

## ALMOST PERIODIC SYNCHRONIZATION OF FUZZY CELLULAR NEURAL NETWORKS WITH TIME-VARYING DELAYS VIA STATE-FEEDBACK AND IMPULSIVE CONTROL

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In this paper, we are concerned with drive-response synchronization for a class of fuzzy cellular neural networks with time varying delays. Based on the exponential dichotomy of linear differential equations, the Banach fixed point theorem and the differential inequality technique, we obtain the existence of almost periodic solutions of this class of networks. Then, we design a state feedback and an impulsive controller, and construct a suitable Lyapunov function to study the problem of global exponential almost periodic synchronization for the drive-response systems considered. At the end of the paper, we provide an example to verify the effectiveness of the theoretical results.

**Keywords:** almost periodic solution, fuzzy cellular neural networks, time-varying delays, synchronization, state feedback, impulsive control.

### 1. Introduction

Neural networks have been extensively studied for their potential applications in many engineering and scientific fields, such as image processing, associative memory, computing technology, nonlinear programming, and so on (Arik, 2002b; 2002a; Sen, 2006; Cao and Liand, 2004; Xia *et al.*, 2007; Xu and Yang, 2005). There are many research results, especially for Hopfield neural networks (Aouiti *et al.*, 2017), cellular neural networks (Park, 2009; Pan and Cao, 2011), bidirectional associative memory neural networks (Aouiti *et al.*, 2017) and Cohen–Grossberg neural networks (Yang *et al.*, 2017). However, when using mathematical modeling to solve real-world problems, the uncertainty of vagueness is encountered. In order to solve the problems of vagueness, Yang and Yang (1996) applied fuzzy logic to the traditional cellular neural networks, which integrates fuzzy logic into the structure of traditional cellular neural networks and maintains local connectedness among cells.

Unlike previous cellular neural network structures,

fuzzy cellular neural networks have fuzzy logic between their template and input and/or output besides the “sum of products” operation. Combining the advantages of fuzzy operations and cellular neural networks, fuzzy neural networks have wide applications in image processing and pattern recognition problems. Therefore, there are a lot of results for fuzzy cellular neural networks; see the works of Li and Wang (2013), Yuan *et al.* (2006), Long and Xu (2011) or Li and Zhang (2009) and the references therein. Moreover, it is well known that periodicity, anti-periodicity and almost periodicity are very important dynamics for non-autonomous neural networks (Li *et al.*, 2009; Pan and Cao, 2011; Li and Fan, 2009). In addition, in the real world, almost periodicity is more realistic and more general than periodicity. Also, time delays are inevitable both in biological and artificial neural networks. Recently, the almost periodic problem of fuzzy cellular neural networks with time delays were considered by Huang (2017a; 2017b).

Pecora and Carroll (1990) proposed the concept of synchronization. Its mechanism is as follows: a chaotic system, called the driver (or master) system,

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generates a signal and sends it to a response (or slave) system over a channel. The response system is identical with the driver system. Then the response system uses this signal to control itself so that it oscillates in a manner synchronized with the drive system. Since then, due to its wide applications in engineering such as in secure communication, biological systems or information processing, much attention has been attracted to this topic. There are many synchronization strategies, such as coupling control, state feedback control, adaptive control and impulsive control (Heagy *et al.*, 1994; Yang, 2001; Cao *et al.*, 2005; Tang *et al.*, 2017a; 2018b; Yuan *et al.*, 2014). In the past few years, the problem of periodic synchronization for systems has been extensively investigated (see Hu *et al.*, 2010; Hong, 2014; Cai *et al.*, 2015; Wu *et al.*, 2015). Very recently, the problems of almost periodic synchronization, almost automorphic synchronization and pseudo almost periodic synchronization for quaternion-valued neural networks have been studied by Li *et al.* (2018b; 2018c; 2018a), respectively.

It should be noted that the impulsive control method has found favor with many researchers, since it allows synchronization of systems to use the impulsive controllers with only small impulses, generated by nodes of the state variables of the drive and the response systems at discrete time instants. Moreover, when the synchronization impulses are sent to the receiving systems at discrete time instants, this can drastically decrease information redundancy in the transmitted signal and increase robustness against disturbances, which can greatly reduce control cost. Recently, some interesting synchronization results for chaotic neural networks have been reported using the impulsive control method, see the works of Cao *et al.* (2009), Lu *et al.*, (2013; 2018), Yang *et al.* (2015; 2018), Guan (2018), Zhang *et al.* (2018) or Lin and Zhang (2018) and the references therein.

Up to now, although there have been few studies on synchronization of fuzzy cellular neural networks (Ding and Han, 2008; Abdurahman *et al.*, 2016; Wang, 2018; Ding *et al.*, 2009; Feng *et al.*, 2011; Yang *et al.*, 2017), almost periodic synchronization of fuzzy cellular neural networks via impulsive controls has not been studied. But it is interesting and challenging.

Inspired by the above analysis, in this paper, we are concerned with the following fuzzy cellular neural network with time-varying delays:

$$x'_i(t) = -a_i(t)x_i(t) + \sum_{j=1}^n b_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^n c_{ij}(t)g_j(x_j(t - \sigma_{ij}(t)))$$

$$+ \sum_{j=1}^n d_{ij}(t)\mu_j(t) + \bigwedge_{j=1}^n T_{ij}(t)\mu_j(t) + \bigwedge_{j=1}^n \alpha_{ij}(t)h_j(x_j(t - \tau_{ij}(t))) + \bigvee_{j=1}^n \beta_{ij}(t)h_j(x_j(t - \eta_{ij}(t))) + \bigvee_{j=1}^n S_{ij}(t)\mu_j(t) + I_i(t), \tag{1}$$

where  $i \in \{1, 2, \dots, n\} := \Pi$ ,  $n$  corresponds to the number of units in a neural network,  $x_i(t)$  is the state variable of the  $i$ -th neuron at time  $t$ ;  $a_i(t) > 0$  represents the rate with which the  $i$ -th neuron will reset its potential to the resting state in isolation when disconnected from the networks and external inputs;  $\alpha_{ij}, \beta_{ij}, T_{ij}, S_{ij}$  are elements of the fuzzy feedback MIN template, fuzzy feedback MAX template, fuzzy feedforward MIN template, fuzzy feedforward MAX template, respectively;  $\bigwedge, \bigvee$  denote the fuzzy AND and OR operation, respectively;  $b_{ij}, d_{ij}$  are elements of the feedback and free-forward template;  $c_{ij}$  denotes the weight strength of the  $i$ -th neuron at time  $t$ ;  $\mu_i$  denotes an input of the  $i$ th neuron;  $I_i(t)$  denotes the external input to the  $i$ th neuron at time  $t$ ;  $f_j, g_j, h_j : \mathbb{R} \rightarrow \mathbb{R}$  are the activation functions of signal transmission;  $\sigma_{ij}(t) > 0, \tau_{ij}(t) > 0, \eta_{ij}(t) > 0$  correspond to transmission delays.

Throughout the paper, for convenience, for a bounded continuous function, we use the notation

$$f^+ = \sup_{t \in \mathbb{R}} |f(t)|, \quad f^- = \inf_{t \in \mathbb{R}} |f(t)|.$$

The initial conditions associated with the system (1) are of the form

$$x_i(s) = \varphi_i(s), \quad s \in [-\theta, 0], \quad i \in \Pi,$$

where

$$\theta = \max\{\sigma, \tau, \eta\}, \quad \sigma = \max_{1 \leq i, j \leq n} \{\sigma_{ij}^+\},$$

$$\tau = \max_{1 \leq i, j \leq n} \{\tau_{ij}^+\}, \quad \eta = \max_{1 \leq i, j \leq n} \{\eta_{ij}^+\},$$

and  $\varphi_i(\cdot)$  is a real-valued bounded continuous function defined on  $[-\theta, 0]$ .

The main purpose of this paper is to study the problem of almost periodic synchronization of (1) via state-feedback and impulsive control. To the best of our knowledge, this is the first paper to study almost periodic synchronization for fuzzy cellular neural networks. Our results are completely new and our methods can be used to study the problem of almost periodic and almost automorphic synchronization for other types of neural networks.

