$\pi(t_b, r_b, n_b) = \frac{\pi P\{N_1(t_b|r_b) \le n_b\}}{\pi P\{N_1(t_b|r_b) \le n_b\} + (1-\pi) P\{N_2(t_b|r_b) \le n_b\}}, \quad (2.3)$$

and the distribution function of X_i given $R = r$ within warranty region is

$$F_i^w(t|r, t_b, r_b) = \frac{F_i((r/r_0)^{\xi_i} t + (r_b/r_0)^{\xi_i} t_b | r_0) - F_i((r_b/r_0)^{\xi_i} t_b | r_0)}{1 - F_i((r_b/r_0)^{\xi_i} t_b | r_0)}, i = 1,2.$$



(a) $r \le U/W$ (b) $r > U/W$

Figure 1. The actual length of warranty time T_r given $R = r$.

Then, the failure rate of X_i given $R = r$ within warranty region is

$$\lambda_i^w(t|r, t_b, r_b) = \frac{\frac{dF_i^w(t|r, t_b, r_b)}{dt}}{1 - F_i^w(t|r, t_b, r_b)}$$

$$= (\frac{r}{r_0})^{\xi_i} \frac{f_i((\frac{r_b}{r_0})^{\xi_i} t_b + (\frac{r}{r_0})^{\xi_i} t | r_0)}{1 - F_i((\frac{r_b}{r_0})^{\xi_i} t_b + (\frac{r}{r_0})^{\xi_i} t | r_0)} \quad (2.4)$$

$$= (\frac{r}{r_0})^{\xi_i} \lambda_i((\frac{r_b}{r_0})^{\xi_i} t_b + (\frac{r}{r_0})^{\xi_i} t | r_0), i = 1,2.$$

Then the number of failures of X_i given $R = r$ follows the NHPP (denoted by $N_i^w(t|r, t_b, r_b)$) with the intensity function $\lambda_i^w(t|r, t_b, r_b)$, $i = 1,2$. Therefore, the mean number of minimal repairs within warranty region given $R = r$ is given by

$$E[N^w(T_r|r, t_b, r_b, n_b)]$$
$$= \pi(t_b, r_b, n_b) E[N_1^w(T_r|r, t_b, r_b, n_b)] + (1 - \pi(t_b, r_b, n_b)) E[N_2^w(T_r|r, t_b, r_b, n_b)]$$
$$= \pi(t_b, r_b, n_b) \int_0^{T_r} \lambda_1^w(t|r, t_b, r_b, n_b)\, dt + (1 - \pi(t_b, r_b, n_b)) \int_0^{T_r} \lambda_2^w(t|r, t_b, r_b, n_b)\, dt$$
$$= \int_0^{T_r} \lambda^w(t|r, t_b, r_b, n_b)\, dt, \quad (2.5)$$

where

$$\lambda^w(t|r, t_b, r_b, n_b) = \pi(t_b, r_b, n_b) \lambda_1^w(t|r, t_b, r_b, n_b) + (1 - \pi(t_b, r_b, n_b)) \lambda_2^w(t|r, t_b, r_b, n_b). \quad (2.6)$$

Hence, the mean number of warranty claims is

$$E[N^w(t_b, r_b, n_b)] = \int_{\Re} E[N^w(T_r|r, t_b, r_b, n_b)] dG(r)$$
$$= \int_{\Re} \int_0^{T_r} \lambda^w(t|r, t_b, r_b, n_b)\, dt dG(r).$$

3. Model Analysis

In practice, cost has always been a concern for manufacturers. The competitive advantage of items can be improved by reducing costs. Therefore, we wish to develop a cost-based burn-in model to obtain the optimal burn-in policy that minimizes the mean total cost per item sold. The mean total cost can also tell manufacturers how much money they need to set aside for burn-in and warranty.

The total cost includes the burn-in cost and the warranty cost. The burn-in cost includes the fixed setup cost c_0, the burn-in operation cost, the replacement cost, and the cost of minimal repairs within burn-in. The burn-in operation cost should be proportional to the operating time T_b. We use S_n with distribution function $H_n(t)$ to express the occurrence time of the nth repair of an item during burn-in, that is

$$H_n(t) = P\{S_n \leq t\} = P\{N(t|r_b) > n-1\}. \tag{3.1}$$

The operating time of one item during burn-in is S_{n_b+1} if $N(t_b|r_b) > n_b$, and it is t_b if $N(t_b|r_b) \leq n_b$. Let $Y_1, \ldots, Y_n$ be the independent and identically distributed random variables with the distribution function $H_{n_b+1}(t|S_{n_b+1} \leq t_b), t \leq t_b$. Let $M-1$ be the random number of replacements until the first item passes burn-in screening. Then the total burn-in operating time to obtain an item that passes the burn-in screening is

$$T_b = \sum_{j=1}^{M-1} Y_j + t_b.$$

It is easy to see that M follows the geometric distribution with mean $E[M] = 1/(1-H_{n_b+1}(t_b))$. Therefore,

$$\begin{aligned}
E[T_b] &= E\left[\sum_{j=1}^{M-1} Y_j + t_b\right] \\
&= E[M-1]E[Y] + t_b \\
&= \frac{H_{n_b+1}(t_b)}{1-H_{n_b+1}(t_b)} \frac{\int_0^{t_b} x h_{n_b+1}(x)dx}{H_{n_b+1}(t_b)} + t_b \qquad (3.2)\\
&= \frac{\int_0^{t_b} x h_{n_b+1}(x)dx + t_b[1-H_{n_b+1}(x)]}{1-H_{n_b+1}(t_b)} \\
&= \frac{\int_0^{t_b} 1-H_{n_b+1}(x)dx}{1-H_{n_b+1}(t_b)},
\end{aligned}$$

where $h_{n_b+1}(x) = H'_{n_b+1}(x)$. From (3.1) and (3.2), the mean operation cost is

$$c_1(r_b)\frac{\int_0^{t_b} P\{N(t|r_b) \leq n_b\}\mathrm{d}t}{P\{N(t_b|r_b) \leq n_b\}}, \tag{3.3}$$

where $c_1(r_b)$ is the burn-in operation cost per unit time for each item. Clearly, $c_1(r_b)$ should be increasing in r_b. For example, when temperature and voltage are used as environmental stresses, higher usage rate usually means higher temperature and voltage, which can result in additional energy costs. Here we give the form of $c_1(r_b)$ as

$$c_1(r_b) = a r_b^{\xi_0},$$

which is also used by Ye et al. (2013) and Li et al. (2019).