**Remark 1.** Following ideas similar to those used by Cai *et al.* (2015), Hu *et al.* (2010) or Li *et al.* (2018b; 2018a; 2018c), in this paper, drive-response synchronization of systems with almost periodic coefficients and almost periodic solutions is called almost periodic synchronization.

We organize the paper as follows. In Section 2, we introduce the notation and definitions, and state some preliminary results which are needed in further sections. In Section 3, sufficient conditions for the existence of almost periodic solutions of system (1) are obtained. In Section 4, the exponential synchronization is investigated. In Section 5, the effectiveness and feasibility of the developed methods are shown with a numerical example. The paper ends with a brief conclusion in Section 6.

## 2. Preliminaries

In this section, we shall recall some fundamental definitions and lemmas.

**Definition 1.** (Fink, 1974) Let  $f \in BC(\mathbb{R}, \mathbb{R}^n)$ ; then  $f$  is said to be *almost periodic* if, for any  $\epsilon > 0$ , it is possible to find a real number  $l = l(\epsilon) > 0$ , such that for any interval of length  $l(\epsilon)$ , there exists a number  $\tau = \tau(\epsilon)$  in this interval satisfying  $|f(t + \tau) - f(t)| < \epsilon$  for all  $t \in \mathbb{R}$ .

We denote by  $AP(\mathbb{R}, \mathbb{R}^n)$  the set of all almost periodic functions from  $\mathbb{R}$  to  $\mathbb{R}^n$ .

**Lemma 1.** (Diagana, 2013) If  $f, g \in AP(\mathbb{R}, \mathbb{R}^n)$ , then  $f + g, fg \in AP(\mathbb{R}, \mathbb{R}^n)$ .

Consider the following linear homogenous system:

$$x'(t) = A(t)x(t) \tag{2}$$

and the linear non-homogenous system

$$x'(t) = A(t)x(t) + f(t), \tag{3}$$

where  $A(t)$  is an almost periodic matrix function and  $f(t)$  is an almost periodic vector function.

**Definition 2.** (Fink, 1974) The system (2) is said to admit an *exponential dichotomy* if there exist a projection  $P$  and positive constants  $\alpha, \beta$  such that the fundamental solution matrix  $X(t)$  satisfies

$$\begin{aligned} |X(t)PX^{-1}(s)| &\leq \beta e^{-\alpha(t-s)}, & t \geq s, \\ |X(t)(I - P)X^{-1}(s)| &\leq \beta e^{-\alpha(s-t)}, & t \leq s. \end{aligned}$$

**Lemma 2.** (Fink, 1974) If the linear system (2) admits an exponential dichotomy, then the system (3) has a unique almost periodic solution

$$x(t) = \int_{-\infty}^t X(t)PX^{-1}(s)f(s) ds$$

$$- \int_t^{+\infty} X(t)(I - P)X^{-1}(s)f(s) ds,$$

where  $X(t)$  is the fundamental solution matrix of (2).

**Lemma 3.** (Fink, 1974) Let  $a_i$  be an almost periodic function on  $\mathbb{R}$  and

$$M[a_i] := \lim_{T \rightarrow \infty} \frac{1}{T} \int_t^{t+T} a_i(s) ds > 0, \quad i \in \Pi.$$

Then the linear system

$$x'(t) = \text{diag}(-a_1(t), -a_2(t), \dots, -a_n(t))x(t)$$

admits an exponential dichotomy on  $\mathbb{R}$ .

**Lemma 4.** (Yang nad Yang, 1996) Let  $x_j, y_j, \alpha_{ij}, \beta_{ij} \in \mathbb{R}, f_j : \mathbb{R} \rightarrow \mathbb{R}$  be continuous functions where  $i, j \in \Pi$ ; then we have

$$\begin{aligned} &\left| \bigwedge_{j=1}^n \alpha_{ij} f_j(x_j) - \bigwedge_{j=1}^n \alpha_{ij} f_j(y_j) \right| \\ &\leq \sum_{j=1}^n |\alpha_{ij}| |f_j(x_j) - f_j(y_j)|, \\ &\left| \bigvee_{j=1}^n \beta_{ij} f_j(x_j) - \bigvee_{j=1}^n \beta_{ij} f_j(y_j) \right| \\ &\leq \sum_{j=1}^n |\beta_{ij}| |f_j(x_j) - f_j(y_j)|. \end{aligned}$$

Similarly to the proof by Huang (2017b, Remark 2.2), we have the following result.

**Lemma 5.** For  $i, j \in \Pi$ , let  $x_j, \mu_j, \tau_{ij}, \eta_{ij}, \alpha_{ij}, \beta_{ij}, T_{ij}, S_{ij} \in AP(\mathbb{R}, \mathbb{R})$  and  $(A_2)$  hold; then we have  $\bigwedge_{j=1}^n \alpha_{ij}(\cdot)h_j(x_j(\cdot - \tau_{ij}(\cdot)))$ ,  $\bigvee_{j=1}^n \beta_{ij}(\cdot)h_j(x_j(\cdot - \eta_{ij}(\cdot)))$ ,  $\bigwedge_{j=1}^n T_{ij}(\cdot)\mu_j$ ,  $\bigvee_{j=1}^n S_{ij}(\cdot)\mu_j \in AP(\mathbb{R}, \mathbb{R})$ .

Throughout the paper, we make the following assumptions:

- (A<sub>1</sub>) Function  $a_i \in AP(\mathbb{R}, \mathbb{R}^+)$  with  $M[a_i] > 0$ ,  $b_{ij}, c_{ij}, d_{ij}, \alpha_{ij}, \beta_{ij}, S_{ij}, T_{ij}, \mu_i, I_i \in AP(\mathbb{R}, \mathbb{R})$ ,  $\sigma_{ij}, \tau_{ij}, \eta_{ij} \in C^1(\mathbb{R}, \mathbb{R}^+) \cap AP(\mathbb{R}, \mathbb{R})$  with

$$\begin{aligned} \gamma &= \inf_{t \in \mathbb{R}} \{1 - \sigma'_{ij}(t)\} > 0, \\ \rho &= \inf_{t \in \mathbb{R}} \{1 - \tau'_{ij}(t)\} > 0, \\ \varrho &= \inf_{t \in \mathbb{R}} \{1 - \eta'_{ij}(t)\} > 0, \end{aligned}$$

where  $i, j \in \Pi$ .

- (A<sub>2</sub>) For any  $u, v \in \mathbb{R}$ , there exist positive constants  $L_j^f, L_j^g, L_j^h$  such that

$$|f_j(u) - f_j(v)| \leq L_j^f |u - v|,$$

$$|g_j(u) - g_j(v)| \leq L_j^g |u - v|,$$

$$|h_j(u) - h_j(v)| \leq L_j^h |u - v|$$

and  $f_j(0) = g_j(0) = h_j(0) = 0$ , where  $j \in \Pi$ .

(A<sub>3</sub>) There exists a positive constant  $r$  such that

$$\max_{i \in J} \left\{ \frac{P_i}{a_i^-} \right\} := r < 1,$$

where  $i \in \Pi$  and

$$P_i = \sum_{j=1}^n \left[ b_{ij}^+ L_j^f + c_{ij}^+ L_j^g + \alpha_{ij}^+ L_j^h + \beta_{ij}^+ L_j^h \right].$$

### 3. Existence of almost periodic solutions

In this section, we shall state and prove sufficient conditions for the existence and uniqueness of almost periodic solutions of the system (1).

Set  $\mathbb{Y} = \{ \varphi = (\varphi_1, \varphi_2, \dots, \varphi_n)^T : \varphi_i \in AP(\mathbb{R}, \mathbb{R}), i \in \Pi \}$  and equip it with the norm  $\|\varphi\| = \max_{i \in \Pi} \{ \sup_{t \in \mathbb{R}} |\varphi_i(t)| \}$ . Then  $\mathbb{Y}$  is a Banach space. Let

$$\varphi^0(t) = (\varphi_1^0(t), \varphi_2^0(t), \dots, \varphi_n^0(t))^T,$$

where

$$\varphi_i^0(t) = \int_{-\infty}^t e^{-\int_s^t a_i(u) du} \left[ \sum_{j=1}^n d_{ij}(s) \mu_j(s) + \sum_{j=1}^n T_{ij}(s) \mu_j(s) + \sum_{j=1}^n S_{ij}(s) \mu_j(s) + I_i(s) \right] ds,$$

$i \in \Pi$  and  $L$  be a constant satisfying  $L \geq \|\varphi^0\|$ .