It is easy to get that the mean replacement cost and the mean cost of minimal repairs within burn-in procedure are

$$c_2 \frac{P\{N(t_b|r_b) > n_b\}}{P\{N(t_b|r_b) \leq n_b\}},$$

and

$$c_3 n_b \frac{P\{N(t_b|r_b) > n_b\}}{P\{N(t_b|r_b) \leq n_b\}} + c_3 E[N(t_b|r_b)|N(t_b|r_b) \leq n_b],$$

respectively, where c_2 is the replacement cost per item during burn-in, c_3 is the cost of each minimal repair during burn-in, and

$$\begin{aligned}
&E[N(t_b|r_b)|N(t_b|r_b) \leq n_b] \\
&= \pi(t_b, r_b, n_b)E[N_1(t_b|r_b)|N_1(t_b|r_b) \leq n_b] + (1 - \\
&\pi(t_b, r_b, n_b))E[N_2(t_b|r_b)|N_2(t_b|r_b) \leq n_b] \qquad (3.4)
\end{aligned}$$

is the mean number of minimal repairs for an item passing burn-in. Denote $C_b(t_b, r_b, n_b)$ as the burn-in cost incurred for obtaining the item passing the burn-in. Then, the mean burn-in cost is

$$\begin{aligned}
E[C_b(t_b, r_b, n_b)] = c_0 &+ a r_b^{\xi_0} \frac{\int_0^{t_b} P\{N(t|r_b) \leq n_b\}\, dt}{P\{N(t_b|r_b) \leq n_b\}} \\
&+ c_2 \frac{P\{N(t_b|r_b) > n_b\}}{P\{N(t_b|r_b) \leq n_b\}} \\
&+ c_3 n_b \frac{P\{N(t_b|r_b) > n_b\}}{P\{N(t_b|r_b) \leq n_b\}} \\
&+ c_3 E[N(t_b|r_b)|N(t_b|r_b) \\
&\leq n_b]. \qquad (3.5)
\end{aligned}$$

The warranty cost is determined by the number and cost of minimal repairs for each item within warranty. Let c_4 be the minimal repair cost per failure within warranty region. The failure during the warranty period not only leads to repair costs, but also incurs additional reputational losses, labor and scheduling costs. Therefore, the minimal repair cost for each failure in the warranty region should be higher than the minimal repair cost within burn-in period, that is $c_3 \leq c_4$. Then the mean warranty cost is

$$\begin{aligned}
E[C_w(t_b, r_b, n_b)] &= c_4 E[N^w(t_b, r_b, n_b)] \\
&= c_4 \int_{\Re}\int_0^{T_r} \lambda^w(t|r, t_b, r_b, n_b)\, dt dG(r). \qquad (3.6)
\end{aligned}$$

Therefore, the mean total cost per item can be obtained from equations (3.5) and (3.6):

$$E[C(t_b,r_b,n_b)] = c_0 + a r_b^{\xi_0} \frac{\int_0^{t_b} P\{N(t|r_b) \le n_b\}\, dt}{P\{N(t_b|r_b) \le n_b\}} + c_2 \frac{P\{N(t_b|r_b) > n_b\}}{P\{N(t_b|r_b) \le n_b\}} + c_3 n_b \frac{P\{N(t_b|r_b) > n_b\}}{P\{N(t_b|r_b) \le n_b\}} + c_3 E[N(t_b|r_b)|N(t_b|r_b) \le n_b] + c_4 \int_{\Re}\int_0^{T_r} \lambda^w(t|r,t_b,r_b,n_b)\, dt dG(r). \quad (3.7)$$

Then our burn-in model is denoted by

$$[t_b^*, r_b^*, n_b^*] = \underset{t_b\in\mathcal{T}_b, r_b\in\mathcal{R}_b, n_b\in\mathcal{N}_b}{\arg\min} E[C(t_b,r_b,n_b)], \quad (3.8)$$

where t_b^*, r_b^* and n_b^* represent the optimal burn-in time, usage rate and screening threshold, respectively.

The following theorem provides the properties of the optimal burn-in policy. It shows that the optimal burn-in time or usage rate should reach its upper limit under some mild conditions. Its proof is given in Appendix I.

Theorem 1. Suppose $\lambda_1(t|r_0)$ increases with t and $\lambda_1^w(t|r,t_b,r_b) \le \lambda_2^w(t|r,t_b,r_b)$.

(i) If $\xi_0 \le \xi_1 \le \xi_2$, then $r_b^* = \overline{r_b}$.

(ii) If $\xi_0 \ge \xi_1 \ge \xi_2$, then $t_b^* = \overline{t_b}$.

Remark 1. The result in Theorem 1 gives the characterizations for the optimal solution of (3.8), which can reduce the number of decision variables and then reduce the difficulty of solving. If the conditions of our result are not satisfied, we can also use some mature algorithms to calculate the optimal policy, such as pattern search, grid search or other derivative free optimization algorithms.

4. Bayesian model for mixed Weibull distribution

In Section 3, Theorem 1 explains the influence of parameters on the proposed burn-in model. In practice, it is difficult to obtain exact knowledge of these parameters, so their uncertainties shall be adequately quantified. Therefore, this section develops a Bayesian method to quantify the uncertainties of these related parameters.

We consider mixed Weibull distribution as the distribution of the item lifetime, because Weibull distribution is widely used to characterize item lifetime owing to its flexibility (Almalki and Nadarajah 2014; Andrzejczak and Bukowski 2021; Shuto and Amemiya 2022). Then X_i follows the Weibull distribution with distribution function

$$F_i(t) = 1 - e^{-\left(\frac{t}{\delta_i}\right)^{\eta_i}}, i = 1,2,$$

where $\delta_i > 0$ and $\eta_i > 0$ are the scale and shape parameters, respectively. Weibull distribution is related to many distributions, for example, when η_i=1, it is an Exponential distribution; when $\eta_i = 2$, it is a Rayleigh distribution.

Other lifetime distributions can be similarly analyzed by using the Bayesian method developed. On the other hand, the Gamma distribution and Uniform distribution are widely used to describe the distribution of customer usage rate (Iskandar and Murthy 2003). Although the Uniform distribution has a simpler form and is easier for analysis, in reality, the usage rate often varies according to different customer habits, especially for durable goods. The Gamma distribution is more suitable for characterizing random variables supported by $(0,\infty)$ rather than finite intervals. Therefore, we assume that the usage rate follows the Beta distribution on $(0,b)$, because it is an extension of the Uniform distribution, which can well describe the non-uniform distribution of random variables on a finite interval. Then the density function of R is

$$g(r) = \frac{\Gamma(\gamma+\lambda)}{\Gamma(\gamma)\Gamma(\lambda)} \frac{r^{\gamma-1}(b-r)^{\lambda-1}}{b^{\gamma+\lambda-1}}, 0 < r < b, \lambda > 0, \gamma > 0.$$

Therefore, the density function of X is

$$f(t|\boldsymbol{\theta}) = \int_0^b \frac{\Gamma(\gamma+\lambda)}{\Gamma(\gamma)\Gamma(\lambda)} \frac{r^{\gamma-1}(b-r)^{\lambda-1}}{b^{\gamma+\lambda-1}} [\pi \frac{\eta_1}{\delta_1} (\frac{r}{r_0})^{\eta_1\xi_1} (\frac{t}{\delta_1})^{\eta_1 - 1} e^{-(\frac{r}{r_0})^{\eta_1\xi_1}(\frac{t}{\delta_1})^{\eta_1}} + (1-\pi)\frac{\eta_2}{\delta_2}\left(\frac{r}{r_0}\right)^{\eta_2\xi_2}\left(\frac{t}{\delta_2}\right)^{\eta_2-1} e^{-\left(\frac{r}{r_0}\right)^{\eta_2\xi_2}\left(\frac{t}{\delta_2}\right)^{\eta_2}}] \mathrm{d}r,$$

where $\boldsymbol{\theta} = (\pi,\gamma,\lambda,\xi_1,\xi_2,\eta_1,\eta_2,\delta_1,\delta_2)$ represents the parameter vector of the model. We consider quantifying the uncertainty of the model parameters through the joint prior distribution. For analysis, it is reasonable to assume that prior knowledge of different parameters is independent of each other. Since the mixture parameter $\pi \in (0,1)$, a reasonable and general choice of its prior distribution is the Beta distribution with parameters $p, q \ge 0$ (Martz and Waller 1982), whose density function is

$$g(\pi) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \pi^{p-1}(1-\pi)^{q-1}.$$