**Theorem 1.** Let (A<sub>1</sub>)–(A<sub>3</sub>) hold. Then the system (1) has a unique almost periodic solution in  $\mathbb{Y}^* = \{ \varphi \in \mathbb{Y} \mid \|\varphi - \varphi^0\| \leq rL/(1-r) \}$ .

*Proof.* For any given  $\varphi \in \mathbb{Y}$ , consider the following linear system:

$$x_i'(t) = -a_i(t)x_i(t) + F_i(t, \varphi(t)) + E_i(t), \quad i \in \Pi, \quad (4)$$

where

$$F_i(t, \varphi(t)) = \sum_{j=1}^n b_{ij}(t) f_j(\varphi_j(t)) + \sum_{j=1}^n c_{ij}(t) \times g_j(\varphi_j(t - \sigma_{ij}(t))) + \sum_{j=1}^n \alpha_{ij}(t) h_j(\varphi_j(t - \tau_{ij}(t))) + \sum_{j=1}^n \beta_{ij}(t) h_j(\varphi_j(t - \eta_{ij}(t))),$$

$$E_i(t) = \sum_{j=1}^n d_{ij}(t) \mu_j(t) + \sum_{j=1}^n T_{ij}(t) \mu_j(t) + \sum_{j=1}^n S_{ij}(t) \mu_j(t) + I_i(t).$$

From Assumption A<sub>1</sub> and Lemma 3 it follows that the linear system

$$x_i'(t) = -a_i(t)x_i(t), \quad i \in \Pi,$$

admits an exponential dichotomy. By Lemma 2, we have that the system (4) has a unique almost periodic solution  $x^\varphi = (x_1^\varphi, x_2^\varphi, \dots, x_n^\varphi)$ , where

$$x_i^\varphi(t) = \int_{-\infty}^t e^{-\int_s^t a_i(u) du} [F_i(s, \varphi(s)) + E_i(s)] ds, \quad i \in \Pi.$$

For every  $\varphi \in \mathbb{Y}^*$ , we find

$$\|\varphi\| \leq \|\varphi^0\| + \|\varphi - \varphi^0\| \leq L + \frac{rL}{1-r} = \frac{L}{1-r}.$$

Now we define a mapping  $\Phi : \mathbb{Y}^* \rightarrow \mathbb{Y}^*$  by setting  $\Phi\varphi = x^\varphi, \varphi \in \mathbb{Y}^*$ .

First, we will prove that, for any  $\varphi \in \mathbb{Y}^*, \Phi\varphi \in \mathbb{Y}^*$ . In fact, we have

$$\begin{aligned} & |F_i(s, \varphi(s))| \\ &= \left| \sum_{j=1}^n b_{ij}(s) f_j(\varphi_j(s)) + \sum_{j=1}^n c_{ij}(s) \times g_j(\varphi_j(s - \sigma_{ij}(s))) + \sum_{j=1}^n \alpha_{ij}(s) h_j(\varphi_j(s - \tau_{ij}(s))) + \sum_{j=1}^n \beta_{ij}(s) h_j(\varphi_j(s - \eta_{ij}(s))) \right| \\ &\leq \sum_{j=1}^n b_{ij}^+ |f_j(\varphi_j(s)) - f_j(0)| \\ &\quad + \sum_{j=1}^n c_{ij}^+ |g_j(\varphi_j(s - \sigma_{ij}(s))) - g_j(0)| \\ &\quad + \sum_{j=1}^n \alpha_{ij}^+ |h_j(\varphi_j(s - \tau_{ij}(s))) - h_j(0)| \\ &\quad + \sum_{j=1}^n \beta_{ij}^+ |h_j(\varphi_j(s - \eta_{ij}(s))) - h_j(0)| \\ &\leq \sum_{j=1}^n b_{ij}^+ L_j^f |\varphi_j(s)| + \sum_{j=1}^n c_{ij}^+ L_j^g |\varphi_j(s - \sigma_{ij}(s))| \\ &\quad + \sum_{j=1}^n \alpha_{ij}^+ L_j^h |\varphi_j(s - \tau_{ij}(s))| \end{aligned}$$

$$+ \sum_{j=1}^n \beta_{ij}^+ L_j^h |\varphi_j(s - \eta_{ij}(s))|, \quad i \in \Pi.$$

From this and the definition of  $\Phi$  it follows that

$$\begin{aligned} & \sup_{t \in \mathbb{R}} |(\Phi\varphi)_i(t) - \varphi_i^0(t)| \\ &= \sup_{t \in \mathbb{R}} \left| \left\{ \int_{-\infty}^t e^{-\int_s^t a_i(u) du} F_i(s, \varphi(s)) \right\} ds \right| \\ &\leq \frac{1}{a_i^-} \sum_{j=1}^n \left[ b_{ij}^+ L_j^f + c_{ij}^+ L_j^g + \alpha_{ij}^+ L_j^h + \beta_{ij}^+ L_j^h \right] \|\varphi\| \\ &\leq r \|\varphi\| \leq \frac{rL}{1-r}, \quad i \in \Pi, \end{aligned}$$

which implies  $\Phi\varphi \in \mathbb{Y}^*$ . Thus, the mapping  $\Phi$  is a selfmapping from  $\mathbb{Y}^*$  to  $\mathbb{Y}^*$ .

Next, we will show that  $\Phi$  is a contraction mapping in  $\mathbb{Y}^*$ . For any  $\varphi, \psi \in \mathbb{Y}^*$ , we have

$$\begin{aligned} & |F_i(s, \varphi(s)) - F_i(s, \psi(s))| \\ &\leq \sum_{j=1}^n b_{ij}^+ |f_j(\varphi_j(s)) - f_j(\psi_j(s))| \\ &\quad + \sum_{j=1}^n c_{ij}^+ |g_j(\varphi_j(s - \sigma_{ij}(s))) \\ &\quad - g_j(\psi_j(s - \sigma_{ij}(s)))| + \sum_{j=1}^n \alpha_{ij}^+ |h_j(\varphi_j(s \\ &\quad - \tau_{ij}(s))) - h_j(\psi_j(s - \tau_{ij}(s)))| \\ &\quad + \sum_{j=1}^n \beta_{ij}^+ |h_j(\varphi_j(s - \eta_{ij}(s))) \\ &\quad - h_j(\psi_j(s - \eta_{ij}(s)))| \\ &\leq \sum_{j=1}^n b_{ij}^+ L_j^f |\varphi_j(s) - \psi_j(s)| \\ &\quad + \sum_{j=1}^n c_{ij}^+ L_j^g |\varphi_j(s - \sigma_{ij}(s)) - \psi_j(s - \sigma_{ij}(s))| \\ &\quad + \sum_{j=1}^n \alpha_{ij}^+ L_j^h |\varphi_j(s - \tau_{ij}(s)) - \psi_j(s - \tau_{ij}(s))| \\ &\quad + \sum_{j=1}^n \beta_{ij}^+ L_j^h |\varphi_j(s - \eta_{ij}(s)) \\ &\quad - \psi_j(s - \eta_{ij}(s))|, \quad i \in \Pi. \end{aligned}$$

From the above inequality and the definition of  $\Phi$ , we obtain

$$\begin{aligned} & \sup_{t \in \mathbb{R}} |(\Phi\varphi)_i(t) - \varphi_i^0(t) - (\Phi\psi)_i(t)| \\ &\leq \sup_{t \in \mathbb{R}} \left\{ \int_{-\infty}^t e^{-\int_s^t a_i(u) du} \sum_{j=1}^n \left[ b_{ij}^+ L_j^f + c_{ij}^+ L_j^g \right. \right. \end{aligned}$$

$$\begin{aligned} & \left. + \alpha_{ij}^+ L_j^h + \beta_{ij}^+ L_j^h \right] \|\varphi - \psi\| ds \Big\} \\ &\leq \frac{P_i}{a_i^-} \|\varphi - \psi\|, \quad i \in \Pi. \end{aligned}$$

Therefore, we get

$$\|\Phi\varphi - \Phi\psi\| \leq r \|\varphi - \psi\|.$$

By Assumption  $A_3$ ,  $\Phi$  is a contraction mapping. Based on the Banach fixed theorem, we obtain that  $\Phi$  has a unique fixed point in  $\Omega$ , which means that the system (1) has a unique almost periodic solution in  $\mathbb{Y}^*$ . The proof is complete. ■

#### 4. Almost periodic synchronization via state-feedback and impulsive control

In this section, by utilizing some analytic techniques and constructing a suitable Lyapunov function, we will investigate the exponential synchronization problem of fuzzy cellular neural networks with time-varying delays and almost periodic coefficients. For this purpose, we consider the system (1) as the drive system, and the response system is designed as

$$\begin{aligned} y_i'(t) &= -a_i(t)y_i(t) + \sum_{j=1}^n b_{ij}(t)f_j(y_j(t)) \\ &\quad + \sum_{j=1}^n c_{ij}(t)g_j(y_j(t - \sigma_{ij}(t))) \\ &\quad + \sum_{j=1}^n d_{ij}(t)\mu_j(t) + \bigwedge_{j=1}^n T_{ij}(t)\mu_j(t) \\ &\quad + \bigwedge_{j=1}^n \alpha_{ij}(t)h_j(y_j(t - \tau_{ij}(t))) \\ &\quad + \bigvee_{j=1}^n \beta_{ij}(t)h_j(y_j(t - \eta_{ij}(t))) \\ &\quad + \bigvee_{j=1}^n S_{ij}(t)\mu_j(t) + I_i(t) \\ &\quad + u_i(t), \quad i \in \Pi, \end{aligned} \tag{5}$$

where  $u_i(t)$  is the control input to be designed. The initial condition of (5) is as follows:

$$y_i(s) = \psi_i(s), \quad s \in [-\nu, 0], \quad i \in \Pi,$$

where  $\nu = \max\{\sigma, \tau, \eta, \xi\}$ ,  $\xi = \max_{1 \leq i, j \leq n} \{\xi_{ij}^+\}$ , and  $\psi_i(\cdot)$  is a real-valued continuous function defined on  $[-\nu, 0]$ .

Let  $z_i(t) = y_i(t) - x_i(t)$  be the synchronization error; then we can obtain the following error dynamical

system of (1) and (5):

$$\begin{aligned}
 z'_i(t) = & -a_i(t)z_i(t) + \sum_{j=1}^n b_{ij}(t)F_j(z_j(t)) \\
 & + \sum_{j=1}^n c_{ij}(t)G_j(z_j(t - \sigma_{ij}(t))) \\
 & + \bigwedge_{j=1}^n \alpha_{ij}(t)H_j(z_j(t - \tau_{ij}(t))) \\
 & + \bigvee_{j=1}^n \beta_{ij}(t)\hat{H}_j(z_j(t - \eta_{ij}(t))) \\
 & + u_i(t), \quad i \in \Pi,
 \end{aligned} \tag{6}$$

where  $F_j(z_j(t)) \triangleq f_j(y_j(t)) - f_j(x_j(t))$ ,  $G_j(z_j(t - \sigma_{ij}(t))) \triangleq g_j(y_j(t - \sigma_{ij}(t))) - g_j(x_j(t - \sigma_{ij}(t)))$ ,  $H_j(z_j(t - \tau_{ij}(t))) \triangleq h_j(y_j(t - \tau_{ij}(t))) - h_j(x_j(t - \tau_{ij}(t)))$ ,  $\hat{H}_j(z_j(t - \eta_{ij}(t))) \triangleq h_j(y_j(t - \eta_{ij}(t))) - h_j(x_j(t - \eta_{ij}(t)))$ .

The impulsive controller  $u_i(t)$  is designed as

$$\begin{aligned}
 u_i(t) = & \sum_{k=1}^{\infty} J_{ik}(z_i(t))\delta(t - t_k) - \kappa_i(t)z_i(t) \\
 & + \sum_{j=1}^n e_{ij}(t)l_j(z_j(t - \xi_{ij}(t))), \quad i \in \Pi,
 \end{aligned} \tag{7}$$

where  $k \in \mathbb{Z}^+$ ,  $\delta(\cdot)$  is the Dirac impulsive function.

With the impulsive controller (7), similarly to the proof by Zhang et al. (2018) and using the property of  $\delta(\cdot)$ , the error system (5) becomes the following

$$\left\{ \begin{aligned}
 z'_i(t) = & -a_i(t)z_i(t) + \sum_{j=1}^n b_{ij}(t)F_j(z_j(t)) \\
 & + \sum_{j=1}^n c_{ij}(t)G_j(z_j(t - \sigma_{ij}(t))) \\
 & + \bigwedge_{j=1}^n \alpha_{ij}(t)H_j(z_j(t - \tau_{ij}(t))) \\
 & + \bigvee_{j=1}^n \beta_{ij}(t)\hat{H}_j(z_j(t - \eta_{ij}(t))) \\
 & - \kappa_i(t)z_i(t) + \sum_{j=1}^n e_{ij}(t) \\
 & \quad \times l_j(z_j(t - \xi_{ij}(t))), \quad t \neq t_k, \\
 \Delta z_i(t_k) = & z_i(t_k^+) - z_i(t_k^-) \\
 = & J_{ik}(z_i(t_k^-)), \quad t = t_k, \\
 z_i(s) = & \psi_i(s) - \varphi_i(s), \quad s \in [-\nu, 0],
 \end{aligned} \right. \tag{8}$$

where  $i \in \Pi$ ,  $k \in \mathbb{Z}^+$ ,  $\Delta z_i(t_k)$  denotes the state jumping at impulsive time instant  $t = t_k$ , while  $z_i(t_k^+)$  and  $z_i(t_k^-)$  are the right-hand and left-hand limits of function  $z_i(t)$  at  $t_k$ . Moreover,  $z_i(t_k^+) = z_i(t_k)$  and

$$J_k(z_i(t_k^-))$$

$$= (J_{1k}(z_1(t_k^-)), J_{2k}(z_2(t_k^-)), \dots, J_{nk}(z_n(t_k^-)))^T$$

represents impulsive perturbation of the  $i$ -th neuron at  $t_k$ .

Set  $\mathcal{B} = \{\{t_k\} : t_k \in \mathbb{R}, t_k < t_{k+1}, k \in \mathbb{Z}^+, \lim_{k \rightarrow \pm\infty} t_k = \pm\infty\}$ . For  $\{t_k\} \in \mathcal{B}$ , let  $PC(\mathbb{R}, \mathbb{R}^n)$  be the space formed by all bounded piecewise continuous functions  $\varphi : \mathbb{R} \rightarrow \mathbb{R}^n$  such that  $\varphi(\cdot)$  is continuous at  $t$  for any  $t \notin \{t_k\}$  and  $\varphi(t_k) = \varphi(t_k^+)$  for all  $k \in \mathbb{Z}^+$ .

**Definition 3.** (Stamov, 2012) The function  $\varphi \in PC(\mathbb{R}, \mathbb{R}^n)$  is said to be *almost periodic* if the following holds:

- (i)  $\{t_k\}$  constitutes a uniformly almost periodic sequence.
- (ii) For any  $\epsilon > 0$ , there exists a positive number  $\delta = \delta(\epsilon)$  such that, if the points  $t'$  and  $t''$  belong to the same interval of continuity of  $\varphi(t)$  and  $|t' - t''| < \delta$ , then  $\|\varphi(t') - \varphi(t'')\| < \epsilon$ .

- (iii) For every  $\epsilon > 0$ , there exists a relatively dense set  $T$  such that, if  $\tau \in T$ , then  $\|\varphi(t+\tau) - \varphi(t)\| < \epsilon$  for all  $t \in \mathbb{R}$  satisfying the condition  $|t - t_k| > \epsilon, k \in \mathbb{Z}^+$ .