Since the parameters γ and λ should be nonnegative, we use Gamma distribution to describe their prior distributions. The prior density functions of γ and λ are

$$g(\gamma)=\frac{k^{z}}{\Gamma(z)}\gamma^{z-1}e^{-k\gamma},$$

and

$$g(\lambda)=\frac{l^{v}}{\Gamma(v)}\lambda^{v-1}e^{-l\lambda},$$

respectively, where k and l are inverse scale parameters, z and v are the shape parameters. If $z=1$, the distribution is an Exponential distribution; if $z=n/2$ and $k=1/2$, the distribution is a Chi-square distribution.

In practice, the virtual age $(r/r_0)^{\xi_i}t$ should be increasing in r and t, that is, the parameters ξ_i should be nonnegative, $i=1,2$. Therefore, we use Gamma distribution to describe the prior distribution of ξ_i, whose density function is

$$g(\xi_i)=\frac{\sigma_i^{\mu_i}}{\Gamma(\mu_i)}\xi_i^{\mu_i-1}e^{-\sigma_i\xi_i}, i=1,2.$$

As motivated in Ulusoy et al. (2011), the Beta distribution and the Gamma distribution are designated as prior distributions of η_i and δ_i, respectively, $i=1,2$. Then the prior density function of η_i is

$$g(\eta_i)=\frac{\Gamma(a_i+b_i)}{\Gamma(a_i)\Gamma(b_i)}\frac{\eta_i^{a_i-1}(d-\eta_i)^{b_i-1}}{d^{a_i+b_i-1}}, i=1,2,$$

and the prior density function of δ_i is

$$g(\delta_i)=\frac{\beta_i^{\alpha_i}}{\Gamma(\alpha_i)}\delta_i^{\alpha_i-1}e^{-\beta_i\delta_i}, i=1,2.$$

Due to the independence assumption between parameters, the joint prior density function of $\boldsymbol{\theta}$ is

$$g(\boldsymbol{\theta})=g(\pi)g(\eta_1)g(\eta_2)g(\delta_1)g(\delta_2)g(\xi_1)g(\xi_2)g(\lambda)g(\gamma), \boldsymbol{\theta}\in\boldsymbol{\Theta},$$

where $\boldsymbol{\Theta}$ represents the parameter space of $\boldsymbol{\theta}$.

Assume that additional sample data can be obtained prior to the burn-in procedure from the field operation or some surveys. In practice, the sample data observed are usually censored. Let $D=(t_1,\dots,t_m,t^*,m,\kappa)$ denote the general form of the sample data, which represents that there are m of $m+\kappa$ items failed before the censoring time t^* and the observed failure times are given by $t_1,\dots,t_m$. Then the likelihood is

$$L(D|\boldsymbol{\theta}) \quad = \quad (1-F(t^*|\boldsymbol{\theta}))^{\kappa}\prod_{i=1}^{m}f(t_i|\boldsymbol{\theta}),$$

where $F(t^*|\boldsymbol{\theta})=\int_0^{t^*}f(t|\boldsymbol{\theta})\,\mathrm{d}t$. By using Bayes' Theorem, the joint posterior distribution of $\boldsymbol{\theta}\in\boldsymbol{\Theta}$ given date D is

$$\begin{aligned} g(\boldsymbol{\theta}|D) \quad &= \quad \frac{g(\boldsymbol{\theta})L(D|\boldsymbol{\theta})}{\int_{\boldsymbol{\Theta}}g(\boldsymbol{\theta})L(D|\boldsymbol{\theta})\mathrm{d}\boldsymbol{\theta}} \\ &\propto \quad g(\boldsymbol{\theta})L(D|\boldsymbol{\theta}). \end{aligned} \tag{4.1}$$

Then the posterior mean total cost per item is

$$E[C(t_b,r_b,n_b)|D] \quad = \quad \int_{\boldsymbol{\Theta}}E[C(t_b,r_b,n_b)|\boldsymbol{\theta}]g(\boldsymbol{\theta}|D)\,\mathrm{d}\boldsymbol{\theta}. \tag{4.2}$$

The marginal posterior densities of all parameters could be inferred from the joint posterior density, which would be used to calculate the posterior estimation of parameters. For instance,

the marginal posterior density of π is

$$g(\pi|D)=\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{d}\int_0^{d}g(\boldsymbol{\theta}|D)d\gamma d\lambda d\xi_1 d\xi_2 d\delta_1 d\delta_2 d\eta_1 d\eta_2. \tag{4.3}$$

Then, we can substitute the posterior estimation of parameters into equation (3.7) to calculate the mean total cost per item of the model under the posterior estimation parameters. It can be seen that the posterior mean total cost per item $E[C(t_b,r_b,n_b)|D]$ and the marginal posterior densities are obtained by multiple integrals. It is usually difficult to obtain an analytical form of multiple integrals, and sometimes it is even impossible to obtain numerical integrals directly. The Markov Chain Monte Carlo (MCMC) simulation is the most common method for obtaining reliable result without calculating integral (Yuan and Kuo, 2010). Therefore, we use OpenBUGS, a special software for MCMC, to calculate these integrals. Other software packages (e.g., RStan, WinBUGS, JAGS, etc.) can also be used to calculate them similarly. The following algorithm gives the steps of MCMC.

Algorithm 1. The steps of MCMC simulation algorithm are as follows:

(i) Simulate parameter vector samples $\boldsymbol{\theta}^{(1)},\boldsymbol{\theta}^{(2)},\dots,\boldsymbol{\theta}^{(s)}$ from posterior distribution $g(\boldsymbol{\theta}|D)$;

(ii) For each $\boldsymbol{\theta}^{(j)}$ vector obtained in Step (i), calculate the corresponding mean total cost per item $E[C(t_b,r_b,n_b)|\boldsymbol{\theta}^{(j)}], j=1,2,\dots,s$;

(iii) The posterior mean total cost per item $E[C(t_b,r_b,n_b)|D]$ and the posterior estimation of parameters can be approximated with the average value $\frac{1}{s}\sum_{j=1}^{s} E[C(t_b, r_b, n_b)|\boldsymbol{\theta}^{(j)}]$ and $\frac{1}{s}\sum_{j=1}^{s}\boldsymbol{\theta}^{(j)}$, respectively.

5. Numerical Example

This section illustrates the proposed burn-in model and Bayesian method with an example of the automotive gearboxes with 2D warranty. We obtain and compare the optimal burn-in policies for the models under the prior estimator, posterior estimator and actual parameters. For convenience, we assumed the actual parameters and generated the required data samples through simulation. Then, we give some sensitivity analysis on cost parameters to provide guidance for manufacturers to develop burn-in and warranty policy. Assume the warranty period for whole automobile shall not be less than 36 months or the mileage of 60,000 kilometers, whichever comes first (Wang and Xie, 2018). In practice, all parameters and data samples can be obtained through the information on the operation of previous items or through surveys. The units of measurement for cost, time and usage are 10^4 dollars, month and 10^3 kilometers, respectively.