**Definition 4.** The response system (5) and the drive system (1) are said to be *globally exponentially synchronized* if there exist positive constants  $M > 0$  and  $\lambda > 0$  such that

$$\|y(t) - x(t)\| \leq M\|\psi - \varphi\|_0 e^{-\lambda t},$$

where

$$\|y(t) - x(t)\| = \max_{i \in \Pi} \{|y_i(t) - x_i(t)|\},$$

$$\|\psi - \varphi\|_0 = \max_{i \in \Pi} \left\{ \sup_{t \in [-\nu, 0]} |\psi_i(t) - \varphi_i(t)| \right\}.$$

**Theorem 2.** Let  $(A_1)$ – $(A_3)$  hold and suppose further the following:

- $(A_4)$   $\kappa_i \in PC(\mathbb{R}, \mathbb{R}^+)$ ,  $e_{ij} \in PC(\mathbb{R}, \mathbb{R})$  and  $\xi_{ij} \in PC^1(\mathbb{R}, \mathbb{R}^+)$  with  $\varsigma = \inf_{t \in \mathbb{R}} \{1 - \xi'_{ij}(t)\} > 0$  are almost periodic, where  $i, j \in \Pi$ .

- $(A_5)$  For any  $u, v \in \mathbb{R}$ , there exists a positive constant  $L_j^l$  such that

$$|l_j(u) - l_j(v)| \leq L_j^l |u - v|$$

and  $l_j(0) = 0$ , where  $j \in \Pi$ .

- $(A_6)$  The impulsive operator  $J_{ik}(z_i(t_k^-))$  satisfies  $J_{ik}(z_i(t_k^-)) = -\lambda_{ik}(z_i(t_k^-))$ ,  $0 < \lambda_{ik} < 2$ ,  $i \in \Pi$ ,  $k \in \mathbb{Z}^+$ .



(A<sub>7</sub>) There exists a positive constant  $\omega$  such that

$$\begin{aligned} \max_{i \in \Pi} \left\{ \omega - (a_i^- + \kappa_i^-) + \sum_{j=1}^n b_{ij}^+ L_j^f \right. \\ \left. + \frac{e^{\omega\sigma}}{\gamma} \sum_{j=1}^n c_{ij}^+ L_j^g + \frac{e^{\omega\tau}}{\rho} \sum_{j=1}^n \alpha_{ij}^+ L_j^h \right. \\ \left. + \frac{e^{\omega\eta}}{\varrho} \sum_{j=1}^n \beta_{ij}^+ L_j^l + \frac{e^{\omega\xi}}{\varsigma} \sum_{j=1}^n e_{ij}^+ L_j^l \right\} < 0. \end{aligned}$$

Then the drive system (1) and the response system (5) are globally exponentially synchronized.

*Proof.* For  $t > 0$ , and  $t \neq t_k, k \in \mathbb{Z}^+$ , we have

$$\begin{aligned} |z_i'(t)| &= |y_i'(t) - x_i'(t)| \\ &= \left| -a_i(t)[y_i(t) - x_i(t)] + \sum_{j=1}^n b_{ij}(t) \right. \\ &\quad \times [f_j(y_j(t)) - f_j(x_j(t))] \\ &\quad + \sum_{j=1}^n c_{ij}(t)[g_j(y_j(t - \sigma_{ij}(t))) \\ &\quad - g_j(x_j(t - \sigma_{ij}(t)))] \\ &\quad + \bigwedge_{j=1}^n \alpha_{ij}(t)[h_j(y_j(t - \tau_{ij}(t))) \\ &\quad - h_j(x_j(t - \tau_{ij}(t)))] \\ &\quad + \bigvee_{j=1}^n \beta_{ij}(t)[h_j(y_j(t - \eta_{ij}(t))) \\ &\quad - h_j(x_j(t - \eta_{ij}(t)))] - \kappa_i(t)z_i(t) \\ &\quad \left. + \sum_{j=1}^n e_{ij}(t)l_j(z_j(t - \xi_{ij}(t))) \right| \\ &\leq -(a_i^- + \kappa_i^-)|z_i(t)| + \sum_{j=1}^n b_{ij}^+ L_j^f |z_j(t)| \\ &\quad + \sum_{j=1}^n c_{ij}^+ L_j^g |z_j(t - \sigma_{ij}(t))| \\ &\quad + \sum_{j=1}^n \alpha_{ij}^+ L_j^h |z_j(t - \tau_{ij}(t))| \\ &\quad + \sum_{j=1}^n \beta_{ij}^+ L_j^h |z_j(t - \eta_{ij}(t))| \\ &\quad + \sum_{j=1}^n e_{ij}^+ L_j^l |z_j(t - \xi_{ij}(t))|, \quad i \in \Pi. \end{aligned}$$

Also, for  $t = t_k, k \in \mathbb{Z}^+$ , we obtain

$$z_i(t_k^+) = z_i(t_k^-) - \lambda_{ik}(z_i(t_k^-))$$

$$= (1 - \lambda_{ik})(z_i(t_k^-)), \quad i \in \Pi. \quad (9)$$

Then, by (A<sub>6</sub>) and (9), we obtain

$$|z_i(t_k^+)| \leq |z_i(t_k^-)|, \quad k \in \mathbb{Z}^+, \quad i \in \Pi.$$

With  $W_i(t) = e^{\omega t}|z_i(t)|$ , for  $t \neq t_k$ , we have

$$W_i'(t) \leq \omega W_i(t) + e^{\omega t}|z_i'(t)|, \quad i \in \Pi$$

and for  $t = t_k$ , we have

$$\begin{aligned} W_i(t_k^+) &= e^{\omega t_k^+}|z_i(t_k^+)| \leq e^{\omega t_k^-}|z_i(t_k^-)| \\ &= W_i(t_k^-), \quad k \in \mathbb{Z}^+, \quad i \in \Pi. \end{aligned}$$

Let us construct a Lyapunov function  $V(t)$  as follows:

$$V(t) = \sum_{i=1}^n \{W_i(t) + \Lambda_i(t)\},$$

where  $i \in \Pi$  and

$$\begin{aligned} \Lambda_i(t) &= \frac{e^{\omega\sigma}}{\gamma} \sum_{j=1}^n c_{ij}^+ L_j^g \int_{t-\sigma_{ij}(t)}^t |z_j(s)| e^{\omega s} ds \\ &\quad + \frac{e^{\omega\tau}}{\rho} \sum_{j=1}^n \alpha_{ij}^+ L_j^h \int_{t-\tau_{ij}(t)}^t |z_j(s)| e^{\omega s} ds \\ &\quad + \frac{e^{\omega\eta}}{\varrho} \sum_{j=1}^n \beta_{ij}^+ L_j^h \int_{t-\eta_{ij}(t)}^t |z_j(s)| e^{\omega s} ds \\ &\quad + \frac{e^{\omega\xi}}{\varsigma} \sum_{j=1}^n e_{ij}^+ L_j^l \int_{t-\xi_{ij}(t)}^t |z_j(s)| e^{\omega s} ds. \end{aligned}$$

For  $t \neq t_k$ , calculating the Dini right-hand derivative of  $V(t)$  along the solutions of the error system (6), we obtain

$$\begin{aligned} D^+V(t) &\leq \sum_{i=1}^n \left\{ \omega W_i(t) + e^{\omega t}|z_i'(t)| + \frac{e^{\omega\sigma}}{\gamma} \sum_{j=1}^n c_{ij}^+ \right. \\ &\quad \times L_j^g |z_j(t)| e^{\omega t} - \frac{e^{\omega\sigma}}{\gamma} \sum_{j=1}^n c_{ij}^+ L_j^g \\ &\quad \times |z_j(t - \sigma_{ij}(t))| e^{\omega(t - \sigma_{ij}(t))} (1 - \sigma'_{ij}(t)) \\ &\quad + \frac{e^{\omega\tau}}{\rho} \sum_{j=1}^n \alpha_{ij}^+ L_j^h |z_j(t)| e^{\omega t} \\ &\quad - \frac{e^{\omega\tau}}{\rho} \sum_{j=1}^n \alpha_{ij}^+ L_j^h |z_j(t - \tau_{ij}(t))| \\ &\quad \times e^{\omega(t - \tau_{ij}(t))} (1 - \tau'_{ij}(t)) + \frac{e^{\omega\eta}}{\varrho} \sum_{j=1}^n \beta_{ij}^+ \\ &\quad \times L_j^h |z_j(t)| e^{\omega t} - \frac{e^{\omega\eta}}{\varrho} \sum_{j=1}^n \beta_{ij}^+ L_j^h |z_j(t) \end{aligned}$$