5.1. Model structure and parameters

The 2D warranty is $W = 36$ and $U = 60$.

The value ranges of t_b, r_b and n_b are $\mathcal{T}_b = [0,2]$, $\mathcal{R}_b = [0,20]$ and $\mathcal{N}_b = [0,10]$, respectively.

The cost parameters are: $c_0 = 0.2$, $c_1 = 0.15 r_b^{0.5}, c_2 = 20, c_3 = 0.6$ and $c_4 = 2$.

The prior distribution of parameters are $\pi \sim Beta(13,2), \gamma \sim Gam(9,5), \lambda \sim Gam(70,3), \xi_1 \sim Gam(2,7), \xi_2 \sim Gam(2,14), \eta_1/6 \sim Beta(2,10), \eta_2/6 \sim Beta(2,10), \delta_1 \sim Gam(14,5), \delta_2 \sim Gam(2,13)$.

In order to obtain data samples, we simulated and generated failure data of 100 items within 10 months, where the actual parameters of the items are shown in Table 2.

Table 2. Parameters.

Parameter	π	γ	λ	η_1	η_2	δ_1	δ_2	ξ_1	ξ_2	r_0
Actual	0.80	2.00	23.00	1.20	1.20	3.00	0.20	0.25	0.15	20.00

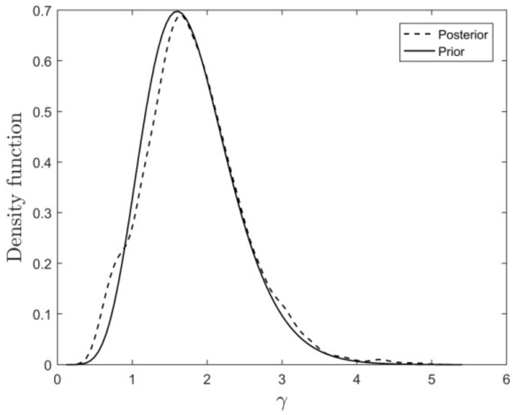
At the end of the test, we observed 88 failure times and 12 items were still operable at the end of 10 months, where the data samples are given in the Table 3.

Table 3. Data samples.

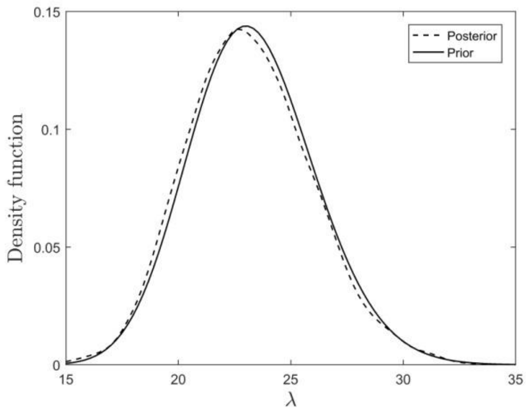
0.03	0.03	0.10	0.11	0.13	0.13	0.16	0.16	0.16	0.18	0.21	0.22	0.22	0.22	0.23
0.28	0.29	0.29	0.32	0.33	0.34	0.39	0.44	0.45	0.47	0.53	0.57	0.58	0.64	0.72
0.88	0.96	0.96	1.09	1.26	1.28	1.45	1.50	1.66	1.78	1.82	1.83	1.92	1.93	2.26
2.40	2.42	2.43	2.57	2.84	2.95	3.08	3.09	3.15	3.27	3.27	3.42	3.54	3.64	4.16
4.17	4.44	5.03	5.09	5.11	5.11	5.18	5.22	5.36	5.38	5.39	5.52	5.63	5.76	6.17
6.38	6.46	6.67	6.80	6.86	7.12	7.32	7.55	7.57	7.80	7.81	8.76	9.93		

5.2 Bayesian calculation

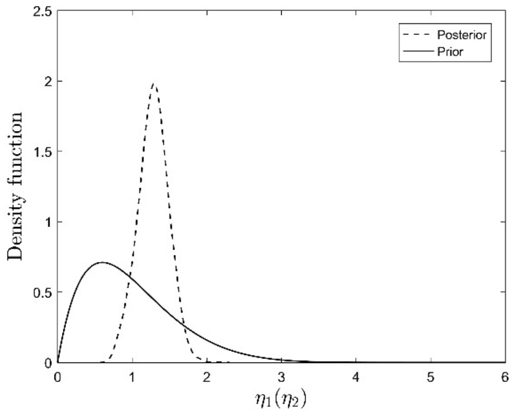
To calculate the mean total cost, we used OpenBUGS simulation to generate 12,000 realizations for each parameter and discarded the first 2,000 observations as warm-up calculations. Then, we can obtain the posterior density functions of all parameters, which are compared with the prior density functions of all parameters in Figure 2.



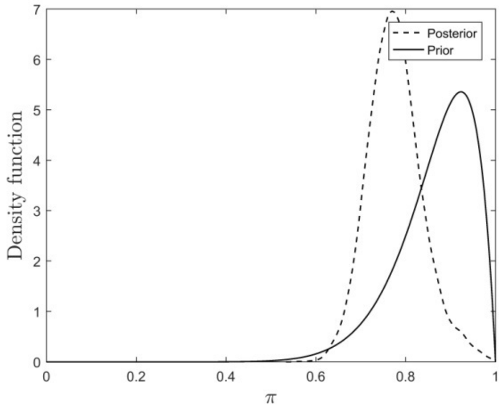
(a) The density functions of γ



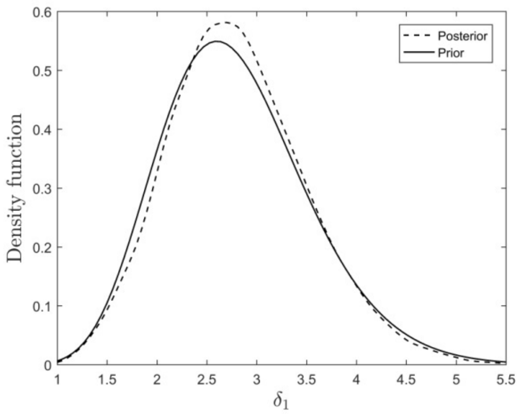
(b) The density functions of λ



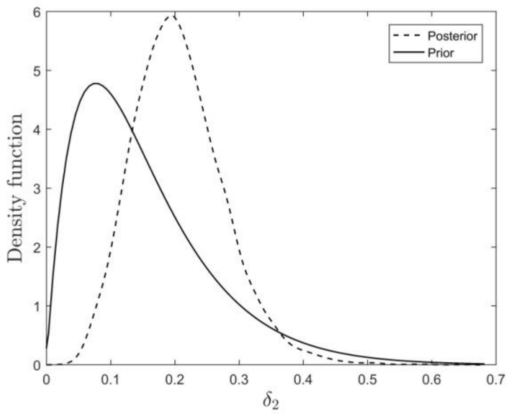
(c) The density functions of η_1 (η_2)



(d) The density functions of π



(e) The density functions of δ_1



(f) The density functions of δ_2

Figure 2. The density functions of prior and posterior parameters.

The posterior mean estimator (also known as posterior expectation estimator) is the most common Bayesian point estimator (Hamada et al. 2008). For convenience, we refer to posterior and prior mean estimators as posterior and prior estimators in this section. From the prior distribution of parameters, we can easily obtain the prior estimators of these parameters, which are given in the second row of Table 4. From the MCMC simulation algorithm, we can also obtain the posterior estimators of these parameters, which are given in the third row of Table 3. It shows that the posterior estimators of all parameters are closer to the actual values of the parameters than the prior estimators of all parameters, which means that the Bayesian calculation method proposed in this paper is effective. In addition, for parameters of the Table 3, we have $\xi_0 > \xi_1 > \xi_2$. Therefore, from Theorem 1 (ii), the optimal burn-in time should be taken as its upper bound $\overline{t_b} = 2$.

Table 4. prior and posterior parameters.

Parameter	π	γ	λ	η_1	η_2	δ_1	δ_2	ξ_1	ξ_2
Prior	0.87	1.80	23.33	1.00	1.00	2.80	0.15	0.29	0.14
Posterior	0.78	1.83	23.17	1.26	1.26	2.81	0.21	0.25	0.15

The effects of burn-in usage rate and screening threshold on the mean total costs under prior, posterior and actual parameters are given in Figure 3. We can obtain that the optimal burn-in policies under prior, posterior and actual parameters are $[t_{b1}^*, r_{b1}^*, n_{b1}^*] = [2,0.13,1]$, $[t_{b2}^*, r_{b2}^*, n_{b2}^*] = [2,0.40,1]$ and $[t_{b3}^*, r_{b3}^*, n_{b3}^*] = [2,0.39,1]$, respectively. The corresponding minimum mean total costs under prior, posterior and actual parameters are $E[C_1(t_{b1}^*, r_{b1}^*, n_{b1}^*)] = 14.3806$, $E[C_2(t_{b2}^*, r_{b2}^*, n_{b2}^*)] = 25.5603$ and $E[C_3(t_{b3}^*, r_{b3}^*, n_{b3}^*)] = 21.7980$, respectively. We can also see that the optimal burn-in policy and the minimum mean total cost under posterior parameters are closer to the actual parameters than those under prior parameters. It means that the Bayesian calculation method proposed in this paper is reasonable.

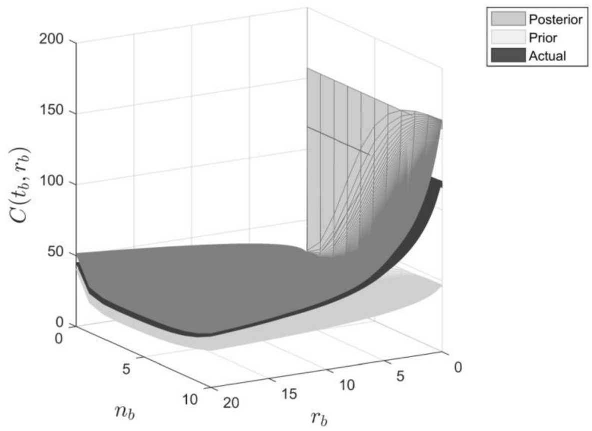


Figure 3. The mean total cost functions under prior, posterior and actual parameters.

5.3. Benefit analysis

5.3.1. The benefit of burn-in

Table 5 gives the benefit of burn-in, where the benefit (the "Reduction" in the table) represents the reduction percentage of the mean total cost for the optimal burn-in policy under the posterior estimator compared with no burn-in. The result indicates that the proposed burn-in model can save up to 74.83% of the cost compared to no burn-in, which demonstrates the rationality and importance of burn-in.

Table 5. The benefit of burn-in.

t_{b2}^*	r_{b2}^*	n_{b2}^*	$E[C_3(t_{b2}^*, r_{b2}^*, n_{b2}^*)]$	$E[C_3(0,0,0)]$	Reduction (%)
2.00	0.40	1.00	21.80	115.70	81.16

On the other hand, the burn-in can also significantly enhance the quality level (i.e., the proportion of strong items) of items delivered by manufacturers. This index will seriously affect the item's competitiveness and public. Figure 4 gives the change of this index with burn-in. It shows that the quality level of items is improved significantly with the burn-in procedure, and finally tends to be flat ($\pi = 1$). This is because the weak item fails more often, and then most of the weak items will be screened out due to their high number of failures during burn-in period.

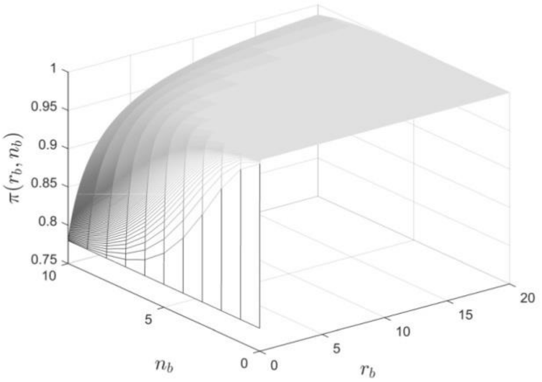


Figure 4. The proportion of strong item.

5.3.2. The benefit of Bayesian calculation

The benefit of Bayesian calculation is given in Table 6, where the benefit (the "Reduction" in the table) of Bayesian calculation represents the reduction percentage of the mean total cost per item for optimal burn-in policy under the posterior estimator compared with that under the prior estimator. The result indicates that the mean total cost of the optimal burn-in policy under the posterior estimator is lower than that under the prior estimator, which illustrates the necessity of Bayesian calculation.

Table 6. The benefit of Bayesian calculation.

t_{b1}^*	r_{b1}^*	n_{b1}^*	t_{b1}^*	r_{b2}^*	n_{b2}^*	$E[C_3(t_{b1}^*, r_{b1}^*, n_{b1}^*)]$	$E[C_3(t_{b2}^*, r_{b2}^*, n_{b2}^*)]$	Reduction(%)
2.00	0.13	1.00	2.00	0.40	1.00	22.40	21.80	2.68

5.4. Comparison with other model

Wei et al. (2022) considered the burn-in model without repairs during burn-in procedure. We consider to compare the model proposed in this paper with the model in Wei et al. (2022). Under the same setup, the optimal burn-in policy for the model of Wei et al. (2022) is $[t_b^*, r_b^*, n_b^*] = [2{,}0.04{,}0]$, and the corresponding minimum mean total cost is 23.45. Obviously, our model saves 7.04% of the cost compared to the model of Wei et al. (2022), which indicates that the proposed model can better save costs for manufacturers.

5.5. Sensitivity analysis

In the following, we conduct sensitivity analysis on the main cost parameters to give more guidance for the manufacturer, where the benefit (the "Reduction" in the table) represents the reduction percentage of the mean total cost for optimal burn-in policy under the prior or posterior estimator compared with no burn-in.

Table 7 provides the sensitivity analysis on $c_1(r_b)$. It shows that the actual mean total cost for optimal burn-in policy under posterior estimator is always less than that under prior estimator, and both of them increase with $c_1(r_b)$. It also shows that the optimal burn-in usage rates under posterior estimator and prior estimator are decreasing in $c_1(r_b)$. We can also see that the benefit of burn-in is decreasing in $c_1(r_b)$ due to the fact that the mean burn-in cost is increasing in $c_1(r_b)$.