$$\begin{aligned}
 & -\eta_{ij}(t)|e^{\omega(t-\eta_{ij}(t))}(1-\eta'_{ij}(t)) \\
 & + \frac{e^{\omega\xi}}{\varsigma} \sum_{j=1}^n e_{ij}^+ L_j^l |z_j(t)|e^{\omega t} - \frac{e^{\omega\xi}}{\varsigma} \sum_{j=1}^n e_{ij}^+ L_j^l \\
 & \times |z_j(t-\xi_{ij}(t))|e^{\omega(t-\xi_{ij}(t))}(1-\xi'_{ij}(t)) \Big\} \\
 \leq & \sum_{i=1}^n \left\{ \omega W_i(t) + e^{\omega t} \left[ -(a_i^- + \kappa_i^-)|z_i(t)| \right. \right. \\
 & + \sum_{j=1}^n b_{ij}^+ L_j^f |z_j(t)| + \sum_{j=1}^n c_{ij}^+ L_j^g |z_j(t-\sigma_{ij}(t))| \\
 & + \sum_{j=1}^n \alpha_{ij}^+ L_j^h |z_j(t-\tau_{ij}(t))| + \sum_{j=1}^n \beta_{ij}^+ L_j^h |z_j(t \\
 & - \eta_{ij}(t))| + \sum_{j=1}^n e_{ij}^+ L_j^l |z_j(t-\xi_{ij}(t))| \Big] \\
 & + \frac{e^{\omega\sigma}}{\gamma} \sum_{j=1}^n c_{ij}^+ L_j^g |z_j(t)|e^{\omega t} - \frac{e^{\omega\sigma}}{\gamma} \sum_{j=1}^n c_{ij}^+ L_j^g \\
 & \times |z_j(t-\sigma_{ij}(t))|e^{\omega(t-\sigma_{ij}(t))}(1-\sigma'_{ij}(t)) \\
 & + \frac{e^{\omega\tau}}{\rho} \sum_{j=1}^n \alpha_{ij}^+ L_j^h |z_j(t)|e^{\omega t} - \frac{e^{\omega\tau}}{\rho} \sum_{j=1}^n \alpha_{ij}^+ L_j^h \\
 & \times |z_j(t-\tau_{ij}(t))|e^{\omega(t-\tau_{ij}(t))}(1-\tau'_{ij}(t)) \\
 & + \frac{e^{\omega\eta}}{\varrho} \sum_{j=1}^n \beta_{ij}^+ L_j^h |z_j(t)|e^{\omega t} - \frac{e^{\omega\eta}}{\varrho} \sum_{j=1}^n \beta_{ij}^+ L_j^h \\
 & \times |z_j(t-\eta_{ij}(t))|e^{\omega(t-\eta_{ij}(t))}(1-\eta'_{ij}(t)) \\
 & + \frac{e^{\omega\xi}}{\varsigma} \sum_{j=1}^n e_{ij}^+ L_j^l |z_j(t)|e^{\omega t} - \frac{e^{\omega\xi}}{\varsigma} \sum_{j=1}^n e_{ij}^+ L_j^l \\
 & \times |z_j(t-\xi_{ij}(t))|e^{\omega(t-\xi_{ij}(t))}(1-\xi'_{ij}(t)) \Big\} \\
 \leq & \sum_{i=1}^n \left\{ W_i(t) \left[ \omega - (a_i^- + \kappa_i^-) + \sum_{j=1}^n b_{ij}^+ L_j^f \right. \right. \\
 & + \frac{e^{\omega\sigma}}{\gamma} \sum_{j=1}^n c_{ij}^+ L_j^g + \frac{e^{\omega\tau}}{\rho} \sum_{j=1}^n \alpha_{ij}^+ L_j^h \\
 & \left. \left. + \frac{e^{\omega\eta}}{\varrho} \sum_{j=1}^n \beta_{ij}^+ L_j^h + \frac{e^{\omega\xi}}{\varsigma} \sum_{j=1}^n e_{ij}^+ L_j^l \right] \right\}. \tag{10}
 \end{aligned}$$

From Assumption  $A_6$ , we can calculate the right limits of Lyapunov functional  $V$  at impulsive moments  $t_k$  as follows:

$$\begin{aligned}
 V(t_k^+) = & \sum_{i=1}^n \left\{ W_i(t_k^+) + \frac{e^{\omega\sigma}}{\gamma} \sum_{j=1}^n c_{ij}^+ L_j^g \right. \\
 & \times \int_{t_k^+ - \sigma_{ij}(t_k^+)}^{t_k^+} e^{\omega s} |z_j(s)| ds + \frac{e^{\omega\tau}}{\rho} \sum_{j=1}^n \alpha_{ij}^+
 \end{aligned}$$

$$\begin{aligned}
 & \times L_j^h \int_{t_k^+ - \tau_{ij}(t_k^+)}^{t_k^+} e^{\omega s} |z_j(s)| ds \\
 & + \frac{e^{\omega\eta}}{\varrho} \sum_{j=1}^n \beta_{ij}^+ L_j^h \int_{t_k^+ - \eta_{ij}(t_k^+)}^{t_k^+} e^{\omega s} |z_j(s)| ds \\
 & \left. + \frac{e^{\omega\xi}}{\varsigma} \sum_{j=1}^n e_{ij}^+ L_j^l \int_{t_k^+ - \xi_{ij}(t_k^+)}^{t_k^+} |z_j(s)| e^{\omega s} ds \right\} \\
 \leq & \sum_{i=1}^n \left\{ W_i(t_k^-) + \frac{e^{\omega\sigma}}{\gamma} \sum_{j=1}^n c_{ij}^+ L_j^g \right. \\
 & \times \int_{t_k^- - \sigma_{ij}(t_k^-)}^{t_k^-} e^{\omega s} |z_j(s)| ds \\
 & + \frac{e^{\omega\tau}}{\rho} \sum_{j=1}^n \alpha_{ij}^+ L_j^h \int_{t_k^- - \tau_{ij}(t_k^-)}^{t_k^-} e^{\omega s} |z_j(s)| ds \\
 & + \frac{e^{\omega\eta}}{\varrho} \sum_{j=1}^n \beta_{ij}^+ L_j^h \int_{t_k^- - \eta_{ij}(t_k^-)}^{t_k^-} e^{\omega s} |z_j(s)| ds \\
 & \left. + \frac{e^{\omega\xi}}{\varsigma} \sum_{j=1}^n e_{ij}^+ L_j^l \int_{t_k^- - \xi_{ij}(t_k^-)}^{t_k^-} |z_j(s)| e^{\omega s} ds \right\} \\
 = & V(t_k^-), \quad k \in \mathbb{Z}^+. \tag{11}
 \end{aligned}$$

In view of Assumption  $A_7$ , (10) and (11), we obtain

$$D^+V(t) \leq 0,$$

which implies  $V(t) \leq V(0)$  for all  $t \geq 0$ .