Table 7. Sensitivity analysis on $c_1(r_b)$.

Estimator	$c_1(r_b)$	$0.05r_b^{0.5}$	$0.10r_b^{0.5}$	$0.15r_b^{0.5}$	$0.20r_b^{0.5}$	$0.25r_b^{0.5}$
Prior	$[t_{b1}^*, r_{b1}^*, n_{b1}^*]$	[2.0,0.14,1.0]	[2.0,0.13,1.0]	[2.0,0.13,1.0]	[2.0,0.12,1.0]	[2.0,0.11,1.0]
	$E[C_3(t_{b1}^*, r_{b1}^*, n_{b1}^*)]$	22.23	22.36	22.40	22.53	22.68
	Reduction(%)	80.79	80.67	80.64	80.53	80.40
Posterior	$[t_{b2}^*, r_{b2}^*, n_{b2}^*]$	[2.0,0.43,1.0]	[2.0,0.42,1.0]	[2.0,0.40,1.0]	[2.0,0.39,1.0]	[2.0,0.38,1.0]
	$E[C_3(t_{b2}^*, r_{b2}^*, n_{b2}^*)]$	21.66	21.73	21.80	21.87	21.93
	Reduction(%)	81.28	81.22	81.16	81.10	81.05

Table 8 provides the sensitivity analysis on c_2. It shows that the actual mean total cost for optimal burn-in policy under posterior estimator is always less than that under prior estimator, and both of them increase with c_2. It also shows that for the models under posterior estimator and prior estimator, the higher the c_2, the smaller the optimal burn-in usage rate or the larger the optimal burn-in screening threshold. We can also see that the benefit of burn-in is decreasing in c_2 due to the fact that the mean burn-in cost is increasing in c_2.

Table 8. Sensitivity analysis on c_2.

Estimator	c_2	$0.05r_b^{0.5}$	$0.10r_b^{0.5}$	$0.15r_b^{0.5}$	$0.20r_b^{0.5}$	$0.25r_b^{0.5}$
Prior	$[t_{b1}^*, r_{b1}^*, n_{b1}^*]$	[2.0,0.02,0.0]	[2.0,0.14,1.0]	[2.0,0.13,1.0]	[2.0,0.11,1.0]	[2.0,0.10,1.0]
	$E[C_3(t_{b1}^*, r_{b1}^*, n_{b1}^*)]$	18.28	21.03	22.40	23.89	25.29
	Reduction(%)	84.20	81.82	80.64	79.35	78.14
Posterior	$[t_{b2}^*, r_{b2}^*, n_{b2}^*]$	[2.0,0.09,0.0]	[2.0,0.43,1.0]	[2.0,0.40,1.0]	[2.0,0.38,1.0]	[2.0,0.36,1.0]
	$E[C_3(t_{b2}^*, r_{b2}^*, n_{b2}^*)]$	17.74	20.45	21.80	23.14	24.48
	Reduction(%)	84.67	82.32	81.16	80.00	78.84

Table 9 provides the sensitivity analysis on c_3. It shows that the actual mean total cost for optimal burn-in policy under posterior estimator is always less than that under prior estimator, and both of them increase with c_3. It also shows that the optimal burn-in usage rates under posterior estimator and prior estimator are decreasing in c_3. We can also see that the benefit of burn-in is decreasing in c_3 due to the fact that the mean burn-in cost is increasing in c_3.

Table 9. Sensitivity analysis on c_3.

Estimator	c_3	$0.05r_b^{0.5}$	$0.10r_b^{0.5}$	$0.15r_b^{0.5}$	$0.20r_b^{0.5}$	$0.25r_b^{0.5}$
Prior	$[t_{b1}^*, r_{b1}^*, n_{b1}^*]$	[2.0,0.13,1.0]	[2.0,0.13,1.0]	[2.0,0.13,1.0]	[2.0,0.12,1.0]	[2.0,0.11,1.0]
	$E[C_3(t_{b1}^*, r_{b1}^*, n_{b1}^*)]$	22.21	22.28	22.40	22.64	22.95
	Reduction(%)	80.80	80.74	80.64	80.43	80.16
Posterior	$[t_{b2}^*, r_{b2}^*, n_{b2}^*]$	[2.0,0.41,1.0]	[2.0,0.41,1.0]	[2.0,0.40,1.0]	[2.0,0.39,1.0]	[2.0,0.38,1.0]
	$E[C_3(t_{b2}^*, r_{b2}^*, n_{b2}^*)]$	21.58	21.67	21.80	21.97	22.18
	Reduction(%)	81.35	81.27	81.16	81.01	80.83

Table 10 provides the sensitivity analysis on c_4. It shows that the actual mean total cost for optimal burn-in policy under posterior estimator is always less than that under prior estimator, and both of them increase with c_4. It also shows that the optimal burn-in usage rates under posterior estimator and prior estimator are increasing in c_4. We can also see that the benefit of burn-in is increasing in c_4 due to burn-in can improve item reliability and then reduce the fact that the mean warranty cost proportional to c_4.

Table 10. Sensitivity analysis on c_4.

Estimator	c_4	$0.05r_b^{0.5}$	$0.10r_b^{0.5}$	$0.15r_b^{0.5}$	$0.20r_b^{0.5}$	$0.25r_b^{0.5}$
Prior	$[t_{b1}^*, r_{b1}^*, n_{b1}^*]$	[2.0,0.07,1.0]	[2.0,0.10,1.0]	[2.0,0.13,1.0]	[2.0,0.15,1.0]	[2.0,0.17,1.0]
	$E[C_3(t_{b1}^*, r_{b1}^*, n_{b1}^*)]$	14.46	18.47	22.40	26.37	30.31
	Reduction(%)	75.00	78.72	80.64	81.77	82.54
Posterior	$[t_{b2}^*, r_{b2}^*, n_{b2}^*]$	[2.0,0.29,1.0]	[2.0,0.36,1.0]	[2.0,0.40,1.0]	[2.0,0.44,1.0]	[2.0,0.47,1.0]
	$E[C_3(t_{b2}^*, r_{b2}^*, n_{b2}^*)]$	13.88	17.85	21.80	25.73	29.66
	Reduction(%)	76.01	79.43	81.16	82.21	82.91

These sensitivity analyses indicate that benefit of burn-in increases as the repair cost within warranty increases. Therefore, burn-in should be performed in the case that the repair cost within warranty is high. Furthermore, the benefit of burn-in procedure is decreasing in the burn-in related costs. Therefore, manufacturers should try to find ways to reduce the costs related to burn-in. In addition, the changes in c_2 and c_4 have a greater impact on mean total costs and benefit than the changes in $c_1(r_b)$ and c_3. Therefore, manufacturers should pay more attention to c_2 and c_4.

On the other hand, the selection of prior distributions has a certain degree of subjectivity and can sometimes have an impact on the model. Table 11 presents the sensitivity analysis on the assumptions of prior distribution, where “U(· , ·)” and “TN(· , ·)” represent the Uniform distribution and Truncated Normal distribution, respectively. The results indicate that the choice of following prior distribution has no significant impact on the the optimal burn-in policy and minimum posterior mean total cost of the model. Therefore, in practice, manufacturers do not need to be too fixated on the selection of prior distributions.