On the other hand, we have

$$\begin{aligned}
 V(0) = & \sum_{i=1}^n \left\{ W_i(0) + \frac{e^{\omega\sigma}}{\gamma} \sum_{j=1}^n c_{ij}^+ L_j^g \right. \\
 & \times \int_{-\sigma_{ij}(0)}^0 e^{\omega s} |z_j(s)| ds + \frac{e^{\omega\tau}}{\rho} \sum_{j=1}^n \alpha_{ij}^+ L_j^h \\
 & \times \int_{-\tau_{ij}(0)}^0 e^{\omega s} |z_j(s)| ds + \frac{e^{\omega\eta}}{\varrho} \sum_{j=1}^n \beta_{ij}^+ L_j^h \\
 & \times \int_{-\eta_{ij}(0)}^0 e^{\omega s} |z_j(s)| ds \\
 & \left. + \frac{e^{\omega\xi}}{\varsigma} \sum_{j=1}^n e_{ij}^+ L_j^l \int_{-\xi_{ij}(0)}^0 |z_j(s)| e^{\omega s} ds \right\} \\
 \leq & \sum_{i=1}^n \left[ 1 + \frac{e^{\omega\sigma} - 1}{\omega\gamma} \sum_{j=1}^n c_{ij}^+ L_j^g \right. \\
 & + \frac{e^{\omega\tau} - 1}{\omega\rho} \sum_{j=1}^n \alpha_{ij}^+ L_j^h \\
 & + \frac{e^{\omega\eta} - 1}{\omega\varrho} \sum_{j=1}^n \beta_{ij}^+ L_j^h \\
 & \left. + \frac{e^{\omega\xi} - 1}{\omega\varsigma} \sum_{j=1}^n e_{ij}^+ L_j^l \right] \|\psi - \varphi\|_0.
 \end{aligned}$$



It is easy to check that

$$\begin{aligned} \|y(t) - x(t)\| &\leq V(t)e^{-\omega t} \leq V(0)e^{-\omega t} \\ &\leq Me^{-\omega t}\|\psi - \varphi\|_0, \quad t \geq 0, \end{aligned}$$

where

$$\begin{aligned} M &= \sum_{i=1}^n \left[ 1 + \frac{e^{\omega\sigma} - 1}{\omega\gamma} \sum_{j=1}^n c_{ij}^+ L_j^g \right. \\ &\quad + \frac{e^{\omega\tau} - 1}{\omega\rho} \sum_{j=1}^n \alpha_{ij}^+ L_j^h + \frac{e^{\omega\eta} - 1}{\omega\rho} \sum_{j=1}^n \beta_{ij}^+ L_j^h \\ &\quad \left. + \frac{e^{\omega\xi} - 1}{\omega\varsigma} \sum_{j=1}^n e_{ij}^+ L_j^l \right] > 0. \end{aligned}$$

Therefore, the drive system (1) and the response system (5) are globally exponentially synchronized. The proof is complete. ■

**Remark 2.** Li *et al.* (2018b; 2018a) considered almost periodic synchronization and pseudo almost periodic synchronization for quaternion-valued cellular neural networks with time-varying delays, but the controllers in those works are only state feedback ones, not the state feedback and the impulsive controller. In addition, the networks considered in these works are deterministic, rather than fuzzy networks. Therefore, Theorem 2 is new.

### 5. Numerical example

In this section, an example is given to show the effectiveness of the result obtained in this paper.

**Example 1.** Consider the following fuzzy cellular neural networks as the drive system:

$$\begin{aligned} x_i'(t) &= -a_i(t)x_i(t) + \sum_{j=1}^2 b_{ij}(t)f_j(x_j(t)) \\ &\quad + \sum_{j=1}^2 c_{ij}(t)g_j(x_j(t - \sigma_{ij}(t))) \\ &\quad + \sum_{j=1}^2 d_{ij}(t)\mu_j(t) + \bigwedge_{j=1}^2 T_{ij}(t)\mu_j(t) \\ &\quad + \bigwedge_{j=1}^2 \alpha_{ij}(t)h_j(x_j(t - \tau_{ij}(t))) \\ &\quad + \bigvee_{j=1}^2 \beta_{ij}(t)h_j(x_j(t - \eta_{ij}(t))) \\ &\quad + \bigvee_{j=1}^2 S_{ij}(t)\mu_j(t) + I_i(t). \end{aligned} \tag{12}$$

The corresponding response system is given by

$$\begin{aligned} y_i'(t) &= -a_i(t)y_i(t) + \sum_{j=1}^2 b_{ij}(t)f_j(y_j(t)) \\ &\quad + \sum_{j=1}^2 c_{ij}(t)g_j(y_j(t - \sigma_{ij}(t))) \\ &\quad + \sum_{j=1}^2 d_{ij}(t)\mu_j(t) + \bigwedge_{j=1}^2 T_{ij}(t)\mu_j(t) \\ &\quad + \bigwedge_{j=1}^2 \alpha_{ij}(t)h_j(y_j(t - \tau_{ij}(t))) \\ &\quad + \bigvee_{j=1}^2 \beta_{ij}(t)h_j(y_j(t - \eta_{ij}(t))) \\ &\quad + \bigvee_{j=1}^2 S_{ij}(t)\mu_j(t) + I_i(t) + u_i(t), \end{aligned} \tag{13}$$

and the controller is as follows:

$$\begin{aligned} u_i(t) &= \sum_{k=1}^{\infty} J_{ik}(z_i(t))\delta(t - t_k) - \kappa_i(t)z_i(t) \\ &\quad + \sum_{j=1}^n e_{ij}(t)l_j(z_j(t - \xi_{ij}(t))), \end{aligned} \tag{14}$$

where  $i = 1, 2$ , and the coefficients are

$$\begin{aligned} f_1(x) &= f_2(x) = 0.3 \sin x, \\ g_1(x) &= g_2(x) = 0.2 \tanh x, \\ h_1(x) &= h_2(x) = 0.5(|x + 1| + |x| - 1), \\ l_1(x) &= l_2(x) = 0.4|x|, \end{aligned}$$

$$\begin{aligned} a_1(t) &= \sin \sqrt{2}t + 3, & a_2(t) &= \cos 2t + 4, \\ I_1(t) &= 0.1 \sin 2t, & I_1(t) &= 0.1 \sin \sqrt{3}t, \\ \mu_1(t) &= \sin t, & \mu_2(t) &= \cos t, \\ b_{11}(t) &= 0.1 \sin 5t, & b_{12}(t) &= 0.2 \cos 2t, \\ b_{21}(t) &= 0.4 \sin t, & b_{22}(t) &= 0.5 \cos t, \\ c_{11}(t) &= 0.3 \sin 7t, & c_{12}(t) &= 0.1 \cos t, \\ c_{21}(t) &= 0.4 \cos 3t, & c_{22}(t) &= 0.2 \sin \sqrt{2}t, \\ d_{11}(t) &= 0.5 \sin \sqrt{2}t, & d_{12}(t) &= 0.5 \sin 5t, \\ d_{21}(t) &= 0.4 \cos \sqrt{3}t, & d_{22}(t) &= 0.5 \sin t, \\ \alpha_{11}(t) &= 0.25 \cos t, & \alpha_{12}(t) &= 0.5 \sin 2t, \\ \alpha_{21}(t) &= 0.6 \sin 5t, & \alpha_{22}(t) &= 0.3 \sin \sqrt{3}t, \\ \beta_{11}(t) &= 0.2 \cos \sqrt{3}t, & \beta_{12}(t) &= 0.1 \sin \sqrt{5}t, \\ \beta_{21}(t) &= 0.3 \cos t, & \beta_{22}(t) &= 0.45 \sin t, \\ T_{11}(t) &= 0.3 \sin 2t, & T_{12}(t) &= 0.4 \cos t, \end{aligned}$$

$$\begin{aligned}
 T_{21}(t) &= \sin \sqrt{2}t, & T_{22}(t) &= \cos \sqrt{3}t, & & \approx \max\{-0.5918, -0.8198\} \\
 S_{11}(t) &= 0.5 \sin t, & S_{12}(t) &= 0.6 \cos t, & & = -0.5918 < 0. \\
 S_{21}(t) &= \sin 3t, & S_{22}(t) &= \sin \sqrt{3}t, & & \\
 \kappa_1(t) &= \cos \sqrt{2}t + 3, & \kappa_2(t) &= \sin t + 4, & & \\
 e_{11}(t) &= 0.1 \sin \sqrt{2}t, & e_{12}(t) &= 0.2 \sin t, & & \\
 e_{21}(t) &= 0.3 \sin 2t, & e_{22}(t) &= 0.2 \cos \sqrt{3}t, & & \\
 \sigma_{ij}(t) &= 0.4 \sin^2 t, & \tau_{ij}(t) &= 0.2 \cos^2 t, & & \\
 \eta_{ij}(t) &= 0.3 |\sin t|, & \xi_{ij}(t) &= 0.5 |\cos t|. & & 
 \end{aligned}$$