Table 11. Sensitivity analysis on prior distribution.

π	ξ_1	ξ_2	δ_1	δ_2	$\eta/6$	$[t_{b2}^*, r_{b2}^*, n_{b2}^*]$	$E[C_3(t_{b2}^*, r_{b2}^*, n_{b2}^*)]$
$Beta(13,2)$	$Gam(2,7)$	$Gam(2,14)$	$Gam(14,5)$	$Gam(2,13)$	$Beta(2,10)$	[2.0,0.40,1.0]	21.80
$U(0.73,1)$	$Gam(2,7)$	$Gam(2,14)$	$Gam(14,5)$	$Gam(2,13)$	$Beta(2,10)$	[2.0,0.33,1.0]	21.81
$Beta(13,2)$	$Beta(1.14, 2.86)$	$Beta(1.57, 9.434)$	$Gam(14,5)$	$Gam(2,13)$	$Beta(2,10)$	[2.0,0.40,1.0]	21.80
$Beta(13,2)$	$Gam(2,7)$	$Gam(2,14)$	$TN(2.8, 0.9)$	$TN(0.1, 0.15)$	$Beta(2,10)$	[2.0,0.46,1.0]	21.81
$Beta(13,2)$	$Gam(2,7)$	$Gam(2,14)$	$Gam(14,5)$	$Gam(2,13)$	$Gam(0.17, 15.6)$	[2.0,0.44,1.0]	21.80

6. Conclusion

This paper establishes a cost-based burn-in model for heterogeneous items under non-renewing 2D warranty, where all failures during the burn-in procedure and warranty period are repaired by the minimal repair at subpopulations level. The failure information of the item during burn-in procedure is used for burn-in screening. Firstly, we show that the optimal burn-in time or optimal usage rate should reach its upper bound under some conditions. Secondly, we propose a Bayesian method to calculate the optimal burn-in policy, which fully considers the uncertainty of parameters in the model. Finally, we give an example to illustrate our results and the effectiveness of Bayesian methods. The sensitivity analyses of the important cost parameters and prior distribution are also elaborated. These results can offer some informative advice to manufacturers for actual production. In addition, we also provide the comparison of the proposed model with the existing burn-in model under 2D warranty to verify that the proposed model can save cost better. The burn-in model with 1D warranty based on age (usage) can be easily obtained by setting the usage limit U (age limit W) to infinite. For arbitrary number of subpopulations, similar results can be conducted by setting the items in the first m subpopulations as strong items and the items in other subpopulations as weak items. This paper considers the items sold with a non-renewing free 2D warranty, the renewing warranty or other more general warranty policies can also be considered in the future research. It is also interesting to consider maintenance measures during the warranty period.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (72071071 and 12271419), the Young Talent Support Plan of Hebei Province, and the Fundamental Research Funds for the Central Universities (YJSJ23003).

Appendix.

Before proceeding, we review some definitions of stochastic orders that need to be used in the proof (see, Shaked and Shanthikumar, 2007). For two discrete random variables, Z_1 is said to be smaller than Z_2 in the usual stochastic order (written as $Z_1 \leq_{\mathrm{st}} Z_2$) if $P\{Z_1 \leq n\} \geq P\{Z_2 \leq n\}$ for all n; Z_1 is said to be smaller than Z_2 in the likelihood order (written as $Z_1 \leq_{\mathrm{lr}} Z_2$) if $P\{Z_1 = n\}/P\{Z_2 = n\}$ is decreasing in n. They satisfy the relationship: $Z_1 \leq_{\mathrm{lr}} Z_2 \Rightarrow Z_1 \leq_{\mathrm{st}} Z_2 \Rightarrow E[Z_1] \leq E[Z_2]$.

Proof of Theorem 1. (i) For any policy $[t_b, r_b, n_b]$, let $t_b^{''} = (\frac{r_b}{\overline{r_b}})^{\xi_2} t_b$. Then

$$(\frac{r_b}{r_0})^{\xi_1} t_b \geq (\frac{\overline{r_b}}{r_0})^{\xi_1} t_b'' \quad \text{and} \quad (\frac{r_b}{r_0})^{\xi_2} t_b = (\frac{\overline{r_b}}{r_0})^{\xi_2} t_b''. \tag{6.1}$$

From equation (2.1), we have

$$\int_0^{t_b} \lambda_i(t|r_b)\,\mathrm{d}t = \int_0^{(r_b/r_0)^{\xi_i} t_b} \lambda_i(t|r_0)\,\mathrm{d}t, i = 1,2.$$

Then,

$$\begin{aligned}\frac{P\{N_i(t_b''|\overline{r_b}) = n\}}{P\{N_i(t_b|r_b) = n\}} &= [\frac{\int_0^{t_b''} \lambda_i(t|\overline{r_b})\,\mathrm{d}t}{\int_0^{t_b} \lambda_i(t|r_b)\,\mathrm{d}t}]^n \exp\{\int_0^{t_b} \lambda_i(t|r_b)\,\mathrm{d}t - \int_0^{t_b''} \lambda_i(t|\overline{r_b})\,\mathrm{d}t\} \\ &\overset{\mathrm{sgn}}{=} [\frac{\int_0^{(\overline{r_b}/r_0)^{\xi_i} t_b''} \lambda_i(t|r_0)\,\mathrm{d}t}{\int_0^{(r_b/r_0)^{\xi_i} t_b} \lambda_i(t|r_0)\,\mathrm{d}t}]^n, i = 1,2.\end{aligned}$$

Therefore, $\frac{P\{N_2(t_b''|\overline{r_b})=n\}}{P\{N_2(t_b|r_b)=n\}} = 1$ and $\frac{P\{N_1(t_b''|\overline{r_b})=n\}}{P\{N_1(t_b|r_b)=n\}}$ is decreasing in n, that is, $N_2(t_b|r_b) \overset{\mathrm{st}}{=} N_2(t_b''|\overline{r_b})$ and $N_1(t_b''|\bar{r}_b) \leq_{\mathrm{lr}} N_1(t_b|r_b)$, hence

$N_2(t_b|r_b) \overset{\mathrm{st}}{=} N_2(t_b''|\overline{r_b})$ and $N_1(t_b''|\overline{r_b}) \leq_{\mathrm{st}} N_1(t_b|r_b)$.(6.2)

Then,

$$P\{N(t_b|r_b) \leq n_b\} = \pi P\{N_1(t_b|r_b) \leq n_b\} + (1 - \pi)P\{N_2(t_b|r_b) \leq n_b \leq \pi P\{N_1(t_b''|\bar{r}_b) \leq n_b\} + (1 - \pi)P\{N_2(t_b''|\bar{r}_b) \leq n_b\} = P\{N(t_b''|\bar{r}_b) \leq n_b\}. \tag{6.3}$$

Note that for $n \leq n_b$,

$$\frac{P\{N_i(t_b''|\bar{r}_b) = n|N_i(t_b''|\bar{r}_b) \leq n_b\}}{P\{N_i(t_b|r_b) = n|N_i(t_b|r_b) \leq n_b\}}$$

$$= \frac{P\{N_i(t_b|r_b) \leq n_b\}}{P\{N_i(t_b''|\bar{r}_b) \leq n_b\}} \left[\frac{\int_0^{t_b''} \lambda_i(t|\bar{r}_b)dt}{\int_0^{t_b} \lambda_i(t|r_b)dt}\right]^n exp\{\int_0^{t_b} \lambda_i(t|r_b)dt$$

$$- \int_0^{t_b''} \lambda_i(t|\bar{r}_b)dt\} \overset{sgn}{=} \left[\frac{\int_0^{(\frac{\bar{r}_b}{r_0})^{\xi_i} t_b''} \lambda_i(t|r_0)dt}{\int_0^{(\frac{r_b}{r_0})^{\xi_i} t_b} \lambda_i(t|r_b)dt}\right]^n, i = 1,2.$$

Therefore, $\frac{P\{N_2(t_b''|\overline{r_b})=n|N_2(t_b''|\overline{r_b})\leq n_b\}}{P\{N_2(t_b|r_b)=n|N_2(t_b|r_b)\leq n_b\}} = 1$ and $\frac{P\{N_1(t_b''|\overline{r_b})=n|N_1(t_b''|\overline{r_b})\leq n_b\}}{P\{N_1(t_b|r_b)=n|N_1(t_b|r_b)\leq n_b\}}$ is decreasing in n. That is, $(N_2(t_b^{''}|\bar{r}_b)|N_2(t_b^{''}|\bar{r}_b) \leq n_b) =_{\mathrm{st}} (N_2(t_b|r_b)|N_2(t_b|r_b) \leq n_b)$ and $(N_1(t_b^{''}|\bar{r}_b)|N_1(t_b^{''}|\bar{r}_b) \leq n_b) \leq_{\mathrm{lr}} (N_1(t_b|r_b)|N_1(t_b|r_b) \leq n_b)$.

Then $E[N_2(t_b^{''}|\bar{r}_b)|N_2(t_b^{''}|\bar{r}_b) \leq n_b]$=E$[N_2(t_b|r_b)|N_2(t_b|r_b) \leq n_b]$ and $E[N_1(t_b^{''}|\bar{r}_b)|N_1(t_b^{''}|\bar{r}_b) \leq n_b] \leq E[N_1(t_b|r_b)|N_1(t_b|r_b) \leq n_b]$. From equations (2.3) and (6.2), we have $\pi(t_b, r_b, n_b) \leq \pi(t_b^{''}, \bar{r}_b, n_b)$. Combined with equation (3.4), we have

$$E[N(t_b''|\bar{r}_b)|N(t_b''|\bar{r}_b) \leq n_b] \leq E[N(t_b|r_b)|N(t_b|r_b) \leq n_b]. \tag{6.4}$$

Note that

$$\begin{aligned}&a r_b^{\xi_0} \frac{\int_0^{t_b} P\{N(t|r_b) \leq n_b\}dt}{P\{N(t_b|r_b) \leq n_b\}} \\ &= a\pi r_b^{\xi_0} \frac{\int_0^{t_b} P\{N_1((r_b/r_0)^{\xi_1} t|r_0) \leq n_b\}dt}{P\{N(t_b|r_b) \leq n_b\}} + a(1 - \pi) r_b^{\xi_0} \frac{\int_0^{t_b} P\{N_2((r_b/r_0)^{\xi_2} t|r_0) \leq n_b\}dt}{P\{N(t_b|r_b) \leq n_b\}} \\ &= a\pi r_b^{\xi_0} \left(\frac{r_b}{r_0}\right)^{-\xi_1} \frac{\int_0^{(\frac{r_b}{r_0})^{\xi_1} t_b} P\{N_1(t|r_0) \leq n_b\}dt}{P\{N(t_b|r_b) \leq n_b\}} + a(1 - \pi) r_b^{\xi_0} \left(\frac{r_b}{r_0}\right)^{-\xi_2} \frac{\int_0^{(\frac{r_b}{r_0})^{\xi_2} t_b} P\{N_2(t|r_0) \leq n_b\}dt}{P\{N(t_b|r_b) \leq n_b\}}.\end{aligned}$$

From equations (6.1) and (6.3), we have

$$\frac{\int_0^{(\frac{\overline{r_b}}{r_0})^{\xi_1} t_b''} P\{N_1(t|r_0) \le n_b\}\,\mathrm{d}t}{P\{N(t_b''|\overline{r_b}) \le n_b\}} \le \frac{\int_0^{(\frac{r_b}{r_0})^{\xi_1} t_b} P\{N_1(t|r_0) \le n_b\}\,\mathrm{d}t}{P\{N(t_b|r_b) \le n_b\}}$$

and

$$\frac{\int_0^{(\frac{\overline{r_b}}{r_0})^{\xi_2} t_b''} P\{N_2(t|r_0) \le n_b\}\,\mathrm{d}t}{P\{N(t_b''|\overline{r_b}) \le n_b\}} \le \frac{\int_0^{(\frac{r_b}{r_0})^{\xi_2} t_b} P\{N_2(t|r_0) \le n_b\}\,\mathrm{d}t}{P\{N(t_b|r_b) \le n_b\}}.$$

Since $\xi_0 \le \xi_1 \le \xi_2, \overline{r_b}^{\xi_0}(\frac{\overline{r_b}}{r_0})^{-\xi_i} \le r_b^{\xi_0}(\frac{r_b}{r_0})^{-\xi_i}, i = 1,2,$

$$a\overline{r_b}^{\xi_0} \frac{\int_0^{t_b''} P\{N(t|\overline{r_b}) \le n_b\}\mathrm{d}t}{P\{N(t_b''|\overline{r_b}) \le n_b\}} \le a r_b^{\xi_0} \frac{\int_0^{t_b} P\{N(t|r_b) \le n_b\}\mathrm{d}t}{P\{N(t_b|r_b) \le n_b\}}. \qquad (6.5)$$

Since $\lambda_1(t|r_0)$ is increasing in t, from equations (2.4) and (6.1), we have $\lambda_1^w(t|r, t_b'', \bar{r}_b) \le \lambda_1^w(t|r, t_b, r_b)$ and $\lambda_2^w(t|r, t_b'', \bar{r}) \le \lambda_2^w(t|r, t_b, r_b)$. Since $\lambda_1^w(t|r, t_b, r_b) \le \lambda_2^w(t|r, t_b, r_b)$ and $\pi(t_b, r_b, n_b) \le \pi(t_b'', \bar{r}_b, n_b)$, combined with equation (2.6), we have $\lambda^w(t|r, t_b'', \bar{r}_b, n_b) \le \lambda^w(t|r, t_b, r_b, n_b)$.

Hence

$$\int_{\mathfrak{R}} \int_0^{T_r} \lambda^w(t|r, t_b'', \bar{r}_b, n_b)\, dt dG(r)$$

$$\le \int_{\mathfrak{R}} \int_0^{T_r} \lambda^w(t|r, t_b, r_b, n_b)\, dt dG(r). \; (6.6)$$

From equations (3.7), (6.3), (6.4), (6.5) and (6.6), we have $E[C(t_b'', \bar{r}_b, n_b)] \le E[C(t_b, r_b, n_b)]$. That is for any policy $[t_b, r_b, n_b]$, policy $[t_b'', \overline{r_b}, n_b]$ is better than it. Therefore, $r_b^* = \overline{r_b}$.

(ii) The proof is very similar to the proof of (i) and therefore is omitted.

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