By simple calculation, we obtain

$$a_1^- = 2, \quad \kappa_1^- = 2, \quad a_2^- = 3, \quad \kappa_2^- = 3,$$

$$\begin{aligned}
 L_1^f &= L_2^f = 0.3, & L_1^g &= L_2^g = 0.2, \\
 L_1^h &= L_2^h = 1, & L_1^l &= L_2^l = 0.4,
 \end{aligned}$$

$$\begin{aligned}
 b_{11}^+ &= 0.1, & b_{12}^+ &= 0.2, & b_{21}^+ &= 0.4, & b_{22}^+ &= 0.5, \\
 c_{11}^+ &= 0.3, & c_{12}^+ &= 0.1, & c_{21}^+ &= 0.4, & c_{22}^+ &= 0.2, \\
 d_{11}^+ &= 0.5, & d_{12}^+ &= 0.5, & d_{21}^+ &= 0.4, & d_{22}^+ &= 0.5, \\
 \alpha_{11}^+ &= 0.25, & \alpha_{12}^+ &= 0.5, & \alpha_{21}^+ &= 0.6, & \alpha_{22}^+ &= 0.3, \\
 \beta_{11}^+ &= 0.2, & \beta_{12}^+ &= 0.1, & \beta_{21}^+ &= 0.3, & \beta_{22}^+ &= 0.45, \\
 T_{11}^+ &= 0.3, & T_{12}^+ &= 0.4, & T_{21}^+ &= 1, & T_{22}^+ &= 1, \\
 S_{11}^+ &= 0.5, & S_{12}^+ &= 0.6, & S_{21}^+ &= 1, & S_{22}^+ &= 1,
 \end{aligned}$$

$$I_1^+ = I_2^+ = 0.1, \quad \mu_1^+ = \mu_2^+ = 1,$$

$$\begin{aligned}
 e_{11}^+ &= 0.1, & e_{12}^+ &= 0.2, & e_{21}^+ &= 0.3, & e_{22}^+ &= 0.2, \\
 \sigma &= 0.4, & \tau &= 0.2, & \eta &= 0.3, & \xi &= 0.5, \\
 P_1 &= 1.22, & P_2 &= 2.04.
 \end{aligned}$$

Then we obtain

$$\max \left\{ \frac{P_1}{a_1^-}, \frac{P_2}{a_2^-} \right\} \approx \max\{0.61, 0.68\} = 0.68 < 1,$$

and therefore all the conditions of Theorems 1 are satisfied, so the drive system (12) has an almost periodic solution. Moreover, take  $t_k = 0.5 + 2(k - 1)\pi$ ,  $k \in \mathbb{Z}^+$  and impulsive functions  $J_{ik}(z_i) = -(1 + \frac{1}{3} \sin(2 + k))z_i$ ,  $i = 1, 2$ ,  $\omega = 1$ ,  $\gamma = 0.6$ ,  $\rho = 0.8$ ,  $\varrho = 0.7$ ,  $\varsigma = 0.5$ ; then we get

$$\begin{aligned}
 \max_{i=1,2} & \left\{ \omega - (a_i^- + \kappa_i^-) + \sum_{j=1}^n b_{ij}^+ L_j^f \right. \\
 & + \frac{e^{\omega\sigma}}{\gamma} \sum_{j=1}^n c_{ij}^+ L_j^g + \frac{e^{\omega\tau}}{\rho} \sum_{j=1}^n \alpha_{ij}^+ L_j^h \\
 & \left. + \frac{e^{\omega\eta}}{\varrho} \sum_{j=1}^n \beta_{ij}^+ L_j^l + \frac{e^{\omega\xi}}{\varsigma} \sum_{j=1}^n e_{ij}^+ L_j^l \right\}
 \end{aligned}$$

Thus, Assumption  $A_7$  is also satisfied. Therefore, the systems (12) and (13) can be globally exponentially synchronized (see Figs. 1–6).

### 6. Conclusion

In this paper, we considered the problem of almost periodic synchronization of fuzzy cellular neural networks with time-varying delays via state-feedback and impulsive control. By applying the Banach fixed point theorem, constructing a suitable Lyapunov function and designing a state-feedback and impulsive control, we obtained that the drive-response structure of fuzzy cellular neural networks with almost periodic coefficients realize exponential synchronization. We know that almost periodic synchronization for fuzzy cellular neural networks with time-varying delays via state-feedback and impulsive control is new. Our results are completely original and our methods can be used to study the problem of almost periodic and automorphic synchronization for other types of neural networks.



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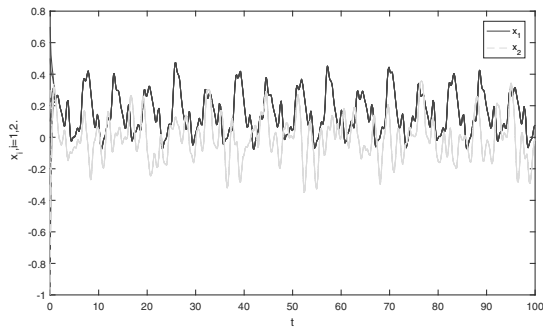


Fig. 1. States of  $x_1$  and  $x_2$ .

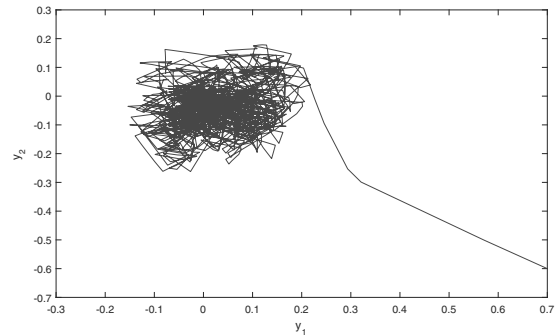


Fig. 4. Phase diagram of  $y_1$  and  $y_2$ .

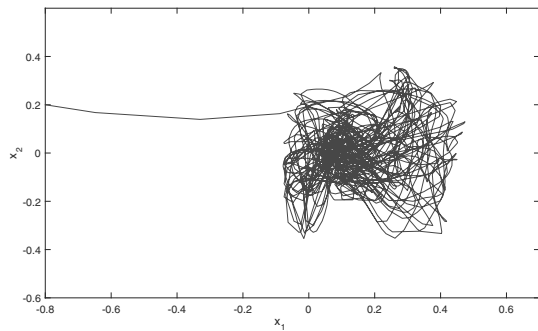


Fig. 2. Phase diagram of  $x_1$  and  $x_2$ .

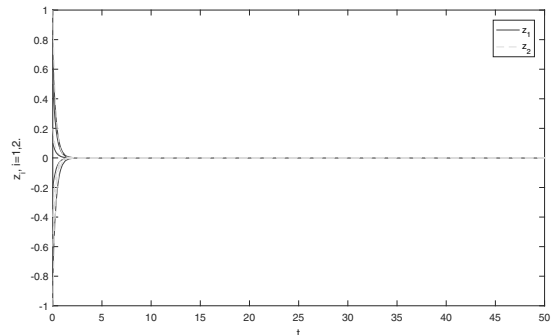


Fig. 5. Drive system (12) and its response system (13) are synchronized.

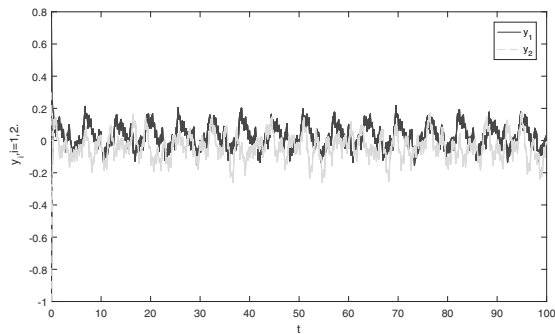


Fig. 3. States of  $y_1$  and  $y_2$ .

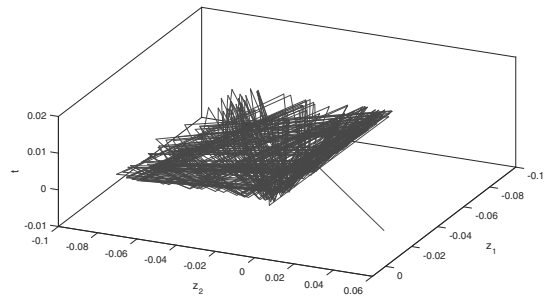


Fig. 6. Curves of  $z$  in a 3-dimensional space for the synchronization case.

